Statistics and Applications {ISSN 2454-7395 (online)}
Volume 23, No. 2, 2025 (New Series), pp 317–321
http://www.ssca.org.in/journal

Resolvability of Some BIB and Regular Group Divisible Designs

Shyam Saurabh[1], Sheet Nihal Topno[2] and Dinesh Prasad[3]
[1] *Department of Mathematics, Tata College, Kolhan University, Chaibasa, Jharkhand, India*
[2] *University Department of Mathematics, Ranchi University, Ranchi, Jharkhand, India*
[3] *Department of Mathematics, Gossner College, Ranchi, Jharkhand, India*

Received: 03 October 2024; Revised: 28 October 2024; Accepted: 29 December 2024

Abstract

$(\mu_1, \mu_2, \ldots \mu_t)$-resolvable solutions of some BIB and regular group divisible designs are obtained by decomposing the $v(= mn) \times b$ incidence matrix into smaller circulant submatrices of orders $m \times t$.

Key words: Regular group divisible design; Balanced incomplete block (BIB) design; Permutation circulant matrix; Resolvability.

AMS Subject Classifications: 62K10, 05B05

1. Introduction

Let the incidence matrix N of a block design $D(v, r, k, b)$ may be decomposed into submatrices as $N = (N_1|N_2|\ldots|N_t)$ such that each row sum of N_i $(1 \le i \le t)$ is μ_i. Then the design is $(\mu_1, \mu_2, \ldots, \mu_t)$-resolvable [see Kageyama (1976), Saurabh (2024b)]. If $\mu_1 = \mu_2 = \cdots = \mu_t = \mu$ then the design is μ-resolvable. Such designs are also denoted as A-resolvable in combinatorial design theory [see Ge and Miao (2007)]. A practical application of such designs may be found in Kageyama (1976). The resolvable solutions obtained here are not earlier reported in the Tables of Clatworthy (1973) and Saurabh *et al.* (2021) [see Table 1].

Let $v = mn$ elements be arranged in an $m \times n$ array. A *regular group divisible* (RGD) design is an arrangement of the $v = mn$ elements in b blocks each of size k such that:

1. Every element occurs at most once in a block;
2. Every element occurs in r blocks;
3. Every pair of elements, which are in the same row of the $m \times n$ array, occur together in λ_1 blocks whereas every other pair of elements occur together in λ_2 blocks;
4. $r - \lambda_1 > 0$ and $rk - v\lambda_2 > 0$.

Further let N be the incidence matrix of a GD design then the structure of $NN^\top$ is given as [see Saurabh and Sinha (2023) for GD association schemes]:

Corresponding Author: Shyam Saurabh
Email: shyamsaurabh785@gmail.com

(i) $NN^\top = (r-\lambda_1)(I_m \otimes I_n) + (\lambda_1-\lambda_2)(I_m \otimes J_n) + \lambda_2(J_m \otimes J_n)$; or
(ii) $NN^\top = (r-\lambda_2)(I_n \otimes I_m) + \lambda_2(J_n \otimes J_m) + (\lambda_1-\lambda_2)\{(J_n - I_n)\otimes I_m\}$,
where J_n is an $n \times n$ matrix all of whose entries are 1.

Notations: I_n is the identity matrix of order n, $A \otimes B$ is the Kronecker product of two matrices A and B, $0_{m\times n}$ is a null matrix of order $m \times n$ and $N^\top$ is transpose of the matrix N. In Section 2, N represents incidence matrix of the design whereas RX and RXa numbers are from Clatworthy (1973) and Freeman (1976) respectively.

2. Resolvable solutions

2.1. $(\mu_1, \mu_2, \ldots, \mu_t)$-resolvable GD designs

Here $\alpha = circ(010\ldots0)$ is a permutation circulant matrix of order m such that $\alpha^m = I_m$.

Table 1: $(\mu_1, \mu_2, \ldots, \mu_t)$-resolvable RGD designs

No.	Design	v	r	k	b	λ_1	λ_2	m	n	Present Status
1	$R89a$	18	10	3	60	4	1	9	2	(1,3,3,3)-resolvable
2	$R109a$	12	7	4	21	1	2	6	2	(2,2,3)-resolvable
3	$R113a$	14	10	4	35	6	2	7	2	(2,2,2,4)-resolvable
4	$R123a$	18	10	4	45	0	2	6	3	(2,4,4)-resolvable
5	$R124a$	22	8	4	44	4	1	11	2	(2,2,4)-resolvable
6	$R126a$	24	9	4	54	5	1	12	2	(1,2,2,4)-resolvable
7	$R128a$	26	10	4	65	6	1	13	2	(2,2,2,4)-resolvable
8	$R152a$	22	10	5	44	11	2	0	2	5-resolvable
9	$R163$	45	10	5	90	0	1	9	5	5-resolvable
10	$R167a$	12	9	6	18	5	4	6	2	(3,6)-resolvable

1. $R89a$: $v = 18$, $r = 10$, $k = 3$, $b = 60$, $\lambda_1 = 4$, $\lambda_2 = 1$, $m = 9$, $n = 2$.
   Consider a block matrix
   $$N = (N_1|N_2|N_3|N_4) =$$
   $$\left(\begin{array}{cc|cc|cc|cc} P & 0_{9\times3} & \alpha+\alpha^2 & \alpha & \alpha+\alpha^8 & \alpha^4 & \alpha+\alpha^6 & \alpha \\ 0_{9\times3} & P & \alpha & \alpha+\alpha^2 & \alpha & \alpha+\alpha^8 & \alpha^4 & \alpha+\alpha^6 \end{array}\right),$$
   where $P = \begin{pmatrix}1\\1\\1\end{pmatrix} \otimes circ(010)$. Then $NN^\top = 9(I_2 \otimes I_9) + (J_2 \otimes J_9) + 3\{(J_2 - I_2)\otimes I_9\}$.
   Hence N represents the incidence matrix of $R89a$. Further since each row sum of N_1, N_2, N_3 and N_4 is 1, 3, 3 and 3 respectively, the design is $(1,3,3,3)$-resolvable.
2. $R109a$: $v = 12$, $r = 7$, $k = 4$, $b = 21$, $\lambda_1 = 1$, $\lambda_2 = 2$, $m = 6$, $n = 2$.
   $$N = (N_1|N_2|N_3) = \left(\begin{array}{c|c|cc} \alpha+\alpha^3 & \alpha+\alpha^4 & \alpha+\alpha^2+\alpha^3 & 0_{6\times3} \\ \alpha+I_6 & \alpha^2+I_6 & \alpha^4 & P \end{array}\right),$$
   where $P = \begin{pmatrix}1\\1\end{pmatrix} \otimes circ(011)$. Since each row sum of N_1, N_2 and N_3 is 2, 2 and 3 respectively, the design is $(2,2,3)$-resolvable.

3. $R113a$: $v = 14$, $r = 10$, $k = 4$, $b = 35$, $\lambda_1 = 6$, $\lambda_2 = 2$, $m = 7$, $n = 2$.

$$N = (N_1|N_2|N_3|N_4) = \left(\begin{array}{c|c|c|cc} \alpha+\alpha^2 & \alpha+\alpha^3 & \alpha+\alpha^4 & \alpha+\alpha^2+\alpha^4 & I_7 \\ \alpha+\alpha^2 & \alpha+\alpha^3 & \alpha+\alpha^4 & I_7 & \alpha+\alpha^2+\alpha^4 \end{array}\right).$$

Since each row sum of N_1, N_2, N_3 and N_4 is 2, 2, 2 and 4 respectively, the design is $(2,2,2,4)$-resolvable.

4. $R123a$: $v = 18$, $r = 10$, $k = 4$, $b = 45$, $\lambda_1 = 0$, $\lambda_2 = 2$, $m = 6$, $n = 3$.

$$N = (N_1|N_2|N_3) =$$

$$\left(\begin{array}{c|ccc|ccc} P_{6\times 9} & 0_6 & \alpha^4 & \alpha+\alpha^2+\alpha^3 & \alpha+\alpha^3 & \alpha^4 & \alpha^5 \\ Q_{6\times 9} & \alpha+\alpha^2+\alpha^3 & 0_6 & \alpha^4 & \alpha^5 & \alpha+\alpha^3 & \alpha^4 \\ R_{6\times 9} & \alpha^4 & \alpha+\alpha^2+\alpha^3 & 0_6 & \alpha^4 & \alpha^5 & \alpha+\alpha^3 \end{array}\right),$$

where $P_{6\times 9} = \left(\begin{array}{c|c|c} I_3 & 0_3 & \beta^2 \\ I_3 & 0_3 & \beta^2 \end{array}\right)$, $Q_{6\times 9} = \left(\begin{array}{c|c|c} \beta^2 & I_3 & 0_3 \\ \beta^2 & I_3 & 0_3 \end{array}\right)$, $R_{6\times 9} = \left(\begin{array}{c|c|c} 0_3 & \beta^2 & I_3 \\ 0_3 & \beta^2 & I_3 \end{array}\right)$ and $\beta = circ(010)$.

Since each row sum of N_1, N_2 and N_3 is 2, 4 and 4 respectively, the design is $(2,4,4)$-resolvable.

5. $R124a$: $v = 22$, $r = 8$, $k = 4$, $b = 44$, $\lambda_1 = 4$, $\lambda_2 = 1$, $m = 11$, $n = 2$.

$$N = (N_1|N_2|N_3) = \left(\begin{array}{c|c|cc} \alpha+\alpha^7 & \alpha+\alpha^8 & \alpha+\alpha^3+\alpha^4 & \alpha \\ \alpha+\alpha^8 & \alpha+\alpha^7 & \alpha & \alpha+\alpha^3+\alpha^4 \end{array}\right).$$

Since each row sum of N_1, N_2 and N_3 is 2, 2 and 4 respectively, the design is $(2,2,4)$-resolvable.

6. $R126a$: $v = 24$, $r = 9$, $k = 4$, $b = 54$, $\lambda_1 = 5$, $\lambda_2 = 1$, $m = 12$, $n = 2$.

$$N = (N_1|N_2|N_3|N_4) = \left(\begin{array}{c|c|c|cc} P & \alpha+\alpha^4 & \alpha+\alpha^{11} & \alpha+\alpha^2+\alpha^9 & \alpha \\ P & \alpha+\alpha^{11} & \alpha+\alpha^4 & \alpha & \alpha+\alpha^2+\alpha^9 \end{array}\right),$$

where $P = \begin{pmatrix} circ(010000) \\ I_6 \end{pmatrix}$. Since each row sum of N_1, N_2, N_3 and N_4 is 1, 2, 2 and 4 respectively, the design is $(1,2,2,4)$-resolvable.

7. $R128a$: $v = 26$, $r = 10$, $k = 4$, $b = 65$, $\lambda_1 = 6$, $\lambda_2 = 1$, $m = 13$, $n = 2$.

$$N = (N_1|N_2|N_3|N_4) = \left(\begin{array}{c|c|c|cc} \alpha+\alpha^2 & \alpha+\alpha^4 & \alpha+\alpha^5 & \alpha+\alpha^6+\alpha^8 & \alpha \\ \alpha+\alpha^4 & \alpha+\alpha^2 & \alpha+\alpha^5 & \alpha & \alpha+\alpha^6+\alpha^8 \end{array}\right).$$

Since each row sum of N_1, N_2, N_3 and N_4 is 2, 2, 2 and 4 respectively, the design is $(2,2,2,4)$-resolvable.

8. $R152a$: $v = 22$, $r = 10$, $k = 5$, $b = 44$, $\lambda_1 = 0$, $\lambda_2 = 2$, $m = 11$, $n = 2$.

$$N = (N_1|N_2) = \left(\begin{array}{cc|cc} \alpha+\alpha^2+\alpha^3+\alpha^6 & \alpha^9 & \alpha+\alpha^3+\alpha^8 & \alpha^2+\alpha^{10} \\ \alpha^9 & \alpha+\alpha^2+\alpha^3+\alpha^6 & \alpha^2+\alpha^{10} & \alpha+\alpha^3+\alpha^8 \end{array}\right).$$

Since each row sum of N_1 and N_2 is 5, the design is 5– resolvable [see Saurabh (2024a)].

9. $R163$: $v = 45$, $r = 10$, $k = 5$, $b = 90$, $\lambda_1 = 0$, $\lambda_2 = 1$, $m = 9$, $n = 5$.

A 5-resolvable solution of the design is presented in Table 2:

Table 2: Resolution classes of $R163$

Resolution class I				
(2, 3, 10, 22, 25)	(15, 17, 19, 32, 36)	(4, 7, 29, 30, 37)	(1, 14, 18, 42, 44)	(23, 25, 30, 40, 44)
(1, 3, 11, 23, 26)	(13, 18, 20, 33, 34)	(5, 8, 28, 30, 38)	(2, 15, 16, 40, 45)	(20, 27, 31, 39, 44)
(1, 2, 12, 24, 27)	(14, 16, 21, 31, 35)	(6, 9, 28, 29, 39)	(3, 13, 17, 41, 43)	(21, 25, 32, 37, 45)
(5, 6, 13, 19, 25)	(11, 18, 22, 30, 35)	(1, 7, 32, 33, 40)	(4, 12, 17, 38, 45)	(19, 26, 33, 38, 43)
(4, 6, 14, 20, 26)	(12, 16, 23, 28, 36)	(2, 8, 31, 33, 41)	(5, 10, 18, 39, 43)	(21, 23, 34, 38, 42)
(4, 5, 15, 21, 27)	(10, 17, 24, 29, 34)	(3, 9, 31, 32, 42)	(6, 11, 16, 37, 44)	(19, 24, 35, 39, 40)
(8, 9, 16, 19, 22)	(12, 14, 25, 29, 33)	(1, 4, 35, 36, 43)	(7, 11, 15, 39, 41)	(20, 22, 36, 37, 41)
(7, 9, 17, 20, 23)	(10, 15, 26, 30, 31)	(2, 5, 34, 36, 44)	(8, 12, 13, 37, 42)	(24, 26, 28, 41, 45)
(7, 8, 18, 21, 24)	(11, 13, 27, 28, 32)	(3, 6, 34, 35, 45)	(9, 10, 14, 38, 40)	(22, 27, 29, 42, 43)

Resolution class II				
(11, 12, 19, 31, 34)	(6, 8, 10, 23, 27)	(5, 9, 33, 35, 37)	(5, 11, 17, 40, 42)	(20, 21, 28, 40, 43)
(10, 12, 20, 32, 35)	(4, 9, 11, 24, 25)	(6, 7, 31, 36, 38)	(6, 12, 18, 40, 41)	(19, 21, 29, 41, 44)
(10, 11, 21, 33, 36)	(5, 7, 12, 22, 26)	(4, 8, 32, 34, 39)	(7, 10, 13, 44, 45)	(19, 20, 30, 42, 45)
(14, 15, 22, 28, 34)	(2, 9, 13, 21, 26)	(3, 8, 29, 36, 40)	(8, 11, 14, 43, 45)	(23, 24, 31, 37, 43)
(13, 15, 23, 29, 35)	(3, 7, 14, 19, 27)	(1, 9, 30, 34, 41)	(9, 12, 15, 43, 44)	(22, 24, 32, 38, 44)
(13, 14, 24, 30, 36)	(1, 8, 15, 20, 25)	(2, 7, 28, 35, 42)	(1, 13, 16, 38, 39)	(22, 23, 33, 39, 45)
(17, 18, 25, 28, 31)	(3, 5, 16, 20, 24)	(2, 6, 30, 32, 43)	(2, 14, 17, 37, 39)	(26, 27, 34, 37, 40)
(16, 18, 26, 29, 32)	(1, 6, 17, 21, 22)	(3, 4, 28, 33, 44)	(3, 15, 18, 37, 38)	(25, 27, 35, 38, 41)
(16, 17, 27, 30, 33)	(2, 4, 18, 19, 23)	(1, 5, 29, 31, 45)	(4, 10, 16, 41, 42)	(25, 26, 36, 39, 42)

Remark 1: Juxtaposing the resolution classes I and II of $R163$ with the resolution class III given below, we obtain a (1,5,5)-resolvable solution of BIB design with parameters: $v = 45$, $r = 11$, $k = 5$, $b = 99$, $\lambda = 1$. The blocks of resolution class III are rows of the GD scheme. This BIB design is listed as $T98$ in the Table of Takeuchi (1962).

Resolution class III: (1, 10, 19, 28, 37); (6, 15, 24, 33, 42); (2, 11, 20, 29,38); (7, 16, 25, 34, 43); (3, 12, 21, 30, 39); (8, 17, 26, 35, 44); (4, 13, 22, 31, 40); (9, 18, 27, 36, 45); (5, 14, 23, 32, 41).

10. $R167a$: $v = 12$, $r = 9$, $k = 6$, $b = 18$, $\lambda_1 = 5$, $\lambda_2 = 4$, $m = 6$, $n = 2$.

$$N = (N_1|N_2) = \left(\begin{array}{c|cc} \alpha+\alpha^2+\alpha^4 & I_6+\alpha+\alpha^2+\alpha^4 & \alpha^2+\alpha^3 \\ \alpha+\alpha^2+\alpha^4 & \alpha^2+\alpha^3 & I_6+\alpha+\alpha^2+\alpha^4 \end{array}\right).$$

Since each row sum of N_1 and N_2 is 3 and 6 respectively, the design is $(3, 6)$-resolvable.

2.2. A(2,2,2,2,6)-resolvable BIB design

The following solution of BIB design $T49 : v = 22, r = 14, k = 4, b = 77, \lambda = 2$ using the method of differences may be found in Takeuchi (1962):
$[(0_0, 1_0, 3_0, 10_1),\ (0_1, 1_1, 3_1, 10_0),\ (0_0, 1_0, 5_0, 8_1),\ (0_1, 3_1, 5_1, 8_0),\ (0_0, 4_0, 5_0, 6_1),\ (0_1, 4_1, 5_1, 6_0),$ $(0_0, 4_0, 0_1, 4_1)]$ mod 11.

The incidence matrix N of the design may be decomposed into block submatrices as follows:

$$N = (N_1|N_2|N_3|N_4|N_5) =$$

$$\left(\begin{array}{c|c|c|c|ccc} I_{11}+\alpha^2 & I_{11}+\alpha^3 & I_{11}+\alpha^4 & I_{11}+\alpha^5 & I_{11}+\alpha^3+\alpha^9+\alpha^{10} & I_{11} & I_{11} \\ \alpha^5+\alpha^8 & \alpha^4+\alpha^7 & \alpha^3+\alpha^9 & \alpha^2+\alpha^6 & 0_{11} & I_{11}+\alpha^2+\alpha^7 & I_{11}+\alpha^9+\alpha^{10} \end{array} \right),$$

where $\alpha = circ(010\ldots0)$ is a permutation circulant matrix of order 11. Since each row sum of N_1, N_2, N_3, N_4 and N_5 is 2, 2, 2, 2 and 6 respectively, the design is $(2,2,2,2,6)$-resolvable.

Further repeating the above solution three times, we obtain a 6-resolvable solution of the BIB design with parameters: $v = 22, r = 42, k = 4, b = 231, \lambda = 6$. The 6-resolvable solution is not reported in the Tables of Kageyama and Mohan (1983) and Subramani (1990).

Acknowledgements

The authors are grateful to anonymous reviewers and Editor-in-chief for their insightful comments and suggestions.

Conflict of interest

The authors do not have any financial or non-financial conflict of interest to declare for the research work included in this article.

References

Clatworthy, W. H. (1973). Tables of two–associate–class partially balanced designs. *National Bureau of Standards (U.S.), Applied Mathematics*, **63**.

Freeman, G. H. (1976). A cyclic method of constructing regular group divisible incomplete block designs. *Biometrica*, **63**, 555–558.

Ge, G. and Miao, Y. (2007). *PBDs, Frames, and Resolvability.* Chapman & Hall/ CRC, New York, 2nd edition.

Kageyama, S. (1976). Resolvability of block designs. *The Annals of Statistics*, **4**, 655–661.

Kageyama, S. and Mohan, R. N. (1983). On μ-resolvable BIB designs. *Discrete Mathematics*, **45**, 113–122.

Saurabh, S. (2024a). A note on cyclic and partial cyclic solutions of block designs. *Gujarat Journal of Statistics and Data Science*, **40**, 45–50.

Saurabh, S. (2024b). Resolvability of a BIB design of Takeuchi (1962). *Statistics and Applications*, **22**, 361–362.

Saurabh, S. and Sinha, K. (2023). Matrix approaches to constructions of group divisible designs. *Bulletin of the ICA*, **97**, 83–105.

Saurabh, S., Sinha, K., and Singh, M. K. (2021). A survey of resolvable solutions of partially balanced designs. *Communications in Statistics-Theory and Methods*, **52**, 1946–1962.

Subramani, J. (1990). A note on μ-resolvable BIB designs. *Journal of the Indian Society of Agricultural Statistics*, **XLII**, 226–233.

Takeuchi, K. (1962). A table of difference sets generating balanced incomplete block designs. *Review of International Statistical Institute*, **30**, 361–366.