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Some Results of Auto-Relevation Transform in Reliability Analysis

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Abstract

In this paper, we study some important reliability characteristics of auto-relevation transform. Various ageing and ordering concepts are discussed. Important results in terms of reliability and information measures are studied. Some characterizations are presented. A new lifetime distribution called auto-relevated Lomax (ARL) is introduced and its practical applicability is illustrated with a real dataset.

Key words: Relevation transform; Hazard rate; Ageing properties; Stochastic orders.

AMS Subject Classifications: 90B25, 60E05.

1. Introduction

Let X and Y be two absolutely continuous non-negative random variables, with survival functions $\bar{F}(.)$ and $\bar{G}(.)$ respectively. Consider an item from a population with survival function $\bar{F}(x)$, which is being replaced at the time of its failure at age x, by another item of the same age x from another population with survival function $\bar{G}(x)$. Then the survival function

$$\bar{T}(x) = \bar{F}\#\bar{G}(x) = \bar{F}(x) - \bar{G}(x)\int_0^x \frac{1}{\bar{G}(t)} d\bar{F}t. \tag{1}$$

is called the relevation transform of X and Y introduced by Krakowski (1973). Let $Y(X)$ denote the total lifetime of the random variable Y given it exceeds a random time X, (i.e $Y(X) \stackrel{d}{=} \{Y|Y > X\}$). Then (1) is the survival function of the random variable $Y(X)$. The probability density function (p.d.f.) of the relevation random variable is obtained as

$$t(x) = T'(x) = g(x)\int_0^x \frac{f(t)}{\bar{G}(t)} dt. \tag{2}$$

Grosswald *et al.* (1980) presented two characterizations of the exponential distribution based on relevation transform. The concept of dependent relevation transform and its importance in reliability analysis is given in Johnson and Kotz (1981). Baxter (1982) discussed

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certain reliability applications of the relevation transform. Shanthikumar and Baxter (1985) provided closure properties of certain ageing concepts in the context of relevation transforms. Improved versions of the results in Grosswald *et al.* (1980) are given by Lau and Rao (1990). Chukova *et al.* (1993) established characterizations of the class of distributions with almost lack of memory property based on the relevation transform. Sankaran and Dileepkumar (2019) studied important reliability properties of the relevation transform in the context of proportional hazards model.

When the random variables X and Y are identically distributed, the tail distribution of the random variable $Y(X)$ can be simplified to

$$\bar{T}^*(x) = \bar{F}(x)(1 - \log(\bar{F}(x))). \tag{3}$$

The survival function (3) is known as the auto-relevation of $\bar{F}(x)$. Kapodistria and Psarrakos (2012) studied properties and applications of a sequence of random variables with weighted tail distribution functions based on the auto-relevation transform. In this paper we focus our attention on various properties, applications and characterizations of the auto-relevation transform in the context of reliability theory.

The rest of the paper is organized as follows. We provide the concept and basic characteristics of auto-relevation transform in Section 2. Section 3 presents some important characterization results based on reliability and information measures. Various ageing properties and stochastic orders of auto-relevation are presented in Section 4 and Section 5 respectively. Finally, in Section 6, we provide major conclusions of the study.

2. Auto-Relevation Transform (ART)

Let X and Y be two non-negative continuous random variables with survival functions $\bar{F}(x)$ and $\bar{G}(x)$ respectively. Then the survival function of the relevation random variable $Y(X)$ is given in (1). When X and Y are identically distributed, the random variable $X(X)$ is known as the auto-relevation of X. Survival function of $X(X)$ is obtained as

$$\begin{aligned}\bar{T}^*(x) &= \bar{F}(x) - \bar{F}(x)\int_{t=0}^{x} \frac{1}{\bar{F}(x)} d\bar{F}(x)\\ &= \bar{F}(x)(1 - \log(\bar{F}(x)).\end{aligned} \tag{4}$$

The probability density function (p.d.f) of $X(X)$ is obtained as

$$t^*(x) = -f(x)\log(\bar{F}(x)). \tag{5}$$

From (4) and (5), we have, $X(X)$ is the auto-relevation of X if and only if

$$\begin{aligned} h_{X(X)}(x) &= \frac{t^*(x)}{\bar{T}^*(x)} \\ \Leftrightarrow h_{X(X)}(x) &= -\frac{f(x)\log(\bar{F}(x))}{\bar{F}(x)(1-\log(\bar{F}(x)))} \\ \Leftrightarrow h_{X(X)}(x) &= h_X(x)\left(\frac{\log(\bar{F}(x))}{\log(\bar{F}(x))-1}\right) = h_X(x)\left(\frac{\Lambda_X(x)}{1+\Lambda_X(x)}\right), \end{aligned} \tag{6}$$

where $\Lambda_X(x) = -\log(\bar{F}(x))$ is the cumulative hazards function of X.

An important class of distributions used in risk theory and queueing theory is the class $\mathscr{L}$ distribution. A distribution F belongs to the class $\mathscr{L}$ if

$$\lim_{x\to\infty} \frac{\bar{F}(x-y)}{\bar{F}(x)} = 1, \forall\, y \in R. \tag{7}$$

Kluppelberg (1988) showed that,
$F \in \mathscr{L}$ if and only if $\lim_{x\to\infty} h_F(x) = 0,$ where $h_F(x)$ is the hazard rate function of $F(x)$.
Proposition 1: if $X \in \mathscr{L}$ then $X(X) \in \mathscr{L}$.

Proof: We have

$$\lim_{x\to\infty} h_{X(X)}(x) = \lim_{x\to\infty} h_X(x) \lim_{x\to\infty}\left(\frac{\log(\bar{F}(x))}{\log(\bar{F}(x))-1}\right). \tag{8}$$

Now by applying L'Hospitals's rule and noting that $\lim_{x\to\infty} h(x) = 0$, we get $\lim_{x\to\infty} h_{X(X)}(x) = 0$. This completes the proof. □

Let $Q_X(.)$ and $Q_{X(X)}(.)$ be the quantile functions of the random variables X and $X(X)$ with respective distribution functions $F(x)$ and $T^*(x)$. In the following, we establish the relation between the quantile functions of X and $X(X)$.
Proposition 2: Suppose $Q_X(.)$ and $Q_{X(X)}(.)$ are the quantile functions of the random variables X and $X(X)$ respectively. Then

$$Q_X(u) = Q_{X(X)}(u + (1-u)\log(1-u)). \tag{9}$$

Proof: From (4), we have

$$T^*(x) = 1 - \bar{F}(x)(1-\log(\bar{F}(x))). \tag{10}$$

By taking $F(x) = u$ where $u \in (0,1)$, we get $X = Q_X(u)$. Using this in (10), we have

$$\begin{aligned} & T^*(Q_X(u)) = 1-(1-u)(1-\log(1-u)) \\ \Longrightarrow \quad & Q_X(u) = Q_{X(X)}(u+(1-u)\log(1-u)). \end{aligned}$$

□

Remark 1: When the cumulative distribution function of $X(X)$ is non-invertibe, we can effectively employ the identity (9) to simulate random samples of $X(X)$ using the quantile function of X.

3. Characterization Results

Glaser (1980) established a general theorem that facilitates the determination of whether $h_X(x)$ is increasing (IHR), decreasing (DHR), Bath-tub (BT) or upside-down bathtub (UBT). He made use of the function $\psi(x) = -\frac{f'(x)}{f(x)}$, known as the Glaser's function. In the next proposition, we present an interesting identity connecting the Glaser's functions of the random variables X and $X(X)$.

Proposition 3: Let X be a non-negative continuous random variable with survival function $\bar{F}(x)$. Then $X(X)$ is the auto-relevation of X if and only if

$$\psi_{X(X)}(x) = \psi_X(x) - \frac{h_X(x)}{\Lambda_X(x)}, \tag{11}$$

where $\psi_X(x)$ and $\psi_{X(X)}(x)$ are the Glaser's function of X and $X(X)$ respectively.

Proof: If $X(X)$ is the auto-relevation of X then we have

$$\begin{aligned}
\psi_{X(X)}(x) &= -\frac{t^{*\prime}(x)}{t^*(x)} \\
\psi_{X(X)}(x) &= -\frac{f'(x)}{f(x)} + \frac{f(x)}{\bar{F}(x)\log(\bar{F}(x))} \\
\psi_{X(X)}(x) &= \psi_X(x) - \frac{h_X(x)}{\log(\bar{F}(x))}.
\end{aligned} \tag{12}$$

Conversly (11) gives

$$\frac{d}{dx}\left(\log(t^*(x)\right) = \frac{d}{dx}\Big(\log(-f(x)\ \log(\bar{F}(x)) + C\Big), \tag{13}$$

where C is a constant. Since $t^*(x)$ is a density function, on integration, we get $C = 0$ and (13) reduces to

$$t^*(x) = -f(x)\ \log(\bar{F}(x)). \tag{14}$$

This completes the proof. $\square$ The odds function of a random variable X is defined by

$$\phi_X(x) = \frac{P(X > x)}{P(X \leq x)} = \frac{\bar{F}_X(x)}{F_X(x)}.$$

Note that the odds function is a decreasing function of x. In the coming proposition, we provide an interesting connection between the odds functions of $X(X)$ and X.

Proposition 4: $X(X)$ is the auto-relevated random variable of X if and only if

$$\phi_{X(X)}(x) = \frac{1 + \Lambda(x)}{\phi_X^{-1}(x) - \Lambda(x)}, \tag{15}$$

where $\phi_{X(X)}(x)$ and $\phi_X(x)$ are the odds functions of $X(X)$ and X respectively.

Proof: Assume $X(X)$ is the auto-relevated random variable of X. From (4), We have

$$\begin{aligned} & \phi_{X(X)}(x) = \frac{\bar{T}^*(x)}{1-\bar{T}^*(x)} \\ \Leftrightarrow \quad & \phi_{X(X)}(x) = \frac{\bar{F}(x)-\bar{F}(x)\log(\bar{F}(x))}{F(x)+\bar{F}(x)\log(\bar{F}(x))} \\ \Leftrightarrow \quad & \phi_{X(X)}(x) = \frac{\phi_X(x)(1-\log(\bar{F}(x)))}{1+\phi_X(x)\log(\bar{F}(x))} \\ \Leftrightarrow \quad & \phi_{X(X)}(x) = \frac{1+\Lambda(x)}{\phi_X^{-1}(x)-\Lambda(x)}, \end{aligned} \tag{16}$$

which completes the proof. □

To measure the distance between two probability distributions, Kullback-Leibler divergence (K-L divergence) has been popularly used in modelling of statistical data. The K-L divergence, which is closely related to relative entropy, information divergence, and information for discrimination is a non-symmetric measure of the difference between two probability distributions $f(x)$ and $g(x)$. When $f(x)$ and $g(x)$ are non-negative continuous distributions, then the K-L divergence $I(f,g)$ is defined as

$$I(f,g) = \int_0^\infty f(x)\log\left(\frac{f(x)}{g(x)}\right)dx. \tag{17}$$

Specifically, the K-L divergence of $g(x)$ from $f(x)$, denoted $I(f,g)$, is a measure of the information lost when $g(x)$ is used to approximate $f(x)$. In the following we present a relationship between $I(t^*,f)$ and $I(f,t^*)$ in the context of ART.

Proposition 5: Let $X(X)$ be the ART random variable corresponding to the non-negative random variable X. Then

$$I(X,X(X)) = 1 - I(X(X),X), \tag{18}$$

where $I(X,X(X))$ is the Kullback-Leibler divergence between X and $X(X)$.

Proof: From (17), we have

$$I(X(X),X) = \int_0^\infty t^*(x)\log\left(\frac{t^*(x)}{f(x)}\right)dx. \tag{19}$$

Since $X(X)$ is the ART random variable corresponding to X, using (5) in (19), we get

$$I(X(X),X) = -\int_0^\infty f(x)\log(\bar{F}(x))\log(-\log(\bar{F}(x)))dx. \tag{20}$$

by taking $u=-\log(\bar{F}(x))$, the integral in (20) became

$$I(X(X),X) = \int_0^\infty u\log(u)e^{-u}du.$$

Now by applying integration by parts, we obtain

$$I(X(X), X) = -\underset{x\to\infty}{Lim}\left(\frac{x\log(x)}{e^x}\right) + \underset{x\to 0}{Lim}\left(\frac{x\log(x)}{e^x}\right) + \int_0^\infty (1+\log(x))e^{-x}dx$$
$$= \int_0^\infty (1+\log(x))e^{-x}dx. \quad (21)$$

Again applying integration by parts on (21), we get

$$I(X(X), X) = 1 + \int_0^\infty \log(x)e^{-x}dx = 1-\gamma, \quad (22)$$

where $\gamma = -\int_0^\infty \log(x)e^{-x}dx$ is the Euler–Mascheroni constant ($\gamma \simeq 0.5772$). Now, we have

$$I(X, X(X)) = \int_0^\infty f(x)\log\left(\frac{f(x)}{t^*(x)}\right)$$
$$= -\int_0^\infty f(x)\log(-\log(\bar{F}(x)))dx. \quad (23)$$

Using the transformation $u = -\log(\bar{F}(x))$, (23) becomes

$$I(X, X(X)) = -\int_0^\infty \log(u)e^{-u}du = \gamma. \quad (24)$$

From (22) and (24), the result follows. □

4. Ageing Properties

We describe ageing properties of the relevation random variable $X(X)$ in connection with the ageing behaviour of the baseline random variable X. Various ageing classes and their properties and applications can be seen in Barlow and Proschan (1975), Shaked and Shanthikumar (2007), and Nair *et al.* (2013). From (6), we have

$$h_{X(X)}(x) = h_X(x)\left(\frac{\log(\bar{F}(x))}{\log(\bar{F}(x))-1}\right). \quad (25)$$

Differentiating (25), we obtain

$$h'_{X(X)}(x) = h'_X(x)\left(\frac{\log(\bar{F}(x))}{\log(\bar{F}(x))-1}\right) + \left(\frac{h_X(x)}{(\log(\bar{F}(x)-1)}\right)^2. \quad (26)$$

Note that $\left(\frac{\log(\bar{F}(x))}{\log(\bar{F}(x))-1}\right) > 0$ and $\left(\frac{h_X(x)}{(\log(\bar{F}(x)-1)}\right)^2 > 0$ for all $x > 0$. Thus when X is IHR, we have $h'_X(x) > 0$ for all $x > 0$, which gives $h'_{X(X)}(x) > 0$ for all $x > 0$. Thus $X(X)$ is also IHR. Hence IHR property is preserved under auto-relevation. When X is an exponential random variable with hazard rate $h_X(x) = C$, where $C > 0$, a constant. Then, from (26) we obtain

$$h'_{X(X)}(x) = \left(\frac{C}{(\log(\bar{F}(x)-1)}\right)^2 \geq 0. \quad (27)$$

Thus auto-relevated exponential distribution is always IHR. However, the case when X is DHR gives different options, which is presented in the next proposition.

Proposition 6: Let X be a non-negative continuous random variable with survival function $\bar{F}(x)$. Suppose X is DHR. Then the auto-relevation random variable $X(X)$ is IHR (DHR) if and only if

$$\frac{h_X'(x)}{(h_X(x))^2} \geq (\leq) \frac{-1}{\Lambda_X(x)(\Lambda_X(x)+1)} \quad \text{for all } \ x > 0. \tag{28}$$

Proof: We have

$$h_{X(X)}'(x) = h_X'(x)\left(\frac{\log(\bar{F}(x))}{\log(\bar{F}(x)) - 1}\right) + \left(\frac{h_X(x)}{(\log(\bar{F}(x) - 1)}\right)^2. \tag{29}$$

$X(X)$ is IHR(DHR) if and only if $h_{X(X)}'(x) \geq (\leq) 0$. Now, since X is DHR, we have $h_X'(x) < 0$ for all $x > 0$. By using the facts that $\left(\frac{\log(\bar{F}(x))}{\log(\bar{F}(x))-1}\right)$ and $\left(\frac{h_X(x)}{(\log(\bar{F}(x)-1)}\right)^2$ are non-negative, we get $X(X)$ is IHR(DHR) if and only if, for all $x > 0$,

$$\begin{aligned}
& - h_X'(x)\left(\frac{\log(\bar{F}(x))}{\log(\bar{F}(x)) - 1}\right) \leq (\geq) \left(\frac{h_X(x)}{(\log(\bar{F}(x) - 1)}\right)^2 \\
\Leftrightarrow & - \frac{h_X'(x)}{(h_X(x))^2} \leq (\geq) \frac{1}{\log(\bar{F}(x)(\log(\bar{F}(x) - 1))} \\
\Leftrightarrow & \ \frac{h_X'(x)}{(h_X(x))^2} \geq (\leq) \frac{-1}{\Lambda_X(x)(\Lambda_X(x)+1)}, \quad \text{for all } \ x > 0.
\end{aligned}$$

□

Remark 2: Note that $X(X)$ accommodates non-monotonic shapes when the equality holds in (28). The change point of the non-monotonic hazard function will be obtained by solving the equality (28).

From Proposition 6, it is clear that the auto-relevation of DHR class of distributions can provide new lifetime models with non-monotonic hazard rate functions. Note that the auto-relevated distribution consists of the same number of parameters as in the parent distribution. Thus we can efficiently use the auto-relevation transformation for developing more flexible lifetime models from the existing ones without introducing additional parameters. To illustrate this, consider the Lomax distribution with survival function

$$\bar{F}(x) = \left(\frac{\alpha}{x+\alpha}\right)^c, \qquad \alpha > 0,\, c > 0 \ \text{ and } \ 0 < x < \infty, \tag{30}$$

and hazard function

$$h_X(x) = \frac{c}{\alpha + x}. \tag{31}$$

We have $h_X(x)$ is non-increasing for all parameter combinations. Thus X is always DHR. The survival function of the auto-relevated Lomax random variable (ARL) $X(X)$ has the form

$$\bar{T}^*(x) = \left(\frac{\alpha}{\alpha + x}\right)^c \left(1 - \log\left(\left(\frac{\alpha}{\alpha + x}\right)^c\right)\right). \tag{32}$$

The corresponding hazard function is obtained as

$$h_{X(X)}(x) = \frac{c}{\alpha + x}\left(\frac{\log\left(\left(\frac{\alpha}{\alpha+x}\right)^c\right)}{\log\left(\left(\frac{\alpha}{\alpha+x}\right)^c\right) - 1}\right). \tag{33}$$

On differentiating, we get

$$h'_{X(X)}(x) = \frac{c\left(c + \log\left(\left(\frac{\alpha}{\alpha+x}\right)^c\right) - \left(\log\left(\left(\frac{\alpha}{\alpha+x}\right)^c\right)\right)^2\right)}{(\alpha+x)^2\left(\log\left(\left(\frac{\alpha}{\alpha+x}\right)^c\right) - 1\right)^2}. \tag{34}$$

Thus the sign of $h'_{X(X)}(x)$ depends only on the function

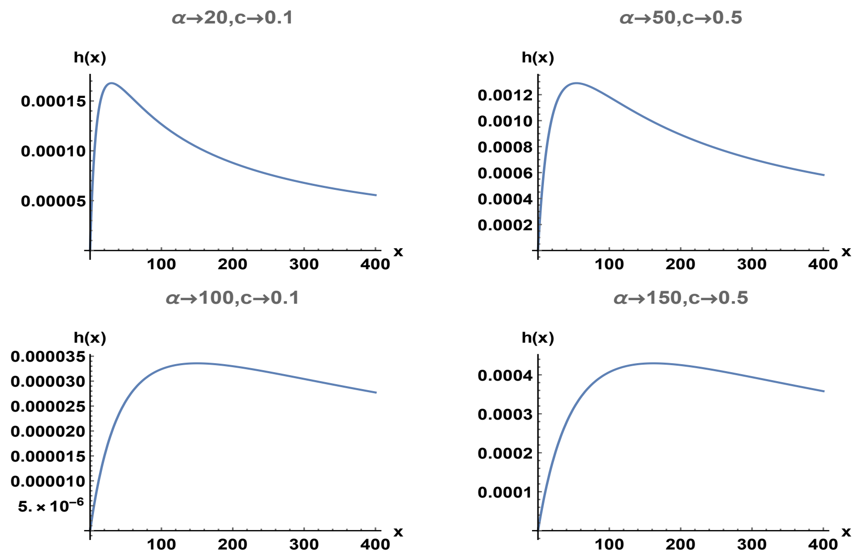


Figure 1: $h_{X(X)}(x)$ of ARL distribution for different parameter combinations.

$$\gamma(x) = \left(c + \log\left(\left(\frac{\alpha}{\alpha+x}\right)^c\right) - \left(\log\left(\left(\frac{\alpha}{\alpha+x}\right)^c\right)\right)^2\right).$$

We can write this as

$$\gamma(x) = c + k(x) - (k(x))^2, \tag{35}$$

where $k(x) = \log\left(\left(\frac{\alpha}{\alpha+x}\right)^c\right)$. We can observe that $k(x) < 0$ for all $x > 0$ and strictly decreasing for all $\alpha, c > 0$. Since $k(0) = 0$, it is clear that $\gamma(x)$ takes a positive sign initially and then became negative as x progresses. Correspondingly, the hazard function first increase then decrease in x for all parameter combinations. Thus the hazard function of ARL distribution is always Bathtub shaped. The change point of $h(x)$ will be attained by solving the equation $\gamma(x) = c + k(x) - (k(x))^2 = 0$, which is obtained as

$$x_0 = \alpha\left(e^{\eta} - 1\right), \text{ where } \eta = \frac{1}{2} + \frac{\sqrt{1+4c}}{2}.$$

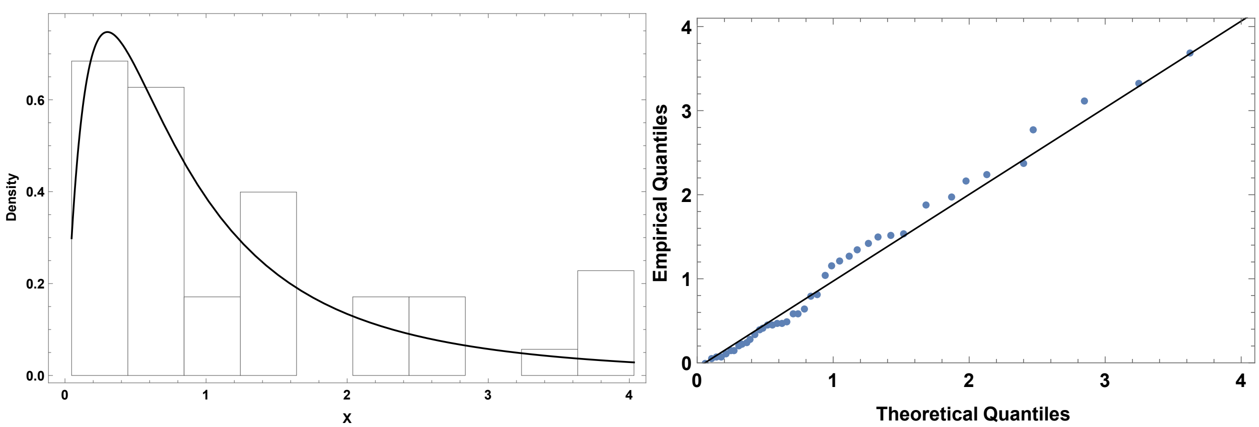
To show the practical importance of the proposed model, we consider a real data reported in Bekker *et al.* (2000), which corresponds to the survival times (in years) of a group of 45 patients given chemotherapy treatment alone. The method of maximum likelihood is employed to estimate the parameters. The estimates obtained are

$$\hat{\alpha} = 0.97003 \quad \text{and} \quad \hat{c} = 2.70067. \tag{36}$$

Recently, Handique and Chakraborty (2016) fitted this data with Beta generalized Kumaraswamy Weibull(BKw-W) distribution and compared with Kumaraswamy Weibull (Kw-W) and Beta generalized Weibull(B-W) distributions. They compared the goodness of fit using the AIC measure. The AIC values of the ARL, BKw-W, Kw-W and B-W models are presented in Table 1.

Table 1: AIC values

Distribution	AIC
ARL	118.831
BKw-W	122.92
Kw-W	123.44
B-W	124.14



(a) Observed histogram and estimated pdf plots **(b) Q-Q plot**

Figure 2

It is evident that the ARL model gives a better fit than the other models concerning the values of AIC. Note that the ARL model contains fewer number of parameters as compared to the competing alternatives. Plot of the fitted density with the histogram of the observed data is given in Figure 2(a). To check the physical closeness of the model, we use the Q-Q plot, which is given in Figure 2(b). We also carry out the Kolmogorov–Smirnov (K–S) goodness of fit test. The K–S test statistic with the associated p-value for the fitted model are 0.093 and 0.80 respectively.

In the context of coherent systems with 'n' identical components, Navarro *et al.* (2013) established that the component survival function $\bar{F}_c(x)$ and the system survival function

$\bar{F}_S(x)$ are connected through the relation

$$\bar{F}_S(x) = q(\bar{F}_c(x)), \tag{37}$$

where $q(u)$ is a distortion function, which is a concave non-decreasing function from $[0, 1]$ to $[0, 1]$, such that $q(0) = 0$ and $q(1) = 1$.

From (5), the survival function $\bar{T}^*(x)$ satisfies

$$\bar{T}^*(x) = q(\bar{F}(x)), \quad \text{where } q(u) = u\,(1 - \log(u))\ u \in [0, 1]. \tag{38}$$

The function $q(u)$ is a concave distortion function. From this, we can infer that $X(X)$ is the distorted random variable obtained from X by the distortion $q(u)$. Distorted random variables have many applications in reliability theory. Navarro *et al.* (2013, 2014) developed various stochastic orders and preservation properties of ageing classes and for the general distorted distributions in the context of coherent systems. For more details on this topic, one could refer to Wang (1996), Sordo and Suarez-Llorens (2011), Sordo *et al.* (2015), and Navarro *et al.* (2016).

Let X and S denotes the lifetimes of the component and system respectively in the context of coherent systems. Then, Navarro *et al.* (2014) showed that If X is NBU (NWU) and $q(u\,v) \leq (\geq)\, q(u)\, q(v)$ for all $0 \leq u,\ v \leq 1$, (submultiplicative (supermultiplicative)) holds then S is NBU (NWU). Similarly, if X is IHRA (DHRA) and $q(u^a) \geq (\leq)\, (q(u))^a$ holds for all $0 \leq u,\ v \leq 1$ and $0 < a < 1$, then S is IHRA (DHRA). Now for the model (4), we have $X(X)$ is the distorted random variable of X, with distortion function $q(u)$ given in (38). We can easily verify that $q(u)$ is submultiplicative and satisfies the condition $q(u^a) \geq (\leq)(q(u))^a$ for all $0 \leq u,\ v \leq 1$ and $0 < a < 1$. Thus, NBU (NWU) and IHRA (DHRA) properties are preserved under auto-relevation transform.

5. Stochastic Orders

There are many situations in practice where we need to compare the characteristics of two distributions. Stochastic orders are used for the comparison of lifetime distributions. In this section, we provide some important stochastic orders between the random variables X and $X(X)$. We shall consider the following stochastic orders. Important properties and interrelations of various stochastic orders can be seen in Shaked and Shanthikumar (2007) and Barlow and Proschan (1975). Suppose $\bar{F}_1(x)$ and $\bar{F}_2(x)$ be the survival functions obtained by distorting $\bar{F}(x)$ using the distortion functions $q_1(u)$ and $q_2(u)$ respectively. Let S_1 and S_2 be the random variables corresponding to $\bar{F}_1(x)$ and $\bar{F}_2(x)$ respectively. Now from Navarro *et al.* (2014) (Theorem 2.5), we have

$$S_1\ \leq_{lr}\ (\geq_{lr})\, S_2\ \text{ if and only if } \frac{q_1'(u)}{q_2'(u)} \text{ is increasing (decreasing) in } u \in (0, 1), \tag{39}$$

where $q_i'(u)$ is the derivative of $q_i(u)$, $i = 1, 2$. To study different stochastic order relations between X and $X(X)$, we take $S_1 = X(X)$ and $S_2 = X$, with distortion functions $q_1(u) = u\,(1 - \log(u))$ and $q_2(u) = u$ respectively.

Note that,

$$\frac{d}{du}\left(\frac{q_1'(u)}{q_2'(u)}\right) = \frac{d}{du}\left(-\log(u)\right) = -\frac{1}{u} \leq 0.$$

Thus $\frac{q_1'(u)}{q_2'(u)}$ is decreasing in $u \in (0,1)$. Now from (39), we get $X \leq_{lr} X(X)$. Moreover, from Shaked and Shanthikumar (2007), we have the following implications,

$$X \leq_{lr} X(X) \implies X \leq_{hr} X(X) \implies X \leq_{st} X(X).$$

Kochar and Wiens (1987) have defined an IHR order by saying that X is more IHR than Y if $X \underset{c}{\leq} Y$. Further, X is more IHRA (NBU) than Y if $G^{-1}(F(x))$ is star-shaped denoted by $X \leq_* Y$ (super additive denoted by $X \leq_{su} Y$). We have also $X \leq_{DMRL} Y$ if $\frac{m_X(x)}{m_Y(x)}$ is non-decreasing, $X \leq_{NBUE} Y$ if $\frac{m_X(x)}{m_Y(x)} \leq \frac{E(X)}{E(Y)}$, $X \leq_{NBUHR} Y$ if $\frac{h_X(x)}{h_Y(x)} \geq \frac{h_X(0)}{h_Y(0)}$, and $X \leq_{NBUHRA} Y$ if $F_Y^{-1}(F_X(x)) \geq x\left(F_Y^{-1}(F(x)\right)_{x=0}$ (Nair *et al.*, 2013). Among these stochastic orders $X \leq_c Y \implies X \leq_{DMRL} Y \implies X \leq_{NBUE} Y$ and $X \leq_{NBU} Y \implies X \leq_{NBUHRA} Y$. Later Sengupta and Deshpande (1994) proved that $X \underset{c}{\leq} Y$ if and only if $\frac{h_X(x)}{h_Y(x)}$ is non-decreasing in x, provided $h_Y(x) \neq 0$. The following proposition establishes various interrelationships among these orderings.

Proposition 7: Let X be a non-negative random variable and $X(X)$ be the auto-relevation of X with survival function (4). Then $X(X) \underset{c}{\leq} X$.

Proof: From (25), we have

$$\frac{h_{X(X)}(x)}{h_X(x)} = \frac{\log(\bar{F}(x))}{\log(\bar{F}(x)) - 1}.$$

Upon differentiating, we obtain

$$\frac{d}{dx}\left(\frac{h_{X(X)}(x)}{h_X(x)}\right) = \frac{f(x)\,\log(\bar{F}(x))}{\bar{F}(x)(\log(\bar{F}(x)) - 1)^2} - \frac{f(x)\,(\log(\bar{F}(x)) - 1)}{\bar{F}(x)(\log(\bar{F}(x)) - 1)^2}$$

$$= \frac{h_X(x)}{\bar{F}(x)(\log(\bar{F}(x)) - 1)^2} \geq 0. \tag{40}$$

Thus $\frac{h_{X(X)}}{h_X(x)}$ is non-decreasing in x and hence $X(X)$ is more IHR than X. □ The implications, consequence of the Proposition 7, are exhibited in the following diagram;

$$\begin{array}{ccccccc} X(X) \leq_c X & \implies & X(X) \leq_* X & \implies & X(X) \leq_{su} X & & \\ \Downarrow & & \Downarrow & & \Downarrow & & \\ X(X) \leq_{DMRL} X & \implies & X(X) \leq_{NBUE} X & \implies & X(X) \leq_{NBUHR} X & \implies & X(X) \leq_{NBUHRA} X. \end{array}$$

Proposition 8: Let Y_1 and Y_2 be the auto-relevated random variables corresponding to X_1 and X_2 respectively. Then the following results hold;

(i) If $X_1 \leq_{st} X_2$ then $Y_1 \leq_{st} Y_2$.

(ii) If $X_1 \leq_{hr} X_2$ then $Y_1 \leq_{hr} Y_2$.

(iii) If $X_1 \leq_{icx} X_2$ then $Y_1 \leq_{icx} Y_2$.

Proof: The proof for (i) is direct from (4). Now to prove (ii), we have

$$\frac{u\,q'(u)}{q(u)} = \frac{-\log(u)}{1-\log(u)}.$$

Note that $\frac{d}{du}\left(\frac{u\,q'(u)}{q(u)}\right) = -\frac{1}{u(1-\log(u))^2} \leq 0$ for all $u \in (0,1)$. Now from Theorem 2.6 of Navarro *et al.* (2014), we get $Y_1 \leq_{hr} Y_2$. From Theorem 2.6 of Navarro *et al.* (2014), (iii) follows since $q(u)$ is concave in (0,1). □

6. Conclusion

In this paper, we have presented the auto-relevation transform, which is useful in the context of lifetime studies. Various properties and characterizations in terms of reliability measures were presented. Ageing and ordering properties, which will be useful in the reliability context were studied. We also introduced the ARL distribution having non-monotonic hazard function and compared the performance with some existing competing alternatives.

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