Statistics and Applications {ISSN 2454-7395 (online)}
Volume 22, No. 2, 2024 (New Series), pp 285–305
http://www.ssca.org.in/journal

Extropy Properties of Ranked Set Sample for Sarmanov Family of Distributions

Manoj Chacko[1] **and Varghese George**[1,2]
[1]*Department of Statistics, University of Kerala, Trivandrum, Kerala 695581, India*
[2]*St. Stephen's College, Maloor College P. O.,Pathanapuram, Kerala 689695, India*

Received: 09 May 2023; Revised: 11 November 2023; Accepted: 25 February 2024

Abstract

In this paper, the extropy of ranked set sample from Sarmanov family of distributions is considered. By deriving the expression for extropy of concomitants of order statistics, the expression for extropy of ranked set sample of the study variable Y in which an auxiliary variable X is used to rank the units in each set, under the assumption that (X, Y) follows Sarmanov family of distributions is obtained.

Key words: Ranked set sampling; Sarmanov family of distributions; Concomitants of order statistics; Extropy.

AMS Subject Classifications: 62B10, 94A20, 62D05

1. Introduction

Let (X, Y) be a random vector with joint probability density function (PDF) $f(x, y)$ and cumulative distribution function (CDF) $F(x, y)$. Let $f_X(x)$ and $f_Y(y)$ be the marginal PDFs and $F_X(x)$ and $F_Y(y)$ be the marginal CDFs of X and Y respectively. Let (X_i, Y_i), $i = 1, 2, ...n$ be a random sample of size n from the population with cdf $F(x, y)$. If these observations are arrange in increasing order of magnitude based on X_i's, then the rth largest observation $X_{r:n}$ is the rth order statistic of X_i's. Then the Y variable associated with $X_{r:n}$ is called concomitant of rth order statistic and it is denoted by $Y_{[r:n]}$. David (1973) introduced the concept of concomitants of statistics which is applicable in various areas like ranked set sampling, double sampling, correlation analysis and in certain selection procedures. More details on this idea was given in David and Nagaraja (1998).

McIntyre (1952) introduced an efficient sampling scheme named ranked set sampling, as an alternative to simple random sampling (see, Chen *et al.* (2004)). The procedure of ranked set sampling is as follows. Select n^2 units randomly from the population. These units are randomly alloted into n sets, each of size n. Then the units in each set are ranked visually, judgement method or using some inexpensive methods. From the first set of n units, choose the unit which has the lowest rank for actual measurement. From the second set of n units the unit ranked second lowest is chosen. The process is continued until choose

Corresponding Author: Varghese George
Email: varghesessc@gmail.com

the unit which has the highest rank in the nth set. Then make measurement on variable of interest of the selected units, which constitute the ranked set sample(RSS).

Ranked set sampling as described in McIntyre (1952) is applicable whenever sample size is small and ranking of a set of sampling units can be done easily by a judgment method. Suppose the variable of interest, say Y, is expensive to measure and difficult to rank the units. In this case as an alternative method, Stokes (1977) modified the method by using an auxiliary variable for ranking the sampling units in each set. Stokes (1977) explained the ranked set sampling procedure as follows. Choose n^2 units randomly from a bivariate population. Arrange these units into n sets, each of size n and measure the auxiliary variable X. In the first set, that unit for which smallest measurment on the auxiliary variable X is chosen and take the measurement of the study variable Y, denoted by $Y_{[1]}$. In the second set, that unit for which second smallest measurement on the auxiliary variable X is chosen and take the measurement of the study variable Y, denoted by $Y_{[2]}$. Finally, in the nth set, that unit for which largest measurment on the auxiliary variable X is chosen and take the measurement of the study variable Y, denoted by $Y_{[n]}$. Clearly $Y_{[r]}$, $r = 1, 2, ...n$ are concomitants of order statistics of the given random sample and are independent.

Bain (2017) give an example for the application of RSS as proposed by Stokes (1977). Here the study variable Y represents the oil pollution of sea water and auxiliary variable X represents the tar deposit in the nearby sea shore. Clearly collecting sea water sample and measuring the oil pollution in it is difficult and costly. However the prevalence of pollution in sea water is much reflected by the tar deposit in the surrounding terminal sea shore. In this example ranking the pollution level of sea water based on the tar deposit in the sea shore is more natural and scientific than ranking it visually or by judgement method. Applying the concepts of concomitant of order statistics in ranked set sampling, Chacko and Thomas (2007, 2008, 2009), Chacko (2017) and Mehta (2022) estimated the parameters of different distributions belonging to Morgenstern family of distributions.

As an alternative to entropy defined by Shannon (1948), Lad *et al.* (2015) introduced a new measure of uncertainty called extropy . Let X be a random variable with PDF $f_X(x)$ and CDF $F_X(x)$. Then the extropy of X is defined as

$$J(X) = \frac{-1}{2}\int_{-\infty}^{\infty}(f_X(x))^2 dx \tag{1}$$

$$= \frac{-1}{2}\int_{0}^{1} f_X(F^{-1}(u))du, \tag{2}$$

where $F^{-1}(u) = inf\{x; F_X(x) \geq u\}, u \in [0, 1]$ is the quantile function of $F_X(x)$.

Lad *et al.* (2015) gave some properties and applications of extropy measure. Qiu (2017) discussed the characterization results, monotone properties, and lower bounds of extropy of order statistics and record values. Zamanzade and Mahdizadeh (2019) discussed the nonparametric estimation of extropy based on ranked set sampling. Eftekharian and Qiu (2022) considered the information content of statified ranked set sampling in terms of extropy. Qiu and Raqab (2022) discussed the properties of weighted extropy using Ranked Set Samples.

Morgenstern (1956)introduced a bivariate family of distributions which can be constructed with specific marginal distributions and the PDF is given by

$$f(x,y) = f_X(x)f_Y(y)[1+\delta(2F_X(x)-1)(2F_Y(y)-1)], -1 \leq \delta \leq 1,$$

where δ is the association parameter, $f_X(x)$ and $f_Y(y)$ are the marginal PDFs and $F_X(x)$ and $F_Y(y)$ are the marginal CDFs of X and Y respectively. One of the important limitations of the Morgenstern family of distributions (MFD) is that the correlation coefficient lies between -1/3 and 1/3. Several authors have modified the MFD to enhance the range of correlation and extended the domain of applications. One of the important modifications of MFD was given by Sarmanov (1966) in the sense that it provides the best improvement in correlation level with only one parameter as in the MFD. The PDF of family of distributions of Sarmanov (1966) is given by

$$\begin{aligned} f(x,y) = f_X(x)f_Y(y)\Big[&1+3\alpha(2F_X(x)-1)(2F_Y(y)-1) \\ &+\frac{5}{4}\alpha^2\Big(3(2F_X(x)-1)^2-1\Big)\Big(3(2F_Y(y)-1)^2-1\Big)\Big], |\alpha| \leq \frac{\sqrt{7}}{5} \end{aligned} \quad (3)$$

where α is the association parameter. When the marginal distributions follow uniform, the distribution attain its maximum correlation coefficient, α.

Alemany *et al.* (2020) give an example for application for Sarmanov family of distributions given in (3). Here the study variable Y follows the average claim cost per insured and X represents the number of claims of individual. This model can be used to obtain the distribution of the total cost of claims based on the collective model, for a policyholder with specific characteristics. If the profiles have larger dependency, the Sarmanov distribution can be used to fit a non- linear dependence between frequency and severity (cost random variable). The different applications of Sarmanov family of distributions are given in Abdallah *et al.* (2016) and Bolancé *et al.* (2020). Barakat *et al.* (2022) discussed the properties of concomitants of order statistics of Sarmanov family of distributions.

It is well known that ranked set sample provides more information than simple random sample(SRS) of the same size about the unknown parameters of the underlying distribution in parametric inferences (see, Chen *et al.* (2004)). Jozani and Ahmadi (2014) explained the concept of information content of RSS data and compared them with their counterparts in SRS data. Raqab and Qiu (2019) described the monotone properties and stochastic orders of ranked set sample and compared the results with their counterpart under SRS design. Husseiny *et al.* (2022) discussed information measures in records and their concomitants arising from Sarmanov family of distributions. Chacko and George (2024, 2023) discussed the extropy properties of RSS for MFD and Cambanis type bivariate distributions. George and Chacko (2023) considered the cumulative residual extropy properties of ranked set samples for Cambanis type bivariate distributions.

In this paper, we derive the extropy of concomitant of order statistic $Y_{[r:n]}$ of a random sample of size n from Sarmanov family of distributions. Since observations of a ranked set sample, in which an auxiliary variable X is used to rank the units in each set, are nothing but concomitant of order statistics, we derive the extropy of RSS when (X, Y) follows Sarmanov family of distributions. The properties and bounds for extropy of RSS are also derived. We also consider the joint extropy of $(X_{RSS}, Y_{[RSS]})$, where $X_{RSS} = \Big(X_{(1)}, X_{(2)}, ..., X_{(n)}\Big)$ is the

RSS of X observations in which ranking in each unit is perfect and $Y_{[RSS]} = \Big(Y_{[1]}, Y_{[2]}, ..., Y_{[n]}\Big)$ is the RSS of Y observations in which ranking in each unit is based on X observations.

The paper is organized as follows. In section 2, the expression for extropy of $Y_{[r:n]}$ and also obtain upper and lower bounds of it. In section 3, we obtain the extropy of the RSS arising from Sarmanov family of distributions and study its properties. Section 4 devotes to obtain extropy of $(X_{r:n}, Y_{[r:n]})$ and thereby obtain the extropy of $(X_{RSS}, Y_{[RSS]})$, where $X_{RSS} = \Big(X_{(1)}, X_{(2)}, ..., X_{(n)}\Big)$ is the ranked set sampling based on X observations in which ranking in each unit is perfect and $Y_{[RSS]} = \Big(Y_{[1]}, Y_{[2]}, ..., Y_{[n]}\Big)$. Finally, in section 5 we give the conclusion.

2. Extropy of concomitant of rth order statistic

Let $Y_{[r:n]}$ $r = 1, 2, ...n$ be the concomitant of rth order statistic of a bivariate random sample arising from Sarmanov family of distributions. If $f_{r:n}(x)$ is the pdf of rth order statistic and $f_{Y|X}(y/x)$ is the conditional pdf of Y given X, then the pdf of concomitant of rth order statistic, $Y_{[r:n]}$ is

$$\begin{aligned}
f_{Y[r:n]}(y) &= \int_{-\infty}^{\infty} f_{Y|X}(y/x) f_{r:n}(x) dx \\
&= \int_{-\infty}^{\infty} f_Y(y)\Big[1 + 3\alpha(2F_X(x) - 1)(2F_Y(y) - 1) \\
&\quad + \frac{5}{4}\alpha^2\Big(3(2F_X(x) - 1)^2 - 1\Big)\Big(3(2F_Y(y) - 1)^2 - 1\Big)\Big] \\
&\quad \times \frac{n!}{(r-1)!(n-r)!}(F_X(x))^{r-1}(1 - F_X(x))^{n-r} dx \\
&= f_Y(y)\left[1 + d_1(2F_Y(y) - 1) + d_2\Big(3(2F_Y(y) - 1)^2 - 1\Big)\right], \qquad (4)
\end{aligned}$$

where

$$d_1 = 3\alpha\frac{2r - n - 1}{n + 1} \qquad (5)$$

and

$$d_2 = \frac{5}{2}\alpha^2\Big(1 - \frac{6r(n - r + 1)}{(n + 1)(n + 2)}\Big). \qquad (6)$$

Then by using (1) the extropy of $Y_{[r:n]}$ is given by

$$\begin{aligned} J(Y_{[r:n]}) &= \frac{-1}{2}\int_y (f_{Y[r:n]}(y))^2 dy \\ &= \frac{-1}{2}\int_y (f_Y(y))^2\left[1+d_1(2F_Y(y)-1)+d_2\Big(3(2F_Y(y)-1)^2-1\Big)\right]^2 dy \\ &= \frac{-1}{2}\int_{u=0}^1 f_Y(F^{-1}(u))\left[1+d_1(2u-1)+d_2\Big(3(2u-1)^2-1\Big)\right]^2 du \\ &= \frac{-1}{2}\int_{u=0}^1 f_Y(F^{-1}(u))\Big(\rho_{(r,n,\alpha)}(u)\Big)^2 du, \end{aligned} \tag{7}$$

where

$$\rho_{(r,n,\alpha)}(u) = 1+d_1(2u-1)+d_2\Big(3(2u-1)^2-1\Big). \tag{8}$$

Theorem 1: Let $Y_{[r:n]}$ be the concomitant of rth order statistic of a random sample of size n arising from Sarmanov family of distributions, then the extropy of $Y_{[r:n]}$ can be written as

$$J(Y_{[r:n]}) = \frac{-1}{2}\sum_{k=0}^{4}\frac{a_k}{k+1}E\Big(F^{-1}(U_k)\Big), \tag{9}$$

where $a_0 = (1-d_1+2d_2)^2$, $a_1 = 2(1-d_1+2d_2)(2d_1-12d_2)$, $a_2 = (2d_1-12d_2)^2+24d_2(1-d_1+2d_2)$, $a_3 = 24d_2(2d_1-12d_2)$, $a_4 = 144d_2^2$ and

$$E\Big(F^{-1}(U_k)\Big) = \int_0^1 (k+1)u^k f_Y(F^{-1}(u))du$$

with U_k follows Beta $(k+1,1)$.

Proof: Since $Y_{[r:n]}$ is the concomitant of rth order statistic of a random sample of size n arising from Sarmanov family of distributions, we have

$$\begin{aligned} (f_{Y[r:n]}(y))^2 &= (f_Y(y))^2\left[1+d_1(2F_Y(y)-1)+d_2\Big(3(2F_Y(y)-1)^2-1\Big)\right]^2 \\ &= (f_Y(y))^2\sum_{k=0}^{4}a_k\Big(F_Y(y)\Big)^k, \end{aligned}$$

where $a_0 = (1-d_1+2d_2)^2$, $a_1 = 2(1-d_1+2d_2)(2d_1-12d_2)$, $a_2 = (2d_1-12d_2)^2+24d_2(1-d_1+2d_2)$, $a_3 = 24d_2(2d_1-12d_2)$ and $a_4 = 144d_2^2$.

Therefore, the extropy of $Y_{[r:n]}$ is given by

$$\begin{aligned} J(Y_{[r:n]}) &= \frac{-1}{2}\int (f_{Y[r:n]}(y))^2 dy \\ &= \frac{-1}{2}\int (f_Y(y))^2 \sum_{k=0}^{4} a_k \Big(F_Y(y)\Big)^k dy \\ &= \frac{-1}{2}\sum_{k=0}^{4} a_k \int_0^1 u^k f_Y(F^{-1}(u))du \\ &= \frac{-1}{2}\sum_{k=0}^{4} \frac{a_k}{k+1} E\Big(F^{-1}(U_k)\Big), \end{aligned}$$

where U_k follows Beta $(k+1, 1)$. Hence the theorem. □

Remark 1: If $r = 1$ and $r = n$ in (4), we get the concomitant of first order statistic and largest order statistic of a random sample of size n. Then the extropy of concomitant of first order statistic $Y_{[1:n]}$ and concomitant of largest order statistic $Y_{[n:n]}$ are given by

$$J(Y_{[1:n]}) = \frac{-1}{2}\sum_{k=0}^{4} \frac{a_k^{(1)}}{k+1} E\Big(F^{-1}(U_k)\Big),$$

where $a_0^{(1)} = (1+q_1+2q_2)^2$, $a_1^{(1)} = -2(1+q_1+2q_2)(2q_1+12q_2)$, $a_2^{(1)} = (2q_1+12q_2)^2 + 24q_2(1+q_1+2q_2)$, $a_3^{(1)} = -24q_2(2q_1+12q_2)$ and $a_4^{(1)} = 144q_2^2$
and

$$J(Y_{[n:n]}) = \frac{-1}{2}\sum_{k=0}^{4} \frac{a_k^{(n)}}{k+1} E\Big(F^{-1}(U_k)\Big),$$

where $a_0^{(n)} = (1-q_1+2q_2)^2$, $a_1^{(n)} = 2(1-q_1+2q_2)(2q_1-12q_2)$, $a_2^{(n)} = (2q_1-12q_2)^2+24q_2(1-q_1+2q_2)$, $a_3^{(n)} = 24q_2(2q_1-12q_2)$ and $a_4^{(n)} = 144q_2^2$
with $q_1 = 3\alpha\frac{n-1}{n+1}$ and $q_2 = \frac{5}{2}\alpha\Big(1-\frac{6n}{(n+1)(n+2)}\Big)$.

Remark 2: If $\alpha = 0$, that is X and Y are independent, then $d_1 = 0$ and $d_2 = 0$ and hence $J(Y_{[r:n]}) = \frac{-1}{2}E\Big(F^{-1}(U_0)\Big) = J(Y)$.

Corollary 1: Let (X_i, Y_i), $i = 1, 2, ...n$ be a bivariate sample of size n arising from Sarmanov family of distributions. Then the extropy of concomitant of rth order statistic for $\alpha > 0$ is same as the extropy of concomitant of $(n-r+1)$th order statistic for $\alpha < 0$.

Proof: Let $J^{(\alpha)}(Y_{[r:n]})$ be the extropy of concomitant of rth order statistic for any α. We have by (5) and (6), $d_{1(n,\alpha)} = d_{1(n-r+1,-\alpha)}$ and $d_{2(n,\alpha)} = d_{2(n-r+1,-\alpha)}$. Therefore by (9),

$$J^{(\alpha)}(Y_{[r:n]}) = J^{(-\alpha)}(Y_{[n-r+1:n]}).$$

□

Example 1: If (X, Y) follows Sarmanov family of distributions given in (3) with $f_X(x) = 1, 0 \le x \le 1$ and $f_Y(y) = 1, 0 \le y \le 1$, then

$$J(Y_{[r:n]}) = \frac{-1}{2}\sum_{k=0}^{4} \frac{a_k}{k+1}.$$

Example 2: If (X, Y) follows Sarmanov family of distributions given in (3) with $f_X(x) = \theta_1 e^{-\theta_1 x}, x \geq 0$ and $f_Y(y) = \theta_2 e^{-\theta_2 y}, y \geq 0$, then

$$J(Y_{[r:n]}) = \frac{-\theta_2}{2} \sum_{k=0}^{4} \frac{a_k}{(k+1)(k+2)}.$$

Theorem 2: Let $Y_{[r:n]}$ be the concomitant of rth order statistic of a random sample of size n arising from Sarmanov family of distributions, the upper bound of J$(Y_{[r:n]})$ can be written as

$$J(Y_{[r:n]}) \leq \frac{-1}{2} \sum_{k=1}^{3} \frac{a_k}{k+1} E\Big(F^{-1}(U_k)\Big), \tag{10}$$

where U_k follows Beta $(k+1, 1)$.

Proof: Since $a_0 \geq 0$ and $a_4 \geq 0$, by using Theorem 1 we can obtain the inequality (10) directly . Hence the proof. □

Example 3: If (X, Y) follows Sarmanov family of distributions given in (3) with $f_X(x) = 1, 0 \leq x \leq 1$ and $f_Y(y) = 1, 0 \leq y \leq 1$, then

$$J(Y_{[r:n]}) \leq \frac{-1}{2} \sum_{k=1}^{3} \frac{a_k}{k+1}.$$

Example 4: If (X, Y) follows Sarmanov family of distributions given in (3) with $f_X(x) = \theta_1 e^{-\theta_1 x}, x \geq 0$ and $f_Y(y) = \theta_2 e^{-\theta_2 y}, y \geq 0$, then

$$J(Y_{[r:n]}) \leq \frac{-\theta_2}{2} \sum_{k=1}^{3} \frac{a_k}{(k+1)(k+2)}.$$

Theorem 3: Let $Y_{[r:n]}$ be the concomitant of rth order statistic of a random sample of size n arising from Sarmanov family of distributions, then the lower bound of $J(Y_{[r:n]})$ is given by

$$J(Y_{[r:n]}) \geq \frac{-1}{2} \Bigg(E[(f_Y(y))^2]\Bigg)^{\frac{1}{2}} \Bigg(\int_0^1 \Big(\sum_{k=0}^{4} a_k u^k\Big)^2 du\Bigg)^{\frac{1}{2}}. \tag{11}$$

Proof: From (7), we have

$$J(Y_{[r:n]}) = \frac{-1}{2} \int_{u=0}^{1} f_Y(F^{-1}(u)) \Big(\rho_{(r,n,\alpha)}(u)\Big)^2 du$$

By applying Cauchy - Schwarz inequality, we have

$$J(Y_{[r:n]}) \geq \frac{-1}{2} \Bigg(\int_{u=0}^{1} (f_Y(F^{-1}(u)))^2 du\Bigg)^{\frac{1}{2}} \Bigg(\int_{u=0}^{1} \Big(\rho_{(r,n,\alpha)}(u)\Big)^4 du\Bigg)^{\frac{1}{2}}. \tag{12}$$

Therefore

$$\begin{aligned}\int_{u=0}^{1}(f_Y(F^{-1}(u)))^2du &= \int_y (f_Y(y))^3dy \\ &= E[(f_Y(y))^2]. \end{aligned} \tag{13}$$

Also

$$\Big(\rho_{(r,n,\alpha)}(u)\Big)^4 = \Big(\sum_{k=0}^{4} a_k u^k\Big)^2. \tag{14}$$

On substituting (13) and (14) in (12) we get (11). Hence the proof. □

Example 5: If (X, Y) follows Sarmanov family of distributions given in (3) with $f_X(x) = 1, 0 \leq x \leq 1$ and $f_Y(y) = 1, 0 \leq y \leq 1$, then

$$J(Y_{[r:n]}) \geq \frac{-1}{2}\left(\int_0^1 \Big(\sum_{k=0}^{4} a_k u^k\Big)^2 du\right)^{\frac{1}{2}}.$$

Example 6: If (X, Y) follows Sarmanov family of distributions given in (3) with $f_X(x) = \theta_1 e^{-\theta_1 x}, x \geq 0$ and $f_Y(y) = \theta_2 e^{-\theta_2 y}, y \geq 0$, then

$$J(Y_{[r:n]}) \geq \frac{-1}{2}\Big(\frac{\theta_2^2}{3}\Big)^{\frac{1}{2}}\left(\int_0^1 \Big(\sum_{k=0}^{4} a_k u^k\Big)^2 du\right)^{\frac{1}{2}}.$$

3. Extropy of ranked set sample

Let $Y_{[1]}, Y_{[2]}, ... Y_{[n]}$ be the RSS of size n arising from Sarmanov family of distributions in which X observations are used to rank the units in each set. Clearly $Y_{[r]}$, $r = 1, 2, ..., n$ are independent and $Y_{[r]} \stackrel{d}{=} Y_{[r:n]}$. If $Y_{[RSS]} = \{Y_{[r]}, r = 1, 2, ...n\}$, then the extropy of $Y_{[RSS]}$ can be written as

$$\begin{aligned} J(Y_{RSS}) &= \frac{-1}{2}\prod_{r=1}^{n}\int_y (f_{Y[r:n]}(y))^2dy \\ &= \frac{-1}{2}\prod_{r=1}^{n}[-2J(Y_{[r:n]})]. \end{aligned}$$

Therefore,

$$J(Y_{RSS}) = \frac{-1}{2}\prod_{r=1}^{n}\sum_{k=0}^{4}\frac{a_k}{k+1}E\Big(F^{-1}(U_k)\Big).$$

Example 7: If (X, Y) follows Sarmanov family of distributions given in (3) with $f_X(x) = 1, 0 \leq x \leq 1$ and $f_Y(y) = 1, 0 \leq y \leq 1$,then

$$J(Y_{RSS}) = \frac{-1}{2}\prod_{r=1}^{n}\sum_{k=0}^{4}\frac{a_k}{k+1}.$$

Example 8: If (X, Y) follows Sarmanov family of distributions given in (3) with $f_X(x) = \theta_1 e^{-\theta_1 x}, x \geq 0$ and $f_Y(y) = \theta_2 e^{-\theta_2 y}, y \geq 0$, then

$$J(Y_{RSS}) = \frac{-1}{2}\theta_2^n \prod_{r=1}^{n} \sum_{k=0}^{4} \frac{a_k}{(k+1)(k+2)}.$$

Definition 1: (Shaked and Shanthikumar (2007)) Let X_1 and X_2 be two random variables with cdfs F_1 and F_2 and pdfs f_1 and f_2 respectively. The left continuous inverses of F_1 and F_2 are given by $F_1^{-1}(u) = inf\{t : F_1(t) \geq u\}$ and $F_2^{-1}(u) = inf\{t : F_2(t) \geq u\}$, $0 \leq u \leq 1$. Then X_1 is said to be smaller than X_2 in dispersive order denoted by $X_1 \leq_{disp} X_2$ if $F_2^{-1}(F_1(x)) - x$ is increasing in $x \geq 0$. Clearly if $X_1 \leq_{disp} X_2$, then $f_1(F_1^{-1}(u)) \leq f_2(F_2^{-1}(u))$, for $0 \leq u \leq 1$.

Theorem 4: Let (X, Y) follows Sarmanov family of distributions given in (3) with marginal cdfs $F_X(x)$ and $F_Y(y)$ and pdfs $f_X(x)$ and $f_Y(y)$ respectively. Let $Y_{RSS} = \{Y_{[r]}, r = 1, 2, ...n\}$ be the ranked set sample of size n arising from Sarmanov family of distributions in which X observations are used to rank the units. Let (V, W) be another pair of random variables follows Sarmanov family of distributions given in (3) with marginal cdfs $G_V(v)$and $G_W(w)$ and pdfs $g_V(v)$ and $g_W(w)$ respectively. Let $W_{RSS} = \{W_{[r]}, r = 1, 2, ...n\}$ be the ranked set sample of size n arising from (V, W) in which V observations are used to rank the units. If $Y \leq_{disp} W$, then $J(Y_{RSS}) \leq J(W_{RSS})$.

Proof: We have

$$J(Y_{RSS}) = \frac{-1}{2} \prod_{r=1}^{n} \int_{u=0}^{1} f_Y(F^{-1}(u))\Big(\rho_{(r,n,\alpha)}(u)\Big)^2 du.$$

Since $Y \leq_{disp} W$, we have $f_Y(F^{-1}(u)) \geq g_W(G^{-1}(u))$ for all u in $(0, 1)$.
Therefore

$$\begin{aligned} J(Y_{RSS}) &\leq \frac{-1}{2} \prod_{r=1}^{n} \int_{u=0}^{1} g_W(G^{-1}(u))\Big(\rho_{(r,n,\alpha)}(u)\Big)^2 du \\ &= J(W_{RSS}). \end{aligned}$$

Hence the proof. □

3.1. Bounds of $J(Y_{RSS})$

In this subsection, we obtain some lower bounds and upper bounds for $J(Y_{RSS})$. Before that we give some properties of $\rho_{(r,n,\alpha)}(u)$ given in (8). We have tabulated the value of $\rho_{(r,n,\alpha)}(u)$ for $r = 1, 2, ...10$ and $\alpha = -0.5, -0.25, 0.25$, and 0.5 and are given in Table 1 and Table 2. We have also drawn the graphs of $\rho_{(r,n,\alpha)}(u)$ for $n = 10$ and for $\alpha > 0$ and $\alpha < 0$ and are given in Figure 1 to Figure 4.

Remark 3: From Table 1 and Table 2, we have for a fixed α , $\rho_{(r,n,\alpha)}(u) = \rho_{(n-r+1,n,\alpha)}(1-u)$. The above inference also be seen from Figures 1, 2, 3 and 4.

Remark 4: From Figures 1 and 2 we have for $\alpha > 0$, $\rho_{(r,n,\alpha)}(u)$ is decreasing in r if $0 \leq u < 0.5$ and is increasing in r if $0.5 < u \leq 1$. Again for $\alpha < 0$, $\rho_{(r,n,\alpha)}(u)$ is increasing in r if $0 \leq u < 0.5$ and is decreasing in r if $0.5 < u \leq 1$.

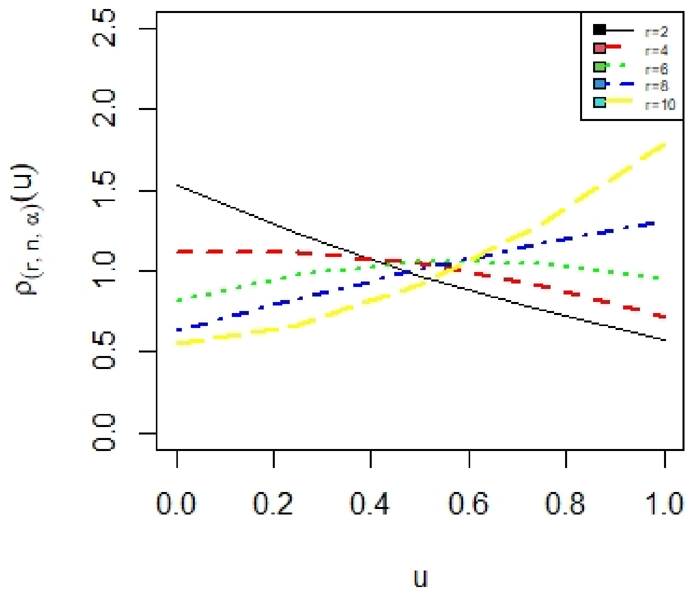

Figure 1: Graph of $\rho_{(r,n,\alpha)}(u)$ against u when $\alpha > 0$

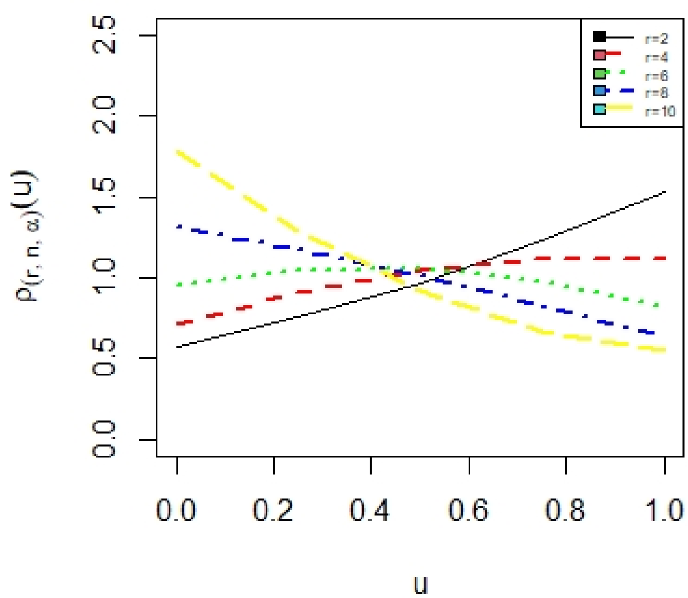

Figure 2: Graph of $\rho_{(r,n,\alpha)}(u)$ against u when $\alpha < 0$

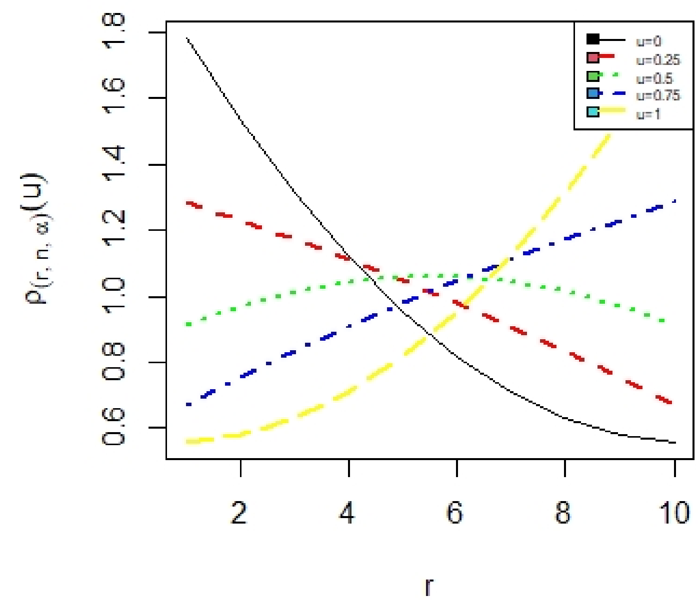

Figure 3: Graph of $\rho_{(r,n,\alpha)}(u)$ against r when $\alpha > 0$

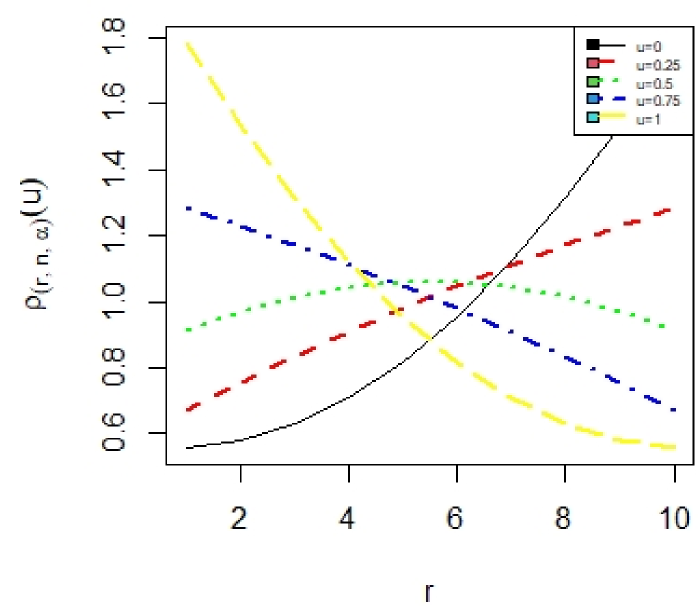

Figure 4: Graph of $\rho_{(r,n,\alpha)}(u)$ against r when $\alpha < 0$

Table 1: $\rho_{(r,n,\alpha)}(u)$ **when** α **is positive for** $n = 10$

	α=0.25				
r	u=0	u=0.25	u=0.5	u=0.75	u=1
1	1.7841	1.2855	0.9148	0.6719	0.5568
2	1.5341	1.2315	0.9716	0.7543	0.5795
3	1.3125	1.1740	1.0142	0.8331	0.6307
4	1.1193	1.1129	1.0426	0.9084	0.7102
5	0.9545	1.0483	1.0568	0.9801	0.8182
6	0.8182	0.9801	1.0568	1.0483	0.9545
7	0.7102	0.9084	1.0426	1.1129	1.1193
8	0.6307	0.8331	1.0142	1.1740	1.3125
9	0.5795	0.7543	0.9716	1.2315	1.5341
10	0.5568	0.6719	0.9148	1.2855	1.7841
	α=0.5				
r	u=0	u=0.25	u=0.5	u=0.75	u=1
1	2.9091	1.5284	0.6591	0.3011	0.4545
2	2.1818	1.4489	0.8864	0.4943	0.2727
3	1.5682	1.3551	1.0568	0.6733	0.2045
4	1.0682	1.2472	1.1705	0.8381	0.2500
5	0.6818	1.1250	1.2273	0.9886	0.4091
6	0.4091	0.9886	1.2273	1.1250	0.6818
7	0.2500	0.8381	1.1705	1.2472	1.0682
8	0.2045	0.6733	1.0568	1.3551	1.5682
9	0.2727	0.4943	0.8864	1.4489	2.1818
10	0.4545	0.3011	0.6591	1.5284	2.9091

Theorem 5: Let $Y_1, Y_2, ...Y_n$ be a simple random sample from a distribution with cdf $F_Y(y)$ and pdf $f_Y(y)$. Let $\{Y_{[r]}, r = 1, 2, ...n\}$ be the RSS of size n arising from Sarmanov family of distributions in which X observations are used to rank the units. If $Y_{SRS} = \{Y_1, Y_2, ..Y_n\}$ and $Y_{[RSS]} = \{Y_{[1]}, Y_{[2]}, ...Y_{[n]}\}$, then for $n \geq 1$,

$$\frac{J(Y_{RSS})}{J(Y_{SRS})} \leq \prod_{r=1}^{n} \Big(\rho_{(r,n,\alpha)}(u_0)\Big)^2,$$

where u_0 is the value of u which maximise $\rho_{(r,n,\alpha)}(u)$.

Proof: We have

$$\begin{aligned} J(Y_{SRS}) &= \frac{-1}{2}\prod_{r=1}^{n}\int_y \Big(f_Y(y)\Big)^2 dy \\ &= \frac{-1}{2}\prod_{r=1}^{n}\int_0^1 f_Y(F^{-1}(u))du. \end{aligned}$$

Then,

$$J(Y_{RSS}) = \frac{-1}{2}\prod_{r=1}^{n}\int_0^1 f_Y(F^{-1}(u))\Big(\rho_{(r,n,\alpha)}(u)\Big)^2 du.$$

Table 2: $\rho_{(r,n,\alpha)}(u)$ when α is negative for $n = 10$

	α=-0.5				
r	u=0	u=0.25	u=0.5	u=0.75	u=1
1	0.4545	0.3011	0.6591	1.5284	2.9091
2	0.2727	0.4943	0.8864	1.4489	2.1818
3	0.2045	0.6733	1.0568	1.3551	1.5682
4	0.2500	0.8381	1.1705	1.2472	1.0682
5	0.4091	0.9886	1.2273	1.1250	0.6818
6	0.6818	1.1250	1.2273	0.9886	0.4091
7	1.0682	1.2472	1.1705	0.8381	0.2500
8	1.5682	1.3551	1.0568	0.6733	0.2045
9	2.1818	1.4489	0.8864	0.4943	0.2727
10	2.9091	1.5284	0.6591	0.3011	0.4545
	α=-0.25				
r	u=0	u=0.25	u=0.5	u=0.75	u=1
1	0.5568	0.6719	0.9148	1.2855	1.7841
2	0.5795	0.7543	0.9716	1.2315	1.5341
3	0.6307	0.8331	1.0142	1.1740	1.3125
4	0.7102	0.9084	1.0426	1.1129	1.1193
5	0.8182	0.9801	1.0568	1.0483	0.9545
6	0.9545	1.0483	1.0568	0.9801	0.8182
7	1.1193	1.1129	1.0426	0.9084	0.7102
8	1.3125	1.1740	1.0142	0.8331	0.6307
9	1.5341	1.2315	0.9716	0.7543	0.5795
10	1.7841	1.2855	0.9148	0.6719	0.5568

Let u_0 be the value of u which maximise $\rho_{(r,n,\alpha)}(u)$. Then,

$$\begin{aligned} J(Y_{RSS}) &\geq \frac{-1}{2}\prod_{r=1}^{n}\int_0^1\left(f_Y(F^{-1}(u))(\rho_{(r,n,\alpha)}(u_0))^2\right)du \\ &= \frac{-1}{2}\prod_{r=1}^{n}\left(\int_0^1 f_Y(F^{-1}(u))du\right)\prod_{r=1}^{n}\left(\rho_{(r,n,\alpha)}(u_0)\right)^2 \\ &= J(Y_{SRS})\prod_{r=1}^{n}\left(\rho_{(r,n,\alpha)}(u_0)\right)^2. \end{aligned}$$

Since $J(Y_{SRS}) < 0$,

$$\frac{J(Y_{RSS})}{J(Y_{SRS})} \leq \prod_{r=1}^{n}\left(\rho_{(r,n,\alpha)}(u_0)\right)^2.$$

Hence the proof. □

Theorem 6: Let $Y_{RSS} = \{Y_{[r]}, r = 1, 2, ...n\}$ be the RSS of size n arising from Sarmanov family of distributions in which X observations are used to rank the units then for all $n \geq 1$, then the lower bound of extropty of Y_{RSS} is given by

$$J(Y_{RSS}) \geq \frac{-1}{2}\Big(Ef_Y(y)^2\Big)^{\frac{n}{2}} \prod_{r=1}^{n}\left(\int_0^1 \Big(\sum_{k=0}^{4} a_k u^k\Big)^2 du\right)^{\frac{1}{2}}.$$

Proof: We have

$$J(Y_{RSS}) = \frac{-1}{2}\prod_{r=1}^{n}\int_{u=0}^{1} f_Y(F^{-1}(u))\Big(\rho_{(r,n,\alpha)}(u)\Big)^2 du.$$

Using Cauchy-Schwarz inequality

$$J(Y_{RSS}) \geq \frac{-1}{2}\prod_{r=1}^{n}\Big(\int_{u=0}^{1} f_Y(F^{-1}(u))^2 du\Big)^{\frac{1}{2}}\Big(\int_{u=0}^{1}\Big(\rho_{(r,n,\alpha)}(u)\Big)^4 du\Big)^{\frac{1}{2}}.$$

We have $\Big(\rho_{(r,n,\alpha)}(u)\Big)^2 = \sum_{k=0}^{4} a_k u^k$.
Therefore

$$J(Y_{RSS}) \geq \frac{-1}{2}\Big(Ef_Y(y)^2\Big)^{\frac{n}{2}} \prod_{r=1}^{n}\left(\int_0^1 \Big(\sum_{k=0}^{4} a_k u^k\Big)^2 du\right)^{\frac{1}{2}}.$$

Hence the proof. □

4. Extropy of $(X_{RSS}, Y_{[RSS]})$

If $X_{(r)}$ is the observation measured on the auxiliary variable X of the unit chosen from the rth set then $X_{(r)}$ is the rth order statistic of a random sample of size n. Since $Y_{[r]}$ is the concomitant of $X_{(r)}$, the joint pdf of $(X_{(r)}, Y_{[r]})$ is given by

$$h(X_{(r)}, Y_{[r]}) = \frac{n!}{(r-1)!(n-r)!} f(x,y)(F_X(x))^{r-1}(1-F_X(x))^{(n-r)}. \quad (15)$$

Then the extropy of $(X_{(r)}, Y_{[r]})$ can be defined as

$$
\begin{aligned}
J(X_{(r)}, Y_{[r]}) &= \frac{-1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(h(X_{(r)}, Y_{[r]})\Big)^2 dydx \\
&= \frac{-1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(\frac{n!}{(r-1)!(n-r)!}\Big)^2\Big(f_X(x)\Big)^2\Big(f_Y(y)\Big)^2 \\
&\quad \times \Bigg[1 + 3\alpha(2F_X(x)-1)(2F_Y(y)-1) \\
&\quad + \frac{5}{4}\alpha^2\Big(3(2F_X(x)-1)^2-1\Big)\Big(3(2F_Y(y)-1)^2-1\Big)\Bigg]^2 \\
&\quad \times \Big(F_X(x)\Big)^{2(r-1)}\Big(1-F_X(x)\Big)^{2(n-r)} dxdy \\
&= -\frac{1}{2}\Big(\frac{n!}{(r-1)!(n-r)!}\Big)^2\Bigg[M_{00}N_{00} + 9\alpha^2 M_{20}N_{20} + \frac{25}{16}\alpha^2 M_{02}N_{02} \\
&\quad + 6\alpha M_{10}N_{10} + \frac{5}{2}\alpha^2 M_{01}N_{01} + \frac{15}{2}\alpha^3 M_{11}N_{11}\Bigg], \qquad (16)
\end{aligned}
$$

where M_{ij} and N_{ij} for $i = 0, 1$ and 2 are given below.

$$
\begin{aligned}
M_{ij} &= \int (f_X(x))^2 (F_X(x))^{2(r-1)}(1-F_X(x))^{2(n-r)}(2F_X(x)-1)^i(3(2F_X(x)-1)^2-1)^j dx \\
&= \frac{(2r-2)!(2n-2r)!}{(2n-1)!}E\Big[f_X(F^{-1}(U))(2U-1)^i(3(2U-1)^2-1)^j\Big], \qquad (17)
\end{aligned}
$$

where U follows beta distribution with parameters $(2r-1, 2n-2r+1)$ and

$$
\begin{aligned}
N_{ij} &= \int (f_Y(y))^2(2F_Y(y)-1)^i(3(2F_Y(y)-1)^2-1)^j dy \\
&= E\Big[f_Y(F^{-1}(V))(2V-1)^i(3(2V-1)^2-1)^j\Big], \qquad (18)
\end{aligned}
$$

where V follows uniform distribution over $(0, 1)$.
If $X_{RSS} = \{X_{(1)}, X_{(2)}, .., X_{(n)}\}$, then X_{RSS} is the RSS of X observations in which ranking of units in each set is perfect. Let $(X_{RSS}, Y_{[RSS]}) = \{(X_{(r)}, Y_{[r]}), r = 1, 2, 3...n\}$ then extropy of

$(X_{RSS}, Y_{[RSS]})$ is given by

$$J(X_{RSS}, Y_{[RSS]}) = \frac{-1}{2}\prod_{r=1}^{n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(h(X_{(r)}, Y_{[r]})\Big)^2 dydx$$
$$= \frac{-1}{2}\prod_{r=1}^{n} -2J(X_{(r)}, Y_{[r]})$$
$$= \frac{-1}{2}\prod_{r=1}^{n}\Big(\frac{n!}{(r-1)!(n-r)!}\Big)^2$$
$$\times\Bigg[M_{00}N_{00} + 9\alpha^2 M_{20}N_{20} + \frac{25}{16}\alpha^2 M_{02}N_{02}$$
$$+ 6\alpha M_{10}N_{10} + \frac{5}{2}\alpha^2 M_{01}N_{01} + \frac{15}{2}\alpha^3 M_{11}N_{11}\Bigg]. \tag{19}$$

Example 9: If (X, Y) follows Sarmanov family of distributions given in (3) with marginal pdfs of X and Y are $f_X(x) = 1, 0 \leq x \leq 1$ and $f_Y(y) = 1, 0 \leq y \leq 1$ respectively, then

$$M_{ij} = \frac{(2r-2)!(2n-2r)!}{(2n-1)!}E\Big[(2U-1)^i(3(2U-1)^2-1)^j\Big]$$
$$= \int_0^1 (2u-1)^i(3(23-1)^2-1)^j u^{2r-2}(1-u)^{2n-2r}du$$

and

$$N_{ij} = E\Big[(2V-1)^i(3(2V-1)^2-1)^j\Big]$$
$$= \int_0^1 (2v-1)^i(3(2v-1)^2-1)^j dv.$$

Therefore,

$$M_{00} = \frac{(2r-2)!(2n-2r)!}{(2n-1)!},$$

$$M_{10} = \frac{(2r-2)!(2n-2r)!}{(2n-1)!}\left[\frac{(2r-1)}{n} - 1\right],$$

$$M_{20} = \frac{(2r-2)!(2n-2r)!}{(2n-1)!}\left[\frac{4(2r)(2r-1)}{(2n)(2n+1)} - \frac{4(2r-1)}{2n} + 1\right],$$

$$M_{01} = \frac{(2r-2)!(2n-2r)!}{(2n-1)!}\left[\frac{12(2r)(2r-1)}{(2n)(2n+1)} - \frac{12(2r-1)}{2n} + 2\right],$$

$$M_{02} = 4\frac{(2r-2)!(2n-2r)!}{(2n-1)!}\Bigg[\frac{36(2r+2)(2r+1)(2r)(2r-1)}{(2n+3)(2n+2)(2n+1)(2n)}$$
$$- \frac{72(2r+1)(2r)(2r-1)}{(2n+2)(2n+1)(2n)} + \frac{48(2r)(2r-1)}{(2n+1)(2n)} - \frac{12(2r-1)}{2n} + 1\Bigg],$$

and

$$M_{11} = 2\frac{(2r-2)!(2n-2r)!}{(2n-1)!}\Bigg[\frac{12(2r+1)(2r)(2r-1)}{(2n+2)(2n+1)(2n)} - \frac{18(2r)(2r-1)}{(2n+1)(2n)} + \frac{8(2r-1)}{2n} - 1\Bigg]$$

Also $N_{00} = 1$, $N_{10} = 0$, $N_{20} = \frac{1}{3}$, $N_{01} = 0$, $N_{02} = \frac{4}{5}$ and $N_{11} = 0$. Then from (16),

$$J(X_{(r:n)}, Y_{[r:n]}) = \frac{-1}{2}\Big(\frac{n!}{(r-1)!(n-r)!}\Big)^2 \frac{(2r-2)!(2n-2r)!}{(2n-1)!} \times \Bigg[\frac{180\alpha^2(2r+2)(2r+1)(2r)(2r-1)}{(2n+3)(2n+2)(2n+1)(2n)} - \frac{360\alpha^2(2r+1)(2r)(2r-1)}{(2n+2)(2n+1)(2n)} + \frac{252\alpha^2(2r)(2r-1)}{(2n+1)(2n)} - \frac{72\alpha^2(2r-1)}{2n} + 8\alpha^2 + 1\Bigg].$$

Therefore,

$$J(X_{RSS}, Y_{[RSS]}) = \frac{-1}{2}\prod_{r=1}^{n}\Big(\frac{n!}{(r-1)!(n-r)!}\Big)^2 \frac{(2r-2)!(2n-2r)!}{(2n-1)!} \times \Bigg[\frac{180\alpha^2(2r+2)(2r+1)(2r)(2r-1)}{(2n+3)(2n+2)(2n+1)(2n)} - \frac{360\alpha^2(2r+1)(2r)(2r-1)}{(2n+2)(2n+1)(2n)} + \frac{252\alpha^2(2r)(2r-1)}{(2n+1)(2n)} - \frac{72\alpha^2(2r-1)}{2n} + 8\alpha^2 + 1\Bigg].$$

Example 10: If (X, Y) follows Sarmanov family of distributions given in (3) with marginal pdfs of X and Y are $f_X(x) = \theta_1 e^{-\theta_1 x}, x \geq 0$ and $f_Y(y) = \theta_2 e^{-\theta_2 y}, y \geq 0$ respectively,then

$$\begin{aligned} M_{ij} &= \frac{(2r-2)!(2n-2r)!}{(2n-1)!}\theta_1 E\Big[(1-U)(2U-1)^i(3(2U-1)^2-1)^j\Big] \\ &= \theta_1\int_0^1 (1-u)(2u-1)^i(3(23-1)^2-1)^j u^{2r-2}(1-u)^{2n-2r}du \end{aligned}$$

and

$$\begin{aligned} N_{ij} &= \theta_2 E\Big[(1-V)(2V-1)^i(3(2V-1)^2-1)^j\Big] \\ &= \theta_2\int_0^1 (1-v)(2v-1)^i(3(2v-1)^2-1)^j dv. \end{aligned}$$

Therefore,

$$M_{00} = \frac{(2r-2)!(2n-2r+1)!}{(2n)!}\theta_1,$$

$$M_{10} = \frac{(2r-2)!(2n-2r+1)!}{(2n)!}\theta_1\left[\frac{2(2r-1)}{(2n+1)} - 1\right],$$

$$M_{20} = \frac{(2r-2)!(2n-2r+1)!}{(2n)!}\theta_1\left[\frac{4(2r)(2r-1)}{(2n+1)(2n+2)} - \frac{4(2r-1)}{(2n+1)} + 1\right],$$

$$M_{01} = \frac{(2r-2)!(2n-2r+1)!}{(2n)!}\theta_1\left[\frac{12(2r)(2r-1)}{(2n+1)(2n+2)} - \frac{12(2r-1)}{(2n+1)} + 2\right],$$

$$M_{02} = \frac{4(2r-2)!(2n-2r+1)!}{(2n)!}\theta_1\left[\frac{36(2r+2)(2r+1)(2r)(2r-1)}{(2n+4)(2n+3)(2n+2)(2n+1)} - \frac{72(2r+1)(2r)(2r-1)}{(2n+3)(2n+2)(2n+1)} + \frac{48(2r)(2r-1)}{(2n+2)(2n+1)} - \frac{12(2r-1)}{2n+1} + 1\right]$$

and

$$M_{11} = \frac{2(2r-2)!(2n-2r+1)!}{(2n)!}\theta_1\left[\frac{122(2r+1)(2r)(2r-1)}{(2n+3)(2n+2)(2n+1)} - \frac{18(2r)(2r-1)}{(2n+2)(2n+1)} - \frac{8(2r-1)}{2n+1} - 1\right].$$

Also, $N_{00} = \frac{\theta_2}{2}$, $N_{10} = \frac{-\theta_2}{6}$, $N_{20} = \frac{\theta_2}{6}$, $N_{01} = 0$, $N_{02} = \frac{2\theta_2}{5}$ and $N_{11} = \frac{-2\theta_2}{15}$. Then from (16),

$$\begin{aligned} J(X_{(r:n)}, Y_{[r:n]}) = \frac{-1}{2}\left(\frac{n!}{(r-1)!(n-r)!}\right)^2 \frac{(2r-2)!(2n-2r+1)!}{(2n)!}\theta_1\theta_2 \\ \times \left[\frac{45\alpha^2(2r+2)(2r+1)(2r)(2r-1)}{4(2n+4)(2n+3)(2n+2)(2n+1)} \right. \\ - \frac{(2r+1)(2r)(2r-1)(45\alpha^2 - 24\alpha^3)}{(2n+3)(2n+2)(2n+1)} + \frac{36(2r)(2r-1)(\alpha^2+\alpha^3)}{(2n+2)(2n+1)} \\ \left. - \frac{(32\alpha^3 + 27\alpha^2 + 4\alpha)}{2}\frac{\alpha^2(2r-1)}{2n+1} + \alpha^3 + \frac{51}{24}\alpha^2 + \alpha + \frac{1}{2}\right]. \end{aligned}$$

Therefore,

$$J(X_{RSS}, Y_{[RSS]}) = \frac{-\theta_1^n \theta_2^n}{2} \prod_{r=1}^{n} \left(\frac{n!}{(r-1)!(n-r)!}\right)^2 \frac{(2r-2)!(2n-2r+1)!}{(2n)!}$$
$$\times \left[\frac{45\alpha^2(2r+2)(2r+1)(2r)(2r-1)}{4(2n+4)(2n+3)(2n+2)(2n+1)} - \frac{(2r+1)(2r)(2r-1)(45\alpha^2-24\alpha^3)}{(2n+3)(2n+2)(2n+1)} + \frac{36(2r)(2r-1)(\alpha^2+\alpha^3)}{(2n+2)(2n+1)} - \frac{(32\alpha^3+27\alpha^2+4\alpha)}{2}\frac{\alpha^2(2r-1)}{2n+1} + \alpha^3 + \frac{51}{24}\alpha^2 + \alpha + \frac{1}{2}\right].$$

5. Conclusion

In this work, we considered the extropy of concomitants of order statistic arising from Sarmanov family of distributions when ranking is subject to error. If we considered a ranked set sampling in which an auxiliary variable is used to rank the units in each set, then the observation of RSS are nothing but concomitants of order statistics. Hence by using the results for extropy of concomitants of order statistics $Y_{[r:n]}$, we derived the extropy of RSS in which units are ranked based on measurements made on an easily and exactly measurable auxiliary variable X which is correlated with the study variable Y, under the assumption that (X, Y) follows Sarmanov family of distributions. The lower and upper bounds of extropy of $Y_{[r:n]}$ were obtained. Moreover, we obtained the lower and upper bound of extropy of RSS. The upper bound for the ratio of extropy of ranked set sample to that of simple random sample were obtained. The extropy of $(X_{RSS}, Y_{[RSS]})$ were also obtained for Sarmanov family of distributions, where X_{RSS} is the RSS of the X observations and $Y_{[RSS]}$ is the RSS of the Y observations in which X observations are used to rank.

Acknowledgements

The authors are thankful to the referee for valuable comments and constructive criticism which lead to an improved version of the manuscript.

References

Abdallah, A., Boucher, J.-P., and Cossette, H. (2016). Sarmanov family of multivariate distributions for bivariate dynamic claim counts model. *Insurance: Mathematics and Economics*, **68**, 120–133.

Alemany, R., Bolancé, C., Rodrigo, R., and Vernic, R. (2020). Bivariate mixed poisson and normal generalised linear models with sarmanov dependence-an application to model claim frequency and optimal transformed average severity. *Mathematics*, **9**, 73.

Bain, L. (2017). *Statistical Analysis of Reliability and Life-testing Models: Theory and Methods*. Routledge.

Barakat, H., Alawady, M., Husseiny, I., and Mansour, G. (2022). Sarmanov family of bivariate distributions: statistical properties-concomitants of order statistics-information measures. *Bulletin of the Malaysian Mathematical Sciences Society*, **45**, 49–83.

Bolancé, C., Guillen, M., and Pitarque, A. (2020). A sarmanov distribution with beta marginals: An application to motor insurance pricing. *Mathematics*, **8**, 2020.

Chacko, M. (2017). Bayesian estimation based on ranked set sample from morgenstern type bivariate exponential distribution when ranking is imperfect. *Metrika*, **80**, 333–349.

Chacko, M. and George, V. (2023). Extropy properties of ranked set sample for cambanis type bivariate distributions. *Journal of the Indian Society for Probability and Statistics*, **24**, 111–133.

Chacko, M. and George, V. (2024). Extropy properties of ranked set sample when ranking is not perfect. *Communications in Statistics-Theory and Methods*, **53**, 3187–3210.

Chacko, M. and Thomas, P. Y. (2007). Estimation of a parameter of bivariate pareto distribution by ranked set sampling. *Journal of Applied Statistics*, **34**, 703–714.

Chacko, M. and Thomas, P. Y. (2008). Estimation of a parameter of morgenstern type bivariate exponential distribution by ranked set sampling. *Annals of the Institute of Statistical Mathematics*, **60**, 301–318.

Chacko, M. and Thomas, P. Y. (2009). Estimation of parameters of morgenstern type bivariate logistic distribution by ranked set sampling. *Journal of the Indian Society of Agricultural Statistics*, **63**, 77–83.

Chen, Z., Bai, Z., and Sinha, B. K. (2004). *Ranked Set Sampling: Theory and Applications*, volume 176. Springer.

David, H. A. (1973). Concomitants of order statistics. *Bulletin of the International Statistical Institute*, **45**, 295–300.

David, H. A. and Nagaraja, H. N. (1998). 18 concomitants of order statistics. *Handbook of Statistics*, **16**, 487–513.

Eftekharian, A. and Qiu, G. (2022). On extropy properties and discrimination information of different stratified sampling schemes. *Probability in the Engineering and Informational Sciences*, **36**, 644–659.

George, V. and Chacko, M. (2023). Cumulative residual extropy properties of ranked set sample for cambanis type bivariate distributions: Cumulative residual extropy properties of ranked set sample. *Journal of the Kerala Statistical Association*, **33**, 50–70.

Husseiny, I., Barakat, H., Mansour, G., and Alawady, M. (2022). Information measures in records and their concomitants arising from sarmanov family of bivariate distributions. *Journal of Computational and Applied Mathematics*, **408**, 114120.

Jozani, M. J. and Ahmadi, J. (2014). On uncertainty and information properties of ranked set samples. *Information Sciences*, **264**, 291–301.

Lad, F., Sanfilippo, G., and Agro, G. (2015). Extropy: Complementary dual of entropy. *Statistical Science*, **30**.

McIntyre, G. (1952). A method for unbiased selective sampling, using ranked sets. *Australian Journal of Agricultural Research*, **3**, 385–390.

Mehta, V. (2022). An improved estimation of parameter of morgenstern-type bivariate exponential distribution using ranked set sampling. In *Ranked Set Sampling Models and Methods*, pages 1–25. IGI Global.

Morgenstern, D. (1956). Einfache beispiele zweidimensionaler verteilungen. *Mitteilingsblatt fur Mathematische Statistik*, **8**, 234–235.

Qiu, G. (2017). The extropy of order statistics and record values. *Statistics & Probability Letters*, **120**, 52–60.

Qiu, G. and Raqab, M. Z. (2022). On weighted extropy of ranked set sampling and its comparison with simple random sampling counterpart. *Communications in Statistics-Theory and Methods*, **53**, 1–18.

Raqab, M. Z. and Qiu, G. (2019). On extropy properties of ranked set sampling. *Statistics*, **53**, 210–226.

Sarmanov, O. V. (1966). Generalized normal correlation and two-dimensional fréchet classes. In *Doklady Akademii Nauk*, volume 168, pages 32–35. Russian Academy of Sciences.

Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic Orders*. Springer.

Shannon, C. E. (1948). A mathematical theory of communication. *The Bell System Technical Journal*, **27**, 379–423.

Stokes, S. (1977). Ranked set sampling with concomitant variables. *Communications in Statistics-Theory and Methods*, **6**, 1207–1211.

Zamanzade, E. and Mahdizadeh, M. (2019). Extropy estimation in ranked set sampling with its application in testing uniformity. In *Ranked set sampling*, pages 259–267. Elsevier.