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Simultaneous Testing Procedure for the Ordered Pair-Wise Comparisons of Location Parameters of Exponential Populations under Heteroscedasticity

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Abstract

Zheng (2013) provided the penalized maximum likelihood estimators (PMLEs) of the location and scale parameters of two-parametric exponential distribution and proved that these estimators are uniformly minimum variance unbiased estimators (UMVUE). In this paper, a test procedure has been proposed, on the basis of the PMLEs of the location and scale parameters of the two-parametric exponential distribution. The purpose of the proposed procedure is to construct the simultaneous confidence intervals (SCIs) for the ordered pair-wise comparisons of location parameters of multi-sample two-parameter exponential distributions under the heteroscedasticity of scale parameters. A Monte Carlo simulation study has revealed that the proposed procedure is better than the existing procedure of Singh and Singh (2013) in terms of coverage probability, average volume, and power. Implementation of the proposed procedure is illustrated through real-life numerical data.

Key words: Simultaneous confidence interval (SCIs); Penalized maximum likelihood estimators (PMLEs); Heteroscedasticity; Simulated power comparison.

1. Introduction

Suppose the k (≥ 3) independent populations are such that the statistical model for the observations from the ith population is a two-parameter exponential distribution, denoted by $E_i(\mu_i, \theta_i)$, with probability density function (pdf)

$$f(x|\gamma_i, \delta_i) = \begin{cases} \dfrac{1}{\theta_i} e^{-(\frac{x-\mu_i}{\theta_i})}, & x \geq \mu_i, \theta_i > 0 \\ 0, & Otherwise, \end{cases}$$

where μ_i and $\theta_i$are the location and the scale parameters respectively, $i = 1, \ldots, k$.

In some of the practical situations, there is prior information of the ordering among the location parameters. For example, in dose-response experiments, the effect of a treatment may be related monotonically to the increasing levels of dose of a drug. Similarly, in against accelerated life testing, the higher stress level may lead to lowering the guaranteed lifetime. Many researchers have proposed statistical tests to test the null hypothesis $H_0: \mu_1 = \cdots =$

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μ_k the simple ordered alternative $H_1: \mu_1 \leq \cdots \leq \mu_k$, with at least one strict inequality, for normal and exponential probability models. This problem of simple ordered alternative is a member of the class of order restricted alternatives. A detailed discussion on order restricted statistical inferences can be found in Barlow *et al.* (1972) and Robertson and Dykstra (1988). Marcus (1976), Hayter (1990), Lee and Spurrier (1995), Liu *et al.* (2000) have also proposed tests for the simple ordered alternatives under normal probability model. Chen (1982) and Dhawan and Gill (1997) inverted the test procedures for testing homogeneity of the location parameters of k (≥ 3) two-parameter exponential distributions to construct simultaneous confidence intervals (SCIs) for the ordered pair-wise differences of location parameters under the assumption of homogeneity of scale parameters. Singh *et al*. (2006) proposed a procedure for successive comparisons of the location parameters of exponential distributions by assuming the equality of scale parameters. Maurya *et al.* (2011) came up with one-stage and two-stage multiple comparison procedures using Lam's (1987,1988) technique and obtained the conservative simultaneous confidence intervals (SCIs) for successive differences of the location parameters of several exponential distributions under the heteroscedasticity of scale parameters, *i.e.*, $\theta_i \neq \theta_j, 1 \leq i < j \leq k$. Later, Singh and Singh (2013) put forward less conservative SCIs by extending Maurya *et al*. (2011) procedure. Kharrati Kopaei (2014) introduced a new lemma and used the same to provide SCIs for the successive differences of the location parameters which were less conservative than the SCIs of Maurya *et al*. (2011). It may be noted that Maurya *et al.* (2011), Singh and Singh (2013) and Kharrati Kopaei (2014) used the maximum likelihood estimator (MLE) of the location parameter. Although, the MLEs have a few desirable properties like efficiency and consistency but may not be unbiased. Zheng (2013) provided the penalized maximum likelihood estimators (PMLEs) of the location and scale parameters of two-parameter exponential distribution which are uniformly minimum variance unbiased estimators (UMVUEs). In this article, we have proposed one-stage and two-stage multiple comparison procedures to construct SCIs using the PMLEs of the location parameters for the ordered pair-wise differences of location parameters under heteroscedasticity of scale parameters. The layout of the paper is as follows.

In this paper, Sections 2 and 3 respectively contain the proposed one-stage and two-stage multiple comparison procedures to construct the simultaneous confidence intervals (SCIs) for the ordered pair-wise differences of location parameters. In Section 4, the results of Monte Carlo simulation studies conducted to compare the power, coverage probabilities (CP), and average volume (AV) of the proposed procedures with the procedure of Singh and Singh (2013), are presented. The implementation and the better performance ability of the proposed procedures over the more conservative procedure of Singh and Singh (2013), is demonstrated by taking a real-life example in Section 5. Finally, a brief conclusion is presented in Section 6.

2. One-Stage Procedure for the Simultaneous Testing of the Ordered Differences of Location Parameters

Let there be k independent exponential populations and that $X_{i1}, X_{i2}, \ldots, X_{im}$ be a random sample of size $m\,(> 2)$ from the ith population $E_i(\mu_i, \theta_i)$, $i = 1, \ldots, k$. The maximum likelihood estimators (MLEs) of μ_i and θ_i are $X_i = \min(X_{i1}, X_{i2}, \ldots, X_{im})$ and $V_i = \sum_j^m (X_{ij} - X_i)/m$, respectively and these MLEs are not unbiased estimators. In literature, an approach exists in which a penalty is added to the regular likelihood function so that the new function no longer remains a monotone function of the location parameter. Let $X_{i[1]} \leq$

$X_{i[2]} \dots \leq X_{i[m]}$ be the ordered values corresponding to the above random sample. Zheng (2013) used the penalty term $x_{i[1]} - \mu_i$ in the regular likelihood function, where $x_{i[1]}$ is the realized value of $X_{i[1]}$, and gave the penalized maximum likelihood function as follows:

$$L(\mu_i, \theta_i) = \left(x_{i[1]} - \mu_i\right) \prod_{j=1}^{m} f\left(x_{i[1]} \middle| \mu_i, \theta_i\right) = \left(x_{i[1]} - \mu_i\right) \frac{1}{\theta_i^m} e^{-\frac{1}{\theta}\sum_{j=1}^{m}(x_{ij}-\mu_i)}, x_{i[1]} \geq \mu_i$$

The penalized maximum likelihood estimators (PMLEs) of μ_i and θ_i obtained from the above likelihood function are $Y_i = \frac{mX_{i[1]}-\bar{X}}{(m-1)}$ and $S_i = \frac{m(\bar{X}-X_{i[1]})}{(m-1)}$ respectively, where $\bar{X} = \sum_{j=1}^{m} X_{ij}/m,$ is the sample mean. It is also proven that these estimators of the location and scale parameters are unique minimum variance unbiased estimators (UMVUEs). Previously, the same estimators have also been obtained by Cohen and Helm (1973) and Sarhan (1954) using different methods of estimation such as modified moment and least square, respectively.

Consider the family of hypotheses for the ordered location parameters

(i) $H_{0i}: \mu_j - \mu_i = 0$ against $H_{1i}: \mu_j - \mu_i > 0,\ 1 \leq i < j \leq k$ (One-sided problem)

(ii) $H_{0i}: \mu_j - \mu_i = 0$ against $H_{2i}: \mu_j - \mu_i \neq 0, 1 \leq i < j \leq k$ (Two-sided problem)

For the testing of these hypotheses, we can use the one-stage multiple comparison procedure given by Lam (1987, 1988) to construct simultaneous confidence intervals (SCIs) for the one-sided and two-sided sets of pair-wise differences of the ordered location parameters when the scale parameters are unknown and $\theta_i \neq \theta_j, 1 \leq i < j \leq k,$ *i.e.*, heteroscedasticity of scale parameters exists. One-stage multiple comparison procedure has the merit over a two-stage procedure (as described in detail in Section 3) in practical situations where the second stage of sampling is not possible due to the shortage of time, budget, and destructive type of experiments or some other factors.

The PMLEs of the location and scale parameters have been utilized instead of the MLEs for the simultaneous testing of the ordered location parameters. It is easy to verify that the PMLEs of the location and scale parameter can be written $Y_i = X_i - S_i/m$ and $S_i = \sum_j^m (X_{ij} - X_i)/(m-1)$. Define a constant $d = \max_{1 \leq i \leq k}(S_i/m)$. The random variables $T_i = (X_i - \mu_i)/\theta_i$ and $2(m-1)S_i/\theta_i$ are stochastically independently distributed as $E(0,1)$ and Chi-square with $2(m-1)$ degree of freedom (d.f.), respectively. Hence, the statistic $W_i^* = m(X_i - \mu_i)/S_i$ is distributed as Snedecor F with $(2, 2m-2)$ degree of freedom (d.f.). Using a one-stage procedure on the similar lines of Lam's (1987, 1988), the proposed one-sided and two-sided simultaneous confidence intervals (SCIs) for the ordered pair-wise differences of location parameters under heteroscedasticity of scale parameters are given in the following theorem.

Theorem 1: Let $q_{k,m,\alpha} = F_{2,2m-2}^{-1}(1-\alpha)^{1/(k-1)} - 1$ and $r_{k,m,\alpha} = F_{2,2m-2}^{-1}(1-\alpha)^{1/k} - 1$, for given $0 < \alpha < 1$

(i) $P\left(\mu_j - \mu_i \geq Y_j - Y_i - dq_{k,m,\alpha},\ 1 \leq i < j \leq k\right) \geq 1 - \alpha.$
Then $\left(Y_j - Y_i - dq_{k,m,\alpha}, \infty\right)$ is the set of one-sided simultaneous confidence intervals for $\mu_j - \mu_i$ with confidence coefficient at least $(1-\alpha)$.

(ii) $P\left(Y_j - Y_i - dr_{k,m,\alpha} \leq \mu_j - \mu_i \leq Y_j - Y_i + dr_{k,m,\alpha},\ 1 \leq i < j \leq k\right) \geq 1 - \alpha.$

Then $(Y_j - Y_i - dr_{k,m,\alpha}, Y_j - Y_i + dr_{k,m,\alpha})$ is the set of two-sided simultaneous confidence intervals for $\mu_j - \mu_i$ with confidence coefficient at least $(1-\alpha)$.

We applied the following lemma of Lam (1987, 1988) to prove Theorem 1.

Lemma 1: Suppose X and Y are two random variables, and a and b are two positive constants; then

$$[aX \geq bY - dmax(a,b)] \supseteq [X \geq -d, Y \leq d \; and \; X \geq Y - d].$$

The proofs of the part (i) and (ii) of Theorem 1 on the basis of Lemma 1 are as follow

Proof of part (i): $P\big(\mu_j - \mu_i \geq \; Y_j - Y_i - dq_{k,m,\alpha}, \; 1 \leq i < j \leq k\big)$
$=P\big(X_i - \mu_i - S_i/m \geq \; X_j - \mu_j - S_j/m - dq_{k,m,\alpha}, \; 1 \leq i < j \leq k\big)$
$= P\big(S_i/m(W_i^* - 1)) \geq \; S_j/m\big(W_j^* - 1\big) - dq_{k,m,\alpha}, \; 1 \leq i < j \leq k\big)$
$= P\big(W_j^* - 1 \leq q_{k,m,\alpha}, \; 1 \leq i < j \leq k\big)$
$= 1 - \alpha \; \big(\text{Since } q_{k,m,\alpha} = F^{-1}_{2,2m-2}(1-\alpha)^{1/(k-1)} - 1 \;\big).$

Proof of part (ii): $P\big(Y_j - Y_i - dr_{k,m,\alpha} \leq \mu_j - \mu_i \leq Y_j - Y_i + dr_{k,m,\alpha}, \; 1 \leq i < j \leq k\big)$

$$= P\big(X_j - S_j/m - X_i + S_i/m - dr_{k,m,\alpha} \leq \mu_j - \mu_i \leq X_j - S_j/m - X_i + S_i/m + dr_{k,m,\alpha}, 1 \leq i < j \leq k\big)$$

$$= P(S_i/m\,(W_i^* - 1) \geq S_j/m\big(W_j^* - 1\big) - dr_{k,m,\alpha} \cap S_j/m\big(W_j^* - 1\big) \geq S_i/m(W_i^* - 1) - \; dr_{k,m,\alpha}, 1 \leq i < j \leq k$$

$$= P\big(W_j^* - 1 \leq r_{k,m,\alpha} \cap W_i^* - 1 \leq r_{k,m,\alpha}, 1 \leq i < j \leq k\big)$$

$$= 1 - \alpha. \; \big(\text{Since } r_{k,m,\alpha} = F^{-1}_{2,2m-2}(1-\alpha)^{1/(k)} - 1 \;\big).$$

Here $F^{-1}_{2,2m-2}(x)$ denotes the xth quantile of the snedecor F distribution with $(2, 2m-2)$ degree of freedom (d.f.).

3. Two-Stage Procedure for the Simultaneous Testing of the Ordered Differences of Location Parameters

A two-stage multiple comparison procedure has been used on the similar lines of Lam's (1987, 1988) to construct one-sided and two-sided simultaneous confidence intervals (SCIs) for the ordered pair-wise comparisons of location parameters of several exponential populations under the heteroscedasticity of scale parameters, which is explained below:

Stage 1: In the first stage, the procedure begins by taking random sample $X_{i1}, X_{i2}, \dots, X_{im}$, of size $m \; (\geq 2)$ from the ith population $E_i(\mu_i, \theta_i)$. Let $\tilde{Y}_i = X_i - S_i/m$ and $S_i = \sum_j^m (X_{ij} - X_i)/(m-1)$ be the PMLEs of μ_i and θ_i, respectively, where $X_i = \min(X_{i1}, X_{i2}, \dots, X_{im})$, $i = 1, \dots, k$. The random variables $T_i = (X_i - \mu_i)/\theta_i$ and $2(m-1)S_i/\theta_i$ are independently distributed as $E(0,1)$ and Chi-square with $2(m-1)$ d.f., respectively.
Stage 2: In the second stage $(N_i - m)$ additional observations are taken, for that we defined $N_i = \max[m, [S_i/c] + 1]$, $i = 1, \dots, k$, where c is an arbitrary positive constant to be chosen to control the width of the confidence intervals and $[x]$ denotes the greatest integer less than or equal to x. If $N_i = m$, we do not take any more sample observations from each population. If $N_i > m$, then take $(N_i - m)$ more/additional sample observations $X_{i,m+1}, \dots, X_{iN_i}$, from the

ith population $E_i(\mu_i, \theta_i)$. This is known as the second stage of the two-stage procedure. Now, based on the combined sample observations $X_{i,1}, \dots, X_{i,m}, X_{i,m+1}, \dots, X_{i,N_i}$, let $\tilde{X}_i = \tilde{X}_{iN_i} = \min(X_{i,1}, \dots, X_{i,m}, X_{i,m+1}, \dots, X_{i,N_i})$ and $\tilde{Y}_i = \tilde{X}_i - S_i/N_i$. It can be noted that $U_i = N_i(\tilde{X}_{iN_i} - \mu_i)/\theta_i$ and $2(m-1)S_i/\theta_i$ are stochastically independently distributed as $E(0,1)$ and Chi-square with $2(m-1)$ d.f., respectively. Hence $W_i = N_i(\tilde{X}_{iN_i} - \mu_i)/S_i$ is distributed as Snedecor F with $(2, 2m-2)$ d.f.

The following theorem will provide us the one-sided and two-sided simultaneous confidence intervals (SCIs) for the ordered pair-wise differences of location parameters under heteroscedasticity of scale parameters.

Theorem 2: Let $u_{k,m,\alpha} = F^{-1}_{2,2m-2}(1-\alpha)^{1/(k-1)} - 1$ and $v_{k,m,\alpha} = F^{-1}_{2,2m-2}(1-\alpha)^{1/k} - 1$, for given $0 < \alpha < 1$

(i) $P(\mu_j - \mu_i \geq \tilde{Y}_j - \tilde{Y}_i - cu_{k,m,\alpha}, 1 \leq i < j \leq k) \geq 1 - \alpha.$
Then $(\tilde{Y}_j - \tilde{Y}_i - cu_{k,m,\alpha}, \infty)$ is the set of one-sided simultaneous confidence intervals for $\mu_j - \mu_i$ with confidence coefficient at least $(1-\alpha)$.

(ii) $P(\tilde{Y}_j - \tilde{Y}_i - cv_{k,m,\alpha} \leq \mu_j - \mu_i \leq \tilde{Y}_j - \tilde{Y}_i + cv_{k,m,\alpha},\ 1 \leq i < j \leq k) \geq 1 - \alpha.$

Then $(\tilde{Y}_j - \tilde{Y}_i - cv_{k,m,\alpha}, \tilde{Y}_j - \tilde{Y}_i + cv_{k,m,\alpha})$ is the set of two-sided simultaneous confidence intervals for $\mu_j - \mu_i$ with confidence coefficient at least $(1-\alpha)$.

Proof: The proof of the Theorem 2 is based on the similar lines of Theorem 1, by replacing c with d.

4. Simulation Study

For the purpose of comparison of the proposed procedures, say Prop, with the procedure of Singh and Singh (2013), say SS, a Monte Carlo simulation study has been performed using 10^5 iterations. The simulated power, coverage probability (CP), and the average volume (AV) of SCIs under each of these procedures have been computed. In each iteration fresh random samples were generated from each of the $k = 4$ exponential distributions with location parameters $(\mu_1, \mu_2, \mu_3, \mu_4)$ and scale parameters $(\theta_1, \theta_2, \theta_3, \theta_4)$. We have used the values of sample size and parametric configuration, i.e., the value of m, $(\mu_1, \mu_2, \mu_3, \mu_4)$ and $(\theta_1, \theta_2, \theta_3, \theta_4)$, as taken by Singh and Singh (2013) so that their simulated results can be incorporated in the comparison Tables 1-4. The simulated coverage probability is the proportion of repetitions in which all the ordered differences of location parameters are contained in the respective confidence intervals among 10^5 repetitions. The volume of simultaneous confidence intervals in a repetition is the product of lengths of all the underlying confidence intervals. The average volume is the average of the volumes obtained under 10^5 repetitions. Thus, the average volume is with respect to two-sided SCIs where the lower and upper limits are finite. Simulated power is the proportion of repetitions in which at least one of the ordered differences $\mu_j - \mu_i$, $1 \leq i < j \leq k$ falls outside the corresponding confidence interval.

Table 1: Simulated powers of one-stage procedure at $1-\alpha=.95$, for varied configuration of $(\mu_1,\mu_2,\mu_3,\mu_4)$ when $(\theta_1,\theta_2,\theta_3,\theta_4)=(1, 1.1, 1.2, 1.3)$

m	$(\mu_1,\mu_2,\mu_3,\mu_4)$	One-sided case		Two-sided case	
		SS	Prop	SS	Prop
10	(0,0,0,.4)	.069	.180	.044	.117
15		.434	.761	.313	.631
16		.666	.933	.425	.746
17		.792	.972	.548	.835
18		.885	.989	.670	.902
19		.943	.996	.772	.942
20		.926	.987	.860	.969
25		.999	1	.986	.998
30		1	1	1	1
10	(0,.2,.3,.4)	.066	.16	.039	.090
15		.369	.629	.218	.424
16		.469	.724	.357	.611
17		.572	.803	.455	.706
18		.672	.863	.559	.787
19		.764	.908	.658	.851
20		.832	.941	.666	.846
25		.981	.995	.944	.982
30		.998	1	.994	.998

Table 2: Simulated powers of one-stage procedure at $1-\alpha=.95$, for varied configuration of $(\mu_1,\mu_2,\mu_3,\mu_4)$ when $(\theta_1,\theta_2,\theta_3,\theta_4)=(1, 1, 1, 1)$

m	$(\mu_1,\mu_2,\mu_3,\mu_4)$	One-sided case		Two-sided case	
		SS	Prop	SS	Prop
10	(0,0,0,.4)	.087	.271	.056	.172
15		.660	.937	.517	.862
16		.791	.973	.667	.933
17		.885	.989	.790	.971
18		.943	.996	.884	.989
19		.976	.999	.943	.996
20		.990	1	.975	.999
25		1	1	1	1
30		1	1	1	1
10	(0,.2,.3,.4)	.080	.225	.050	.145
15		.524	.812	.402	.718
16		.641	.878	.530	.811
17		.750	.924	.646	.877

18		.833	.955	.775	.925
19		.892	.973	.831	.954
20		.932	.984	.892	.972
25		.994	.999	.990	.998
30		1	1	.999	1

Table 3: The Coverage Probabilities (CP) and Average Volumes (AV) of two-sided SCIs under one-stage procedure for$1-\alpha=.95$

m	$(\theta_1,\theta_2,\theta_3,\theta_4)$	SS		Prop	
		CP	AV	CP	AV
10	(1,1,1,1)	.996	24.853	.987	7.66
15		.993	.724	.977	.196
16		.993	.428	.976	.114
17		.992	.263	.974	.069
18		.991	.167	.972	.063
19		.990	.108	.971	.028
20		.990	.073	.969	.019
25		.989	.013	.963	.003
30		.986	.004	.957	.001
10	(1,1.1,1.2,1.3)	.996	68.961	.987	21.255
15		.992	2.031	.978	.551
16		.992	1.195	.975	.319
17		.992	.742	.974	.195
18		.991	.469	.973	.122
19		.991	.308	.971	.079
20		.991	.327	.972	.083
25		.988	.061	.965	.015
30		.987	.016	.962	.004

Table 4: Simulated powers of two-stage procedure for varied configurations of $(\mu_1,\mu_2,\mu_3,\mu_4)$ and $(\theta_1,\theta_2,\theta_3,\theta_4)$ for $1-\alpha=.95$

L	m	$(\theta_1,\theta_2,\theta_3,\theta_4)$	$(\mu_1,\mu_2,\mu_3,\mu_4)$	One-sided case		Two-sided case	
				SS	Prop	SS	Prop
0.6	10	(1,1.1,1.2,1.3)	(0,0,0,.3)	.755	.748	.757	.749
	20			.772	.718	.767	.724
	30			.782	.691	.783	.692
	10		(0,.1,.2,.3)	.555	.546	.552	.545
	20			.580	.514	.570	.518
	30			.581	.478	.578	.475
	10	(1,1.1,1.2,1)	(0,0,0,.3)	.751	.757	.754	.756
	20			.739	.773	.742	.771
	30			.734	.784	.734	.783
	10		(0,.1,.2,.3)	.549	.550	.547	.546

	20			.537	.535	.540	.539
	30			.519	.516	.519	.517
	10	(1,1,1,1)	(0,0,0,.3)	.755	.755	.754	.754
	20			.750	.754	.749	.750
	30			.751	.752	.751	.751
	10		(0,.1,.2,.3)	.543	.544	.542	.544
	20			.530	.530	.525	.525
	30			.509	.510	.509	.509

Tables 1-2 show that the power of the proposed one-stage procedure using the PMLEs is substantially higher for small and moderate sample sizes than the power of the MLEs based procedure of Singh and Singh (2013). The analysis of Table 3 also suggests that the simulated coverage probability (CP) of the proposed procedure is closer to the nominal level .95 for moderate and large sample sizes whereas it is too high (close to .99) under the procedure of Singh and Singh (2013). Further, the average volume is also substantially smaller under the proposed procedure than the Singh and Singh (2013) procedure and it indicates that the length of the SCIs under the proposed procedure is smaller than the Singh and Singh (2013) procedure. The simulated powers under a two-stage setup are the same for the Proposed and Singh and Singh (2013) procedures.

5. Real Life Example

We have taken the same data set as illustrated in Maruya *et al.* (2011) and Singh and Singh (2013), presented in Table 5. The data is about the survival times of inoperable lung cancer patients, categorized on the basis of histological type of tumor (squamous, small, adeno and large), who were subjected to standard chemotherapeutic agents.

Singh and Singh (2013) have constructed one-sided and two-sided simultaneous confidence (SCIs) by taking $c = 11.862$. Note that the choice of c determines the size of the sample from each population. In this numerical example, the choice of $c = 11.862$, gives the same sample sizes (9, 9, 9, 9) from all the four populations under the proposed and Singh and Singh (2013) procedures so that the comparison is feasible. Therefore for $c = 11.862$, the length $l = 2cu_{k,m,\alpha}$ of SCIs under the proposed two-stage procedure are 187.656, 143.981 and 113.92 at $\alpha = .01$, $\alpha = .025$ and $\alpha = .05$, respectively. The lengths of these SCIs are smaller than those reported in Singh and Singh (2013).

Table 5: Survival time (days) of inoperable lung cancer patients

	Type of Tumor			
	Squamous	Small	Adeno	Large
Survival Days	72	30	8	177
	10	13	92	162
	81	23	35	553
	110	16	117	200
	100	21	132	156
	42	18	12	182
	8	20	162	143

	25	27	3	105
	11	31	95	103

We have constructed simultaneous confidence intervals (SCIs) using Theorem 1, since for $c = 11.862$ the sample sizes are same under both the one-stage and two-stage procedures. The estimates of the scale and location parameters respectively, for the above reported data in Table 5 are, $S_1 = 48.375, S_2 = 10.25, S_3 = 78.265, S_4 = 106.7$ and $Y_1' = 8 - \frac{48.375}{8} = 2.625, Y_2' = 13 - \frac{10.25}{8} = 11.861, Y_3' = 3 - \frac{78.265}{8} = -5.696, Y_4' = 103 - \frac{106.7}{8} = 91.144$. The required values of the critical constants for $m = 9, k = 4$ and at the level of significance $\alpha = .01$, $.025$ and $.05$ are $q_{k,m,.05} = 4.318, q_{k,m,.025} = 5.539, q_{k,m,.01} = 7.314$, and $r_{k,m,.05} = 4.080, r_{k,m,.025} = 6.069, r_{k,m,.01} = 7.910$. The constructed one-sided and two-sided simultaneous confidence intervals are presented in Table 6.

Table 6: Simultaneous confidence intervals (SCIs) under the proposed (Prop) and Singh and Singh (2013) (SS) procedures

	Difference	SS	Prop
		$\alpha = .01$	$\alpha = .01$
One-Sided SCI	$\mu_2 - \mu_1$	(-93.620, ∞)	(-77.522, ∞)
	$\mu_3 - \mu_2$	(-108.620, ∞)	(-104.315, ∞)
	$\mu_3 - \mu_1$	(-103.690, ∞)	(-95.079, ∞)
	$\mu_4 - \mu_3$	(1.379, ∞)	(10.076, ∞)
	$\mu_4 - \mu_2$	(-8.620, ∞)	(-7.480, ∞)
	$\mu_4 - \mu_1$	(-3.620, ∞)	(1.755, ∞)
Two-Sided SCI	$\mu_2 - \mu_1$	(-100.690,110.690)	(-84.592, 103.064)
	$\mu_3 - \mu_2$	(-115.690,95.690)	(-111.385, 76.271)
	$\mu_3 - \mu_1$	(-110.690,100.690)	(-102.149,85.507)
	$\mu_4 - \mu_3$	(-5.690,205.690)	(3.006,190.663)
	$\mu_4 - \mu_2$	(-15.690,195.690)	(-14.550,173.106)
	$\mu_4 - \mu_1$	(-10.690,200.690)	(-5.314,182.342)

A pair-wise difference is declared to be significant if the corresponding simultaneous confidence interval (SCI) does not contain zero. Accordingly, at the level $\alpha = .01$, we infer that: (i) Under one-sided SCIs the Singh and Singh (2013) procedure declares the difference $\mu_4 - \mu_3$ as significant whereas the proposed procedure declares two differences $\mu_4 - \mu_3$ and $\mu_4 - \mu_1$ as significant (the corresponding SCIs do not contain zero); (ii) Under two-sided SCIs, the proposed procedure declares the difference $\mu_4 - \mu_1$ as significant whereas the Singh and Singh (2013) procedure does not declare any difference as significant.

6. Conclusion

We have observed that lengths of SCIs of the proposed one-stage and two-stage procedures, based on the PMLEs, are significantly smaller and that their coverage probability is also close to the nominal level as compared to the MLEs based procedure of Singh and Singh (2013). Thus, the Singh and Singh (2013) procedure is too conservative than the proposed procedure. Further, the power of the proposed one-stage procedure is higher than the one-stage procedure of Singh and Singh (2013) and both procedures have almost the same power under the two-stage setup. Keeping in view the dominance of the proposed procedures in terms of lengths of SCIs, coverage probability, and average volume, we recommend the use of proposed procedures, particularly, the one-stage procedure when there are smaller samples from the populations. In most of the practical situations we get smaller samples on life lengths and the use of the proposed one-stage procedure, based on the PMLEs, is recommended since it dominates the procedure of Singh and Singh (2013) in terms of lengths of SCIs, power, coverage probability and average volume.

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