



On the Robustness of LSD Layouts in the Presence of Neighbor Effects

Sobita Sapam¹ and Bikas K Sinha²

¹*Manipur University, Canchipur, Imphal, Manipur, India*

²*Indian Statistical Institute, Kolkata (Retired Professor), India*

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Abstract

In this paper we study the status of four non isomorphic Latin Square Designs (LSDs) of order four while finding out the optimal covariate matrices underlying the LSDs with and without Neighbor Effects (NEs). In these LSDs we consider the four sided NEs viz., left-sided, right-sided, top-sided and bottom-sided in the presence of covariates' effects. We utilize a circular model as was introduced by Kunert. Without NEs each of the four LSDs has six optimal covariate matrices whereas in the presence of the four-sided NEs the results are not as expected, for all the four LSDs.

Key words: Non isomorphic LSDs; Optimal designs for covariates effects; Neighbor effects; Circular models.

AMS Subject Classifications: 62K10

1. Introduction

As we know, there is a long history of use of ANCOVA Models for effective data analysis involving standard and non-standard experimental designs. Troya (1982a, 1982b) introduced the concept of Optimal Covariates Designs and presented optimality results in the context of CRDs. Inspired by Troya's formulation of optimality problems involving covariates effects, Das *et al.* (2003) got interested in this area of research and provided some combinatorial solutions. That was a modest beginning and much of the contents of the Monograph on Optimal Covariate Designs by Das *et al.* (2015) were motivated and inspired by 2003 paper. Prominent contributors in this area of research found their place and citations in the list of references of the monograph. This fascinating topic still holds rich rewards for serious researchers.

The first author (Sapam) got interested in this area of research and the recent works by Sapam *et al.* (2021) hold the key references for this paper. Optimal Covariate Designs (OCDs) are the designs which provide optimal or most efficient estimation of the covariates' effects in terms of the parameters in an assumed linear model. Lopes Troya (1982a, 1982b), Das *et al.* (2003), Shah and Sinha (1989), Dutta *et al.* (2014) are some of the related

references on the OCDs. Sapam *et al.* (2021) focused on OCDs incorporating the neighbor effects in four directions viz., left-sided, right-sided, top-sided and bottom sided in the assumed linear model in different RBD set ups. Sinha and Dutta (2017) worked on three different seasons of LSDs of order four without any NEs. We consider the four non-isomorphic LSDs across the three seasons, as considered in Sinha and Dutta (2017), with and without NEs. We crosscheck the earlier results and provide a few optimal matrices in the presence of the NEs. The notions of NEs are widely studied in the literature; some relevant references are Bailey (2003), Jaggi *et al.* (2007), Jaggi *et al.* (2018), Varghese *et al.* (2014), Sapam *et al.* (2019a, 2019b).

As the readers can realize, this area of research blends (block) designs [such as CRDs, RBDs, LSDs, GLSDs, BIBDs, *etc.*] and neighbour effects (introduced through what are known as circular models) and on the top, there are combinatorial arrangements of (+1/ -1)'s. Most of the reference papers bear testimony to the authors' interest in these areas. An interested reader will benefit by reading Das *et al.* (2003) before proceeding to venture in complicated set-ups. Not to obscure the essential steps of reasoning and understanding, we describe the linear model in simple terms with quantitative covariates and with/without neighbour effects. Since we will be primarily dealing with LSDs in this paper, we restrict to the LSD of order 4 shown in Table 1. There are altogether $4 \times 4 = 16$ plots and we have an LSD laid out there. We describe the linear model for some special plots - covering all diverse positions of the treatments - with/without neighbour effects of the treatments [in all the four positions].

Confining to LSD - S-1, wrt the treatment 1 in the first row and first column, the model specifications is as follows.

$$y(1,1;1) = \mu + \rho_1 + \gamma_1 + \tau_1 + e_{11}$$

$$y(1,1;1) = \mu + \rho_1 + \gamma_1 + \tau_1 + LN4 + RN2 + TN4 + BN2 + e_{11}$$

The parameters involved in the model are obvious. In linear model with four-sided neighbor effects, we have inserted LN, RN, TN and BN effects on the (Direct) Effects ('s) of the treatments.

In order that the readers can accompany and comprehend the thought process we refer to Das *et al.* (2003) wherein the conditions for existence of OCDs has been explicitly laid down.

2. LSDs of order 4 with covariates without neighbor effects

Taking the four treatments 1, 2, 3, 4 let us perform the complete enumeration principle to obtain all the possible forms of Standard Latin Squares. The following four non-isomorphic Standard Latin Square designs viz., S-1, S-2, S-3, S-4 are the only possible LSDs. The matrix X is the general form of covariate matrix wrt the four LSDs.

There should be three conditions for all optimal X- matrices wrt each of the above LSDs, viz., S-1, S-2, S-3 and S-4 without any neighbor effects, the elements of X-matrices being (+1/-1):

- (i) Row totals of the optimal X - matrices = 0
- (ii) Column totals of the optimal X - matrices = 0

(iii) Each treatment totals of the optimal X - matrices = 0

Table 1: Four non-isomorphic Standard Latin Square designs

LSDs				
S-1	1	2	3	4
	2	1	4	3
	3	4	1	2
	4	3	2	1
S-2	1	2	3	4
	2	3	4	1
	3	4	1	2
	4	1	2	3
S-3	1	2	3	4
	2	1	4	3
	3	4	2	1
	4	3	1	2
S-4	1	2	3	4
	2	4	1	3
	3	1	4	2
	4	3	2	1

Table 2: General X-matrix

a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

Sinha and Dutta (2017) studied the LSDs of order 4 in 3 Different SEASONS, viz., S-1, S-2, S-3 and worked out forms of some optimal covariate matrices in the absence of neighbor effects. Below we are showing six optimal covariate matrices for each of S-1, S-2, S-3 and for another additional design S-4 as well. We denote the six optimal X-matrices by S-1C1 to S-1C6 wrt S-1 design, with the same notation S-2C1 to S-2C6 wrt S-2 design and so on.

3. LSDs S-1, S-2, S-3 and S-4 with covariates in the presence of neighbor effects

The following conditions should hold in the presence of the four-sided neighbor effects viz., Left Neighbor (LN), Right Neighbor (RN), Top Neighbor (TN) and Bottom Neighbor (BN), involving the elements (+1/-1) of each of the X-matrices for the LSDs.

- (i) row sum of the optimal X - matrices = 0
- (ii) column sum of the optimal X - matrices = 0
- (iii) each treatment sum of the optimal X - matrices = 0
- (iv) sum of covariate-values of the optimal X - matrices corresponding to the LN of each

Table 3: Six optimal covariates matrix of S-1

	1	-1	1	-1
	1	-1	1	-1
S-1C1	-1	1	-1	1
	-1	1	-1	1
	1	-1	-1	1
	-1	1	1	-1
S-1C2	1	-1	-1	1
	-1	1	1	-1
	1	1	-1	-1
	-1	-1	1	1
S-1C3	1	1	-1	-1
	-1	-1	1	1
	1	-1	1	-1
	-1	1	-1	1
S-1C4	-1	1	-1	1
	1	-1	1	-1
	1	-1	1	-1
	-1	1	-1	1
S-1C5	-1	1	-1	1
	-1	1	-1	1
	1	1	-1	-1
	-1	-1	1	1
S-1C6	1	1	-1	-1
	-1	-1	1	1

treatment = 0

(v) sum of covariate-values of the optimal X - matrices corresponding to the RN of each treatment =0

(vi) sum of covariate-values of the optimal X - matrices corresponding to the TN of each treatment =0

(vii) sum of covariate-values of the optimal X - matrices corresponding to the BN of each treatment = 0.

When we consider the four-sided neighbor effects for each of the designs S-1, S-2, S-3 and S-4, the above $6 \times 4 = 24$ covariate matrices [optimal in the absence of neighbor effects (NEs)] do not all satisfy all the properties listed in (i)-(vii). The designs S-1 and S-2 has each six optimal covariate matrices in the presence of four-sided NEs, viz., S-1C1, S-1C2, S-1C3, S-1C4, S-1C5, S-1C6 and S-2C1, S-2C2, S-2C3, S-2C4, S-2C5 and S-2C6, satisfying all the conditions (i)- (vii) mentioned above for being optimal X -matrices in the presence of all the four-sided NEs. On the other hand, the designs S-3 and S-4 there is not even a single optimal X-matrix in the presence of four sided neighbor effects.

Table 4: Six optimal covariates matrix of S-2

	1	1	-1	-1
	-1	-1	1	1
S-2C1	1	1	-1	-1
	-1	-1	1	1
<hr/>				
	1	-1	1	-1
	-1	1	-1	1
S-2C2	-1	1	-1	1
	1	-1	1	-1
<hr/>				
	1	-1	-1	1
	-1	1	1	-1
S-2C3	1	-1	-1	1
	-1	1	1	-1
<hr/>				
	1	-1	1	-1
	1	-1	1	-1
S-2C4	-1	1	-1	1
	-1	1	-1	1
<hr/>				
	1	1	-1	-1
	-1	1	1	-1
S-2C5	-1	-1	1	1
	1	-1	-1	1
<hr/>				
	1	-1	-1	1
	1	1	-1	-1
S-2C6	-1	1	1	-1
	-1	-1	1	1

4. Existence and non-existence of optimal X-matrices with/without NEs: Status of the LSDs S-1 and S-4

Consider the LSD S-1 in the presence of four sided NEs and examine all the eight combinations corresponding to the choices of (b,e,f) , setting $a=1$. If there exists a solution satisfying the above conditions (i)- (vii) with the solution space $[1,-1]$, an optimal X- matrix will be available in the presence of four-sided neighbor effects.

Case1: $b=e=f=1$: no solution,

Case 2: $b= -1, e=f=1$: no solution,

Case 3: $b=f= -1, e=1$: two solutions viz., S-1C1 & S-1C5,

Case 4: $e= -1, b=f=1$: no solution,

Case 5: $f= -1, b=e=1$: no solution,

Case 6: $b=e= -1, f=1$: two solutions, viz., S-1C2 & S-1C4,

Case 7: $e=f= -1, b=1$: one solution, viz., S-1C3 & S-1C6,

Case 8: $b=e=f= -1$: no solution.

Therefore, total number of optimal X- matrices obtained wrt S-1 is six [viz., S-1C1, S-1C2, S-1C3, S-1C4, S-1C5, S-1C6] in the presence of four-sided NEs.

Table 5: Six optimal covariates matrix of S-3

S-3C1	1	1	-1	-1
	-1	-1	1	1
	1	1	-1	-1
	-1	-1	1	1
S-3C2	1	1	-1	-1
	-1	-1	1	1
	-1	-1	1	1
	1	1	-1	-1
S-3C3	1	-1	-1	1
	-1	1	1	-1
	1	-1	1	-1
	-1	1	-1	1
S-3C4	1	-1	1	-1
	-1	1	-1	1
	-1	1	1	-1
	1	-1	-1	1
S-3C5	1	-1	1	-1
	1	-1	1	-1
	-1	1	-1	1
	-1	1	-1	1
S-3C6	1	-1	-1	1
	1	-1	-1	1
	-1	1	1	-1
	-1	1	1	-1

Next, consider LSD S-4 in the presence of four sided NEs. If there exists a solution satisfying the above conditions (i)- (vii) with the solution space $[1,-1]$, an optimal X- matrix exists. WOLG, using the notations of the general covariate matrix given above, we set, $a=1$ and examine all the eight combinations corresponding to the choices of (b, e, f) . The following are the cases:

Case1: $b=e=f=1$; there is no solution [since, treatment 2 sum cannot be zero]

Case 2: $b=e=1, f=-1$; there is no solution [since, 2nd column sum cannot be zero]

Case 3: $b=f=1, e=-1$; there is no solution, [since, LN of Tr. 1 sum cannot be zero]

Case 4: $b=1, e=f=-1$; there is no solution, [since, LN of Tr. 1 sum cannot be zero]

Case 5: $b=-1, e=f=1$; there is no solution, [since, LN of Tr. 1 sum cannot be zero]

Case 6: $b=-1, e=-1, f=1$; there is no solution, [here two subcases arise: in one case TN of Tr1 sum is not equal to zero and in another subcase LN of Tr. 1 sum is not equal to zero]

Case 7: $b=-1, e=1, f=-1$; there is no solution, [since, LN of Tr. 2 sum is not equal to zero]

Case 8: $b=e=f=-1$; there is no solution [since, 2nd column sum cannot be zero].

This shows that there is not even a single optimal X- matrix in the presence of four sided NEs for the LSD S-4. Further, consider the LSD S-4 without NEs. If there exists a solution satisfying the above conditions (i)- (iii) of section 2 with the solution space $[1,-1]$, an

Table 6: Six optimal covariates matrix of S-4

	1	1	-1	-1
	-1	1	-1	1
S-4C1	1	-1	1	-1
	-1	-1	1	1
	1	1	-1	-1
	-1	-1	1	1
S-4C2	-1	-1	1	1
	1	1	-1	-1
	1	-1	1	-1
	1	1	-1	-1
S-4C3	-1	-1	1	1
	-1	1	-1	1
	1	-1	-1	1
	-1	1	1	-1
S-4C4	1	-1	-1	1
	-1	1	1	-1
	1	-1	1	-1
	-1	1	-1	1
S-4C5	-1	1	-1	1
	1	-1	1	-1
	1	-1	-1	1
S-4C6	-1	1	1	-1
	-1	1	1	-1

optimal X- matrix exists. WOLG, using the notations of the general covariate matrix given above, we set, $a=1$ and examine all the eight combinations corresponding to the choices of (b, e, f) .

Case1: $b=e=f=1$; there is no solution [since, treatment 2 total cannot be zero].

Case 2: $b=e=1, f=-1$; there is no solution [since, second column total cannot be zero].

Case 3: $b=f=1, e=-1$; there is one solution, viz., S-4C1

Case 4: $b=1, e=f=-1$; there is one solution, viz., S-4C2

Case 5: $b=-1, e=f=1$; there is one solution, viz., S-4C3

Case 6: $b=-1, e=-1, f=1$; there are two solutions, viz., S-4C4 and S-4C5

Case 7: $b=-1, e=1, f=-1$; there is one solution, viz., S-4C6

Case 8: $b=e=f=-1$; there is no solution [since, 1st column sum cannot be zero]

These eight cases show the existence of six optimal X-matrices in the absence of NEs wrt S-4 design.

5. Concluding remarks

In the study of four non isomorphic LSDs of order four with and without NEs we can summarize that in the absence of NEs, for each of the designs S-1, S-2, S-3 and S-4 of LSD

of order 4, we can find out all the possible (six) optimal X-matrices. On the other hand, in all the four LSDs, these optimal matrices fail to be optimal when we incorporate the four sided NEs. Only for the designs S-1 and S-2 all the six optimal X-matrices continue to be so even in the presence of NEs. The other two LSDs S-3 and S-4 has no X-matrix. Now we can sum up in the following table as Annexure I and II, the reasons of disqualification and their corresponding optimal X-matrices with respect to the design S-4 in the presence of neighbor effects.

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ANNEXURE I

Table 7: Reasons for disqualification of the X-matrices in the presence of NE wrt S-4

Sl no.	X- matrices*	Reasons for disqualification in the presence of NEs
1	$\begin{matrix} 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{matrix}$	LN=RN=2 for Treatment 1
2	$\begin{matrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{matrix}$	LN= - 4 and RN= 4 for Treatment 1
3	$\begin{matrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{matrix}$	LN= -2 and RN =2 for Treatment 1
4	$\begin{matrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{matrix}$	LN = 4 and RN = -2 for Treatment 1
5	$\begin{matrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{matrix}$	TN of Tr1 = 4 and BN = - 4
6	$\begin{matrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{matrix}$	LN= 4 and RN = 4 wrt Tr. 2

X- Matrices* are without NEs

ANNEXURE II

The design S-4 in the presence of four sided the neighbor effects the above six covariate matrices wrt S-4, we can see that there is no optimal matrix. The reasons are shown as below.

Table 8: Reasons for disqualification of the X-matrices in the presence of four-sided NEs wrt S-4C1 to S-4C6

Sl no.	Treatments	LN effects	RN effects
1 S-4C1	1	$-1+1+1+1= 2$	$1+1+1-1 = 2$
	2	$1+1+1-1 = 2$	$-1+1+1+1= 2$
	3	$1-1-1-1 = -2$	$-1-1-1+1= -2$
	4	$-1-1-1+1 =-2$	$1-1-1-1 = -2$
2 S-4C2	1	$-1-1-1-1 = -4$	$1+1+1+1 = 4$
	2	$1+1+1+1 = 4$	$-1 -1 -1 -1 = -4$
	3	$1+1+1+1 = 4$	$-1 -1 -1 -1 = -4$
	4	$-1 -1 -1 -1 = -4$	$1+1+1+1 = 4$
3 S-4C3	1	$-1 + 1 - 1 - 1 = -2$	$-1 - 1 + 1 - 1 = -2$
	2	$1 - 1 + 1 + 1 = 2$	$1 + 1 - 1 + 1 = 2$
	3	$-1 - 1 + 1 - 1 = -2$	$-1 + 1 - 1 - 1 = -2$
	4	$1 + 1 - 1 + 1 = 2$	$1 - 1 + 1 + 1 = 2$
4 S-4C4	1	$1+1+1+1 =4$	$-1 -1 -1 -1 =4$
	2	$1 -1 - 1 +1 =0$	$-1 +1 +1 -1 =0$
	3	$-1 +1 +1-1 =0$	$1 -1 -1 +1 =0$
	4	$-1 -1 -1 -1 = -4$	$1+ 1+1+1 = 4$
5 S-4C6	1	$1 -1 -1 +1 =0$	$-1 +1 +1 -1 = 0$
	2	$1+1 +1+1= 4$	$-1 -1 -1 -1 = - 4$
	3	$-1 -1 -1 -1 = - 4$	$1+ 1+1+1 = 4$
	4	$-1 +1 +1 -1 = 0$	$1 - 1 -1 +1 = 0$
	Treatments	TN effects	BN effects
6 S-4C5**	1	$1+1+1+1 = 4$	$-1 -1 -1 -1 = - 4$
	2	$-1+1+1 -1 = 0$	$1 - 1 - 1 +1 = 0$
	3	$1 - 1 -1 + 1 = 0$	$-1 +1 +1 -1 = 0$
	4	$-1 -1 -1 -1 = - 4$	$1 +1 + 1+1 =4$

In the case of S-4C5** conditions for both LN and RN effects for each treatment are satisfied whereas for Top Neighbor (TN) and Bottom Neighbor (BN) effects conditions are not satisfied, hence we take up only TN and BN effects in the Sl no. 6.