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Topp-Leone Generated q-Weibull Distribution and its Applications

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Abstract

In this paper, we introduce a new generated distribution called the Topp-Leone Generated q-Weibull(TLqW) Distribution. The described distribution's many distributional attributes and reliability traits are covered. Some well-known special cases of the mentioned model are also listed. When the lifetimes follow this distribution, it is better to establish a new reliability test plan, which aids in picking the best choices. The maximum likelihood method is investigated for parameter estimation in models. Using actual data sets, we used empirical evidence to demonstrate the value and adaptability of the new model in the model building process. The new test plan is then used to demonstrate how it may be used for creating dependable software in commercial settings.

Key words: Topp-Leone generated q-Weibull distribution; Quantile function; Reliability test plan; Maximum likelihood estimation.

AMS Subject Classifications: 60E05, 33B20, 62G05, 62N05, 62G07

1. Introduction

Numerous statistical distributions, including exponential, Weibull, logistic, and others, are significant in modelling survival and life-time data. The support for almost all of these distributions is unbounded. However, there are instances in real life where observations can only represent values in a small range, such percentages, proportions, or fractions. According to Papke and Wooldridge (1996), the variable is limited between zero and one in many economic scenarios, including the percentage of total weekly hours spent working, pension plan participation rates, industry market shares, percentage of land area given to agriculture, *etc*. As a result, for models to produce results that make sense, the unit interval must be used as the definition. Additionally, some writers use continuous models with finite support to characterise lifetime data while conducting reliability analysis. The most prevalent distribution for modelling continuous variables in the unit interval is the beta distribution, as is widely known. Due to the excellent flexibility of its density function, this distribution is well-liked in the fields of engineering, economics, biology, and ecology,

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among others. However, the mathematical formulation is found to be challenging because its distribution function cannot be written in closed form and it incorporates the incomplete beta function. In contrast, a number of scholars have suggested alternatives to the beta distribution by reviving the one Kumaraswamy suggested in 1980.

Topp and Leone's new distribution, known as the Topp Leone (TL) distribution, defined on finite support, was introduced in 1955. Several authors researched this distribution. The Topp Leone distribution offers closed variants of the probability density function (pdf) and cumulative density function (cdf), and it describes empirical data with a J-shaped histogram, such as powered tool band failures and automatic adding machine failure. Prior to being identified by Nadarajah and Kotz (2003), the Topp Leone distribution has gotten little attention. They examined various aspects of TL distribution and supplied its moments, central moments, and characteristic functions. Some reliability metrics of the TL distribution, including a hazard function, mean residual life, reversed hazard rate, predicted inactivity time, and its stochastic orderings were presented by Ghitany *et al.* (2005). Kotz and Seier (2002) reported a discussion on the TL distribution's kurtosis.

If a random variable X belongs to the TL distribution, it can have either finite ($0 < x < b$) or infinite ($0 < x < b < \infty$) support. To avoid using any additional functions for creating a new family of produced distributions, we here largely concentrate on the TL distribution with $b = 1$ (see Zografos and Balakrishnan (2009), Alzaatreh *et al.* (2013), Lee *et al.* (2013)).

Topp and Leone concentrated on creating J-shaped histogram distributions for empirical data. A random variable X is distributed as the TL, bounded on (0,1) with cdf

$$F_{TL}(x) = x^{\alpha}(2-x)^{\alpha}; 0 < x < 1, \tag{1}$$

where $\alpha > 0$. Its pdf associated with equation (1) is

$$f_{TL}(x) = 2\alpha x^{\alpha-1}(1-x)(2-x)^{\alpha-1}. \tag{2}$$

This distribution can alternatively be seen as one in which the failure rate is proportional to a power of time, assuming the random variable X represents the failure times. The survival and hazard functions are the other crucial traits. They are respectively

$$s(x) = 1 - x^{\alpha}(2-x)^{\alpha},$$

and

$$h(x) = \frac{2\alpha x^{\alpha-1}(1-x)(2-x)^{\alpha-1}}{1-x^{\alpha}(2-x)^{\alpha}}.$$

Life time distributions' hazard rate functions can be monotonically increasing, monotonically decreasing, or U-shaped (bath tub shaped). Each example has applications in the real world. In the case of the TL distribution, the failure rate decreases over time if the shape parameter's value is less than one, remains constant over time if it is equal to one, and rises over time if it is more than one. Additionally, Nadarajah and Kotz noted that the bathtub shape of the hazard function is provided by the TL distribution when $0 < \alpha < 1$.

To get the Topp-Leone generated (TLG) family of distribution, use the TL distribution as the generating distribution. Then relation of a random variable X having the TLG distribution and a random variable T having TL distribution is $X = G^{-1}(T)$, with $T \sim TL(\alpha)$. This relationship depicts how the function $G(\cdot)$ transforms the TL distribution's pdf (2) into a new probability function.

$$F_{TLG}(x) = 2\alpha \int_0^{G(x)} t^{\alpha-1}(1-t)(2-t)^{\alpha-1}dt = G(x)^{\alpha}(2-G(x))^{\alpha}. \tag{3}$$

By differentiating, we get the corresponding pdf,

$$f_{TLG}(x) = 2\alpha g(x)(1-G(x))G(x)^{\alpha-1}(2-G(x))^{\alpha-1}. \tag{4}$$

The Topp-Leone generated exponential (TLE) distribution was introduced by Sangsanit and Bodhisuwan (2016) as an illustration of the Topp-Leone generated distribution. Even though exponential distribution is frequently used in reliability analysis, its constant hazard rate still remains a limitation of this distribution. The two-parameter Weibull distribution is one of the most well-known generalisations of the exponential distribution. Weibull distribution has many applications in real data analysis. Aryal *et al.* (2017) discussed characterizations and applications of Topp-Leone generated Weibull distribution. We can generalize TLE distribution into TLW distribution using a transformation. If X follows TLE distribution then the distribution of $Y = X^{\frac{1}{\gamma}}, \gamma > 0$ follows TLW distribution. Hence, a random variable X is said to follow TLW distribution if it has the cdf and the pdf as in the form

$$F_{TLW}(x) = 1 - exp(-2(\nu x)^{\gamma})^{\alpha}, \gamma, \alpha, \nu > 0, \tag{5}$$

$$f_{TLW}(x) = 2\alpha\gamma\nu^{\gamma}x^{\gamma-1}exp(-2(\nu x)^{\gamma})(1-exp(-2(\nu x)^{\gamma})^{\alpha-1}, \tag{6}$$

where α, γ are shape parameter and ν is the scale parameter.

Authors have recently examined a variety of q-type distributions, including q-exponential, q-Weibull, q-logistic, etc. Since the exponential form can be attained gradually as $q \to 1$, the q-exponential distribution can be seen as a stretched model of the exponential distribution (see Beck (2006), Beck and Cohen (2003), Mathai(2005)). According to Tsallis statistics and many research based on q-type distributions, including q-Weibull, Wilk and Wlodarczyk (2000, 2001) and Tsallis (1988). Costa *et al.* (2006) described a research of dielectric breakdown in electronic device oxides and demonstrated that a q-Weibull distribution provides a satisfactory fit for the data. For $x > 0$ and for $q > 1$ the distribution function and the density function of the q-Weibull distribution is,

$$F_1(x) = 1 - [1 + (q-1)(\lambda x)^{\gamma}]^{\frac{q-2}{q-1}}, \tag{7}$$

$$f_1(x) = \gamma\lambda^{\gamma}(2-q)x^{\gamma-1}[1 + (q-1)(\lambda x)^{\gamma}]^{\frac{-1}{q-1}}. \tag{8}$$

where, $\gamma, \lambda > 0, 1 < q < 2$. For $x > 0$ and $q < 1$,the cdf and the density function of q - Weibull distribution becomes

$$F_2(x) = 1 - [1 - (1-q)(\lambda x)^{\gamma}]^{\frac{2-q}{1-q}}, \quad 0 \leq x \leq \frac{1}{\lambda(1-q)^{\frac{1}{\gamma}}}, \tag{9}$$

$$f_2(x) = \gamma\lambda^\gamma(2-q)x^{\gamma-1}[1-(1-q)(\lambda x)^\gamma]^{\frac{1}{1-q}}. \tag{10}$$

Clearly, as q tends to 1 $f_1(x)$ and $f_2(x)$ tend to the usual Weibull distribution with two parameters γ, λ.

The rest of the paper is organized as follows. In section 2 we will introduce the Topp-Leone q-Weibull Distribution (TLqW) and further properties. In section 3, we described a new reliability test plan for TLqW distribution. In section 4, we study the estimation of parameters of the TLqW distribution, using the method of maximum likelihood. Simulation studies, real data illustrations, and reliability test applications of TLqW distribution are also discussed in section 5. Concluding remarks are addressed in section 6.

2. Topp Leone q - Weibull distribution

The applications of the q-weibull distribution have recently been studied by a number of researchers in the contexts of information theory, statistical mechanics, reliability modelling, *etc.* In terms of reliability, the TL distribution is a fairly adaptable distribution. We therefore use the origin of the TLG distribution to merge these two distributions inspired by this. As a result, we present the TLqW distribution.

A random variable X possessing the TLqW distribution with $q > 1$ has the cdf and pdf respectively are

$$F_{1TLqW} = \left(1-(1+(q-1)(\lambda x)^\gamma)^{\frac{2q-4}{q-1}}\right)^\alpha, x>0, \lambda, \alpha, \gamma > 0, 1<q<2. \tag{11}$$

and

$$f_{1TLqW}(x) = 2\alpha\gamma\lambda^\gamma(2-q)x^{\gamma-1}[1+(q-1)(\lambda x)^\gamma]^{\frac{q-3}{q-1}}\{1-[1+(q-1)(\lambda x)^\gamma]^{\frac{2q-4}{q-1}}\}^{\alpha-1}, \tag{12}$$

where $x > 0, \lambda, \alpha, \gamma > 0, 1 < q < 2$.

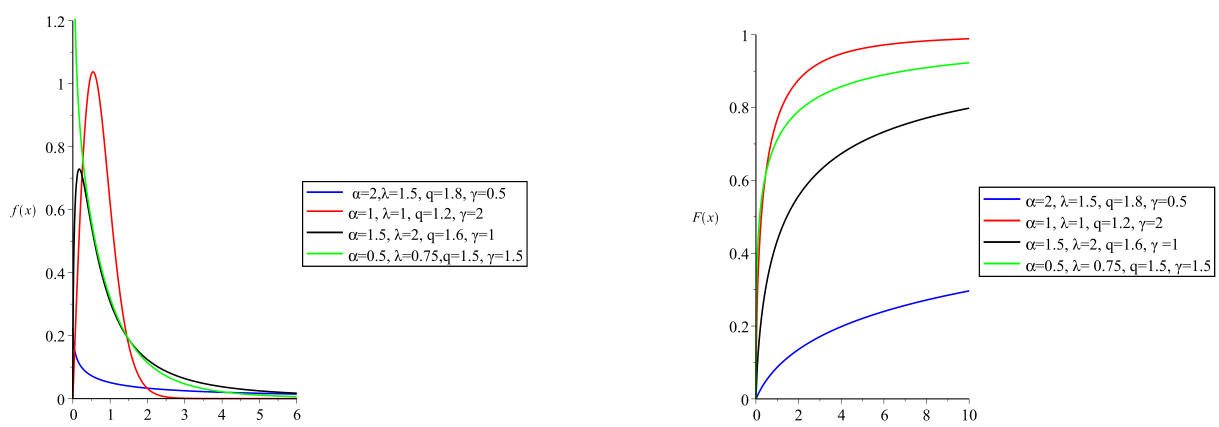

Figure 1: Plots of pdf and cdf of TLqW distribution

The plots of pdf and cdf of TLqW for various values of the shape parameters α, γ and q are shown in Figure 1. Survival function is the probability that a system will survive beyond a given time. The survival function S(x) for TLqW distribution is

$$\begin{aligned} S(x) &= 1 - F_{TLqW}(x) \\ &= 1 - \{1 - (1 + (q-1)(\lambda x)^{\gamma})^{2\frac{q-2}{q-1}}\}^{\alpha}. \end{aligned} \tag{13}$$

The TLqW distribution's hazard function is

$$\begin{aligned} h(x) &= \frac{f_{TLqW}(x)}{1 - F_{TLqW}(x)} \\ &= \frac{2\alpha\gamma\lambda^{\gamma}(2-q)x^{\gamma-1}[1+(q-1)(\lambda x)^{\gamma}]^{\frac{q-3}{q-1}}\{1-[1+(q-1)(\lambda x)^{\gamma}]2\frac{q-2}{q-1}\}^{\alpha-1}}{1-\{1-(1-[1+(q-1)(\lambda x)^{\gamma}]^{2\frac{q-2}{q-1}})\}^{\alpha}}. \end{aligned} \tag{14}$$

One can see the behaviour of hazard function using Figure 2. The Cumulative hazard

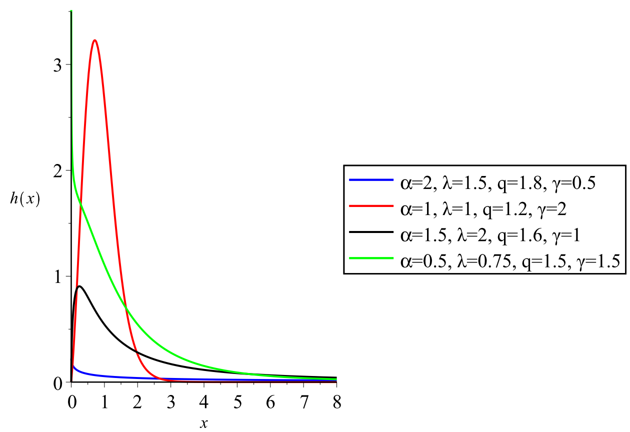

Figure 2: Plot of h(x) of TLqW distribution

function H(x)is defined as

$$\begin{aligned} H(x) &= \int_0^t h(t)dt \\ &= -\ln\{1 - \{1 - (1 + (q-1)(\lambda x)^{\gamma})^{2\frac{q-2}{q-1}}\}^{\alpha}\}. \end{aligned} \tag{15}$$

There are several new as well as well known distributions that can be obtained from the TLqW distributions. The sub-models include the following distributions:

1. When $q \to 1$, we obtain Topp Leone Weibull (TPW) distribution
2. When γ =1, we obtain Topp Leone q Exponential (TPqE) distribution
3. If $q \to 1$ and γ =1, we have the Topp Leone Exponential (TLE) distribution

2.1. L - Class property

The class L distributions are a significant class of distributions utilised in queuing theory and risk theory.

A distribution F belongs to the class L if

$$\lim_{x\to\infty} \frac{1-F(x-y)}{1-F(x)} = 1, \forall y \in R.$$

2.2. Quantile function

Probability distributions can be defined in terms of distribution functions or quantile functions when modelling and analysing statistical data. Quantile functions are more practical for analysis since they possess a number of intriguing characteristics that distributions do not share. The quantile function $Q(u)$ is defined as for a non-negative random variable X with distribution function $F(x)$ (see Nair *et al.* (2013)),

$$Q(u) = F^{-1}(u) = inf\{x : F(x) \geq u\}, \quad 0 \leq u \leq 1.$$

For every -$\infty < x < \infty$ and $0 < u < 1$, we have

$$F(x) \geq u \text{ if and only if } Q(u) \leq x.$$

As a result, $Q(u)$ is the smallest value of x satisfying $F(x) = u$ and $F(Q(u)) = u$ if there is an x such that $F(x) = u$. By solving the equation $F(x) = u$, we may get x in terms of u, which is the quantile function of X, if $F(x)$ is continuous and strictly growing. Moreover, if $Q(u)$ is the only value of x such that $F(x) = u$, then $F(x) = u$. The quantile function of TLqW distribution when $1 < q < 2$ is obtained as,

$$Q(u) = \left(\frac{\left(\sqrt{1-u^{\frac{1}{\alpha}}}\right)^{(\frac{q-1}{q-2})} - 1}{(q-1)\lambda^{\gamma}} \right)^{1/\gamma}, 1 < q < 2.$$

where u is chosen at random from the uniform distribution throughout the range $(0, 1)$. By matching population features with comparable sample characteristics, quantile-based measures of distributional properties such as location, dispersion, skewness, and kurtosis can be used to estimate model parameters. We can obtain the median as

$$M = Q(\frac{1}{2}) = \left(\frac{1-(1-0.50^{\frac{1}{\alpha}})^{\frac{1-q}{4-2q}}}{(1-q)\lambda^{\gamma}} \right)^{\frac{1}{\gamma}}.$$

The inter-quartile-range (IQR) of the TLqW model is,

$$IQR = Q(\frac{3}{4}) - Q(\frac{1}{4}) = \left(\frac{1-(1-0.75^{\frac{1}{\alpha}})^{\frac{1-q}{4-2q}}}{(1-q)\lambda^{\gamma}} \right)^{\frac{1}{\gamma}} - \left(\frac{1-(1-0.25^{\frac{1}{\alpha}})^{\frac{1-q}{4-2q}}}{(1-q)\lambda^{\gamma}} \right)^{\frac{1}{\gamma}}.$$

The Galton's coefficient of skewness (S) of the TLqW model is,

$$
\begin{aligned}
S &= \frac{Q(\frac{3}{4}) + Q(\frac{1}{4}) - 2Median}{IQR} \\
&= \frac{\left(\frac{1-(1-0.75^{\frac{1}{\alpha}})^{\frac{1-q}{4-2q}}}{(1-q)\lambda^{\gamma}}\right)^{\frac{1}{r}} - \left(\frac{1-(1-0.25^{\frac{1}{\alpha}})^{\frac{1-q}{4-2q}}}{(1-q)\lambda^{\gamma}}\right)^{\frac{1}{\gamma}} - 2\left(\frac{1-(1-0.50^{\frac{1}{\alpha}})^{\frac{1-q}{4-2q}}}{(1-q)\lambda^{\gamma}}\right)^{\frac{1}{\gamma}}}{\left(\frac{1-(1-0.75^{\frac{1}{\alpha}})^{\frac{1-q}{4-2q}}}{(1-q)\lambda^{\gamma}}\right)^{\frac{1}{\gamma}} - \left(\frac{1-(1-0.25^{\frac{1}{\alpha}})^{\frac{1-q}{4-2q}}}{(1-q)\lambda^{\gamma}}\right)^{\frac{1}{\gamma}}}.
\end{aligned}
$$

Moor's coefficient of kurtosis (T) of the TLqW model is,

$$
\begin{aligned}
T &= \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) + Q(\frac{3}{8}) - Q(\frac{1}{8})}{IQR} \\
&= \frac{\left(\frac{1-(1-\frac{7}{8}^{\frac{1}{\alpha}})^{\frac{1-q}{4-2q}}}{(1-q)\lambda^{\gamma}}\right)^{\frac{1}{\gamma}} - \left(\frac{1-(1-\frac{5}{8}^{\frac{1}{\alpha}})^{\frac{1-q}{4-2q}}}{(1-q)\lambda^{\gamma}}\right)^{\frac{1}{\gamma}} - \left(\frac{1-(1-\frac{3}{8}^{\frac{1}{\alpha}})^{\frac{1-q}{4-2q}}}{(1-q)\lambda^{\gamma}}\right)^{\frac{1}{\gamma}} - \left(\frac{1-(1-\frac{1}{8}^{\frac{1}{\alpha}})^{\frac{1-q}{4-2q}}}{(1-q)\lambda^{\gamma}}\right)^{\frac{1}{\gamma}}}{\left(\frac{1-(1-\frac{3}{4}^{\frac{1}{\alpha}})^{\frac{1-q}{4-2q}}}{(1-q)\lambda^{\gamma}}\right)^{\frac{1}{r}} - \left(\frac{1-(1-\frac{1}{4}^{\frac{1}{\alpha}})^{\frac{1-q}{4-2q}}}{(1-q)\lambda^{\gamma}}\right)^{\frac{1}{\gamma}}}.
\end{aligned}
$$

2.3. Simulation

A random variable Y having TLqW distribution can be simulated, for $1 < q < 2$ as,

$$
Y = \left\{ \frac{[1 - U^{\frac{1}{\alpha}}]^{\frac{q-1}{2(q-2)}} - 1}{(q-1)(\lambda)^{\gamma}} \right\}^{\frac{1}{\gamma}},
$$

where $U \sim U(0,1)$.

3. Reliability test plan

The acceptance sampling plan inspection method, which is used to decide whether to accept or reject a specific quantity of material (see Kantam *et al.* (2001), Rao *et al.* (2011), Jose and Joseph (2018), *etc.*), is prescribed. If it is applied to a series of lots, the method will give a specific probability of accepting lots of a given quality. Here we establish the reliability test, with its operating characteristic function plan for accepting or rejecting a lot where the lifetime of the product follows Topp-Leone generated q-Weibull distribution. The process in a life testing experiment is to call the test off at a predetermined time t and record the number of failures. We accept the lot with a specified probability of at least p if the number of failures at the end of time t does not exceed a predetermined number c, known as the acceptance number. However, we reject the lot if the failure rate reaches c before time t. We are interested in obtaining the smallest sample size possible in order to arrive at a decision for such a truncated life test and the accompanying decision rule.

Although many distributions from the Topp-Leone produced family have been created with a variety of uses, none of them have been used in acceptance sampling to create reliability test plans. Assume that the lifetime of a product T follows the Topp-Leone generated

q -Weibull distribution with cdf

$$F(t) = \{1 - [1 + (q-1)(\frac{t}{\lambda})^{\gamma}]^{2(\frac{q-2}{q-1})}\}^{\alpha}, t > 0, \lambda, \alpha, \gamma > 0, 1 < q < 2. \quad (16)$$

Let λ_0 be the required minimum average life time and the shape parameters α,γ and q are known. Then

$$F_{TLqW}(t; \alpha, q, \gamma, \lambda) \leq G_{TLqW}(t; \alpha, q, \gamma, \lambda_0) \Leftrightarrow \lambda \geq \lambda_0. \quad (17)$$

The number of units under test n, the acceptance number c, the maximum test time t, and the minimum average lifetime λ_0 are used to define a sampling plan. The consumer's risk (chance of accepting a bad lot) shouldn't be higher than the value $1 - p^*$, where p^* is a lower bound on the likelihood that the sampling plan will reject a lot with a true value of λ below λ_0 . The sampling plan is defined by $(n, c, t/\lambda_0)$ for fixed p^* . For sufficiently large lots, the acceptance probability can be determined using the binomial distribution. For given values of c and t/λ_0, the goal is to find the smallest positive integer n such that

$$L(p_0) = \sum_{i=0}^{c} \binom{n}{i} {p_0}^i (1 - p_0)^{n-i} \leq 1 - p^*. \quad (18)$$

The operational characteristic function is increasing in λ, as indicated by the fact that the product's average lifespan increases with λ and the failure probability $p(\lambda)$ decreases. Where $p_0 = F_{TLqW}(t; \alpha, q, \gamma, \lambda_0)$ is given in (16) and denotes the failure probability before time t, which solely depends on the ratio t/λ_0. For $\alpha = 2, q = 1.1$,$\gamma = 1.2$ and p^*=0.75 and $t/\lambda_0 = 0.248$, 0.361, 0.482, 0.602, 0.903, 1.204, 1.505 and 1.806, the minimal values of n fulfilling (18) are obtained. Table 1 presents the findings.

The binomial probability can be approximated by the Poisson probability with the parameter $\theta = np_0$ if $p_0 = F_{TLqW}(t; \alpha, q, \gamma, \lambda_0)$ is small and n is very large. As a result, (18) becomes true.

$$L_1(p_0) = \sum_{i=0}^{c} \frac{\theta^i}{i!} e^{-\theta} \leq 1 - p^*. \quad (19)$$

For the same set of values for α,γ, q, p^* and t/λ_0, the minimum values of n satisfying (19) are obtained and shown in Table 2. In the beginning equation,

$$L(p_0) = \sum_{i=0}^{c} \binom{n}{i} {p_0}^i (1 - p_0)^{n-i}, \quad (20)$$

and in the end equation, $p = F(t, \lambda)$, where λ is taken into consideration, the probability $L(p_0)$ of accepting the lot is given by the operating characteristic function of the sampling plan $(n, c, t/\lambda_0)$. When p^* and t/λ_0 are given values, the values of n and c are calculated using the operating characteristics (OC) function, taking into account the fact that$p = F(\frac{t}{\lambda_0}/\frac{\lambda}{\lambda_0})$, and the results are displayed in Table 3.

The probability of rejecting a lot when $\lambda > \lambda_0$ is the producer's risk. By first determining that $p = F(t; \lambda)$ and then employing the binomial distribution function, we may calculate the producer's risk. For illustration, we generate p from the sample plan provided in Table 1 for the given value of producer's risk, say 0.05, under the constraint that

$$\sum_{i=0}^{c} \binom{n}{i} {p_0}^i (1-p_0)^{n-i} \geq 0.95. \tag{21}$$

The minimal value of meeting (21) λ/λ_0 for the sampling plan $(n, c, t/\lambda_0)$ and for the specified p^* are reported in Table 6.

3.1. Explanation of the tables

Assume that q=1.1 and α=2 correspond to the TLqW distribution throughout the lifespan. Let us say the experimenter wants to confirm that the true unknown average life is at least 1000 hours with a $p^* = 0.75$ level of confidence. At $t = 903$ hours, the experiment should come to an end. The required n is hence 9 for an acceptance number $c = 4$ (Table 1). With a confidence level of 0.75, the experimenter can claim that the average life is at least 1000 hours if, during the course of 903 hours, no more than 4 failures out of 9 are detected. The value of n is 11 if the Poisson approximation to binomial probability is utilized (Table 2). The operational characteristic values from Table 3 are reported in Table 4 for this sample plan $(n = 9, c = 4, t/\lambda_0 = 0.903)$ under the TLqW distribution. The operational characteristic values from Table 3 are reported in Table 5 for the sample plan $(n = 7, c = 4, t/\lambda_0 = 1.806)$ with the consumer's risk of 0.05 under the TLqW distribution. This demonstrates that producers' risk is 0.05 when $\lambda/\lambda_0 = 3$ and insignificant when $\lambda/\lambda_0 = 4$. According to Table 3 for this plan, the minimum value of λ/λ_0, which represents the producer's risk as 0.05, is 3. When the consumer's risk is 0.25 or p^* =0.75, $c = 4$ and $t/\lambda_0 = 0.903$, the minimum ratio, $\lambda/\lambda_0 = 1.9619$ (from Table 6) which indicates that if $\lambda \geq 1.9619 \times (t/0.903) = 2.1726t = 1961.9$ hours, then, with sample size $n = 9$ and $c = 4$, the lot will be rejected with probability less than or equal to 0.05.

4. Maximum likelihood estimation

Let $x_1, x_2, ..., x_n$ be an observed random sample from TLqW distribution with $1 < q < 2$ unknown parameter vector $\theta = (\alpha, \gamma, \lambda, q)^T$. The likelihood function is then expressed as

$$L(\theta) = \prod_{i=1}^{n} 2\alpha\gamma\lambda^{\gamma}(2-q)x_i^{\gamma-1}(1+(q-1)(\lambda x_i)^{\gamma})^{\frac{q-3}{q-1}}\{1-(1+(q-1)(\lambda x_i)^{\gamma})^{2\frac{q-2}{q-1}}\}^{\alpha-1}.$$

The log-likelihood function is given by,

$$l(\theta) = \ln L(\theta) = n\ln 2 + n\ln\alpha + n\ln\gamma + n\gamma\ln\lambda + n\ln(2-q) + (\gamma-1)\sum_{i=1}^{n}\ln x_i +$$
$$\frac{q-3}{q-1}\sum_{i=1}^{n}\ln(1+(q-1)(\lambda x_i)^{\gamma}) + (\alpha-1)\sum_{i=1}^{n}\ln\{1-(1+(q-1)(\lambda x_i)^{\gamma})^{2\frac{2-q}{1-q}}\}.$$

Let $z_i(x) = 1 + (q-1)(\lambda x_i)^{\gamma}$and $k = 2\frac{(q-2)}{(q-1)}$, then $l(\theta)$ can be written as,

Table 1: Using the Binomial approximation, minimum sample size

p^*	c	t/λ_0							
		0.248	0.361	0.482	0.602	0.903	1.204	1.505	1.806
0.75	0	18	8	5	3	2	1	1	1
	1	35	16	10	7	4	3	2	2
	2	51	21	14	10	6	4	4	3
	3	66	30	18	13	8	6	5	5
	4	82	37	22	16	9	7	6	6
	5	60	33	22	17	11	9	8	7
	6	70	38	25	19	13	10	9	8
	7	79	43	29	22	14	12	10	9
	8	88	48	32	24	16	13	11	11
	9	97	52	35	27	18	14	13	12
	10	106	57	39	30	20	16	14	13
0.90	0	29	13	8	5	3	2	2	1
	1	50	23	13	9	5	4	3	3
	2	69	31	18	13	7	5	4	4
	3	86	39	23	16	9	7	6	5
	4	103	47	28	19	11	8	7	6
	5	120	54	32	23	13	10	8	7
	6	136	62	37	26	15	11	9	9
	7	152	69	41	29	17	13	11	10
	8	168	76	46	32	19	14	12	11
	9	184	84	50	35	21	16	13	12
	10	200	91	55	38	22	17	14	13
0.95	0	39	17	10	7	4	3	2	2
	1	61	27	16	11	6	4	3	3
	2	81	36	21	15	8	6	5	4
	3	100	45	27	18	10	8	6	5
	4	118	53	32	22	12	9	7	7
	5	135	61	36	25	14	11	9	8
	6	153	69	41	29	16	12	10	9
	7	170	77	46	32	18	14	11	10
	8	186	84	50	35	20	15	13	11
	9	203	92	55	38	22	16	14	12
	10	219	99	59	42	24	18	15	14
0.99	0	58	26	15	10	5	4	3	2
	1	85	38	22	15	8	6	4	4
	2	107	48	28	19	10	7	6	5
	3	128	58	34	23	13	9	7	6
	4	149	67	39	27	15	11	9	7
	5	168	75	45	31	17	12	10	9
	6	187	84	50	34	19	14	11	10
	7	205	92	55	38	21	15	13	11
	8	224	101	60	41	23	17	14	12
	9	242	109	65	45	25	18	15	14
	10	259	117	70	47	27	20	17	15

Table 2: Using the Poisson approximation, the minimal sample size

p^*	c	t/λ_0							
		0.248	0.361	0.482	0.602	0.903	1.204	1.505	1.806
0.75	0	19	9	6	4	3	2	2	2
	1	36	17	10	7	5	4	4	3
	2	52	24	15	11	7	6	5	5
	3	67	31	19	14	9	7	6	6
	4	83	38	24	17	11	9	8	7
	5	98	45	28	20	13	10	9	8
	6	113	52	32	23	14	12	10	10
	7	127	59	36	26	16	13	12	11
	8	142	66	40	29	18	14	13	12
	9	157	72	45	32	20	16	14	13
	10	171	79	49	35	22	17	15	15
0.90	0	31	14	9	7	4	3	3	4
	1	51	24	15	11	7	6	5	6
	2	70	33	20	15	9	7	7	7
	3	88	41	25	18	11	9	8	9
	4	105	49	30	22	14	11	10	10
	5	122	57	35	25	16	12	11	12
	6	138	64	39	28	18	14	13	13
	7	155	72	44	32	20	16	14	15
	8	171	79	49	35	22	17	15	16
	9	187	86	53	38	24	19	17	17
	10	202	94	57	41	26	20	18	18
0.95	0	40	19	12	8	5	4	4	4
	1	63	29	18	13	8	7	6	6
	2	83	39	24	17	11	9	8	7
	3	102	47	29	21	13	11	9	9
	4	120	56	34	25	15	12	11	10
	5	138	64	39	28	18	14	13	12
	6	156	72	44	32	20	16	14	13
	7	173	80	49	35	22	17	16	15
	8	190	88	54	39	24	19	17	16
	9	206	95	59	42	26	21	18	17
	10	223	103	63	45	28	22	20	19
0.99	0	61	28	18	13	8	6	6	5
	1	87	41	25	18	11	9	8	8
	2	111	51	32	23	14	11	10	10
	3	132	61	38	27	17	13	12	11
	4	152	71	43	31	19	15	14	13
	5	172	80	49	35	22	17	16	15
	6	191	89	54	39	24	19	17	16
	7	210	97	60	43	27	21	19	18
	8	228	106	65	47	29	23	20	19
	9	246	114	70	50	31	25	22	21
	10	264	122	75	54	33	27	24	22

Table 3: Values for the sample plan's operating characteristic function $(n, c, t/\lambda_0)$

				λ/λ_0						
p^*	n	c	t/λ_0	2	2.5	3	3.5	4	4.5	5
0.75	82	4	0.241	0.9997	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	37	4	0.361	0.9991	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	22	4	0.482	0.9980	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999
	16	4	0.602	0.9952	0.9996	0.9999	0.9999	0.9999	0.9999	0.9999
	9	4	0.903	0.9856	0.9984	0.9997	0.9999	0.9999	0.9999	0.9999
	7	4	1.204	0.9587	0.9938	0.9989	0.9998	0.9999	0.9999	0.9999
	6	4	1.505	0.9162	0.9838	0.9968	0.9993	0.9998	0.9999	0.9999
	6	4	1.806	0.7663	0.9360	0.9838	0.9958	0.9988	0.9996	0.9998
0.90	103	4	0.241	0.9993	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	47	4	0.361	0.9975	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999
	28	4	0.482	0.9940	0.9995	0.9999	0.9999	0.9999	0.9999	0.9999
	19	4	0.602	0.9894	0.9990	0.9998	0.9999	0.9999	0.9999	0.9999
	11	4	0.903	0.9621	0.9951	0.9993	0.9998	0.9999	0.9999	0.9999
	8	4	1.204	0.9192	0.9864	0.9976	0.9995	0.9998	0.9999	0.9999
	7	4	1.505	0.8195	0.9587	0.9910	0.9979	0.9994	0.9998	0.9999
	6	4	1.806	0.7663	0.9360	0.9838	0.9958	0.9988	0.9996	0.9998
0.95	118	4	0.241	0.9987	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	53	4	0.361	0.9958	0.9997	0.9999	0.9999	0.9999	0.9999	0.9999
	32	4	0.482	0.9894	0.9991	0.9999	0.9999	0.9999	0.9999	0.9999
	22	4	0.602	0.9801	0.9980	0.9997	0.9999	0.9999	0.9999	0.9999
	12	4	0.903	0.9448	0.9925	0.9988	0.9998	0.9999	0.9999	0.9999
	9	4	1.204	0.8659	0.9747	0.9953	0.9990	0.9997	0.9999	0.9999
	7	4	1.505	0.8195	0.9587	0.9910	0.9979	0.9994	0.9998	0.9999
	7	4	1.806	0.5795	0.8572	0.9587	0.9884	0.9966	0.9989	0.9996
0.99	149	4	0.241	0.9965	0.9997	0.9999	0.9999	0.9999	0.9999	0.9999
	67	4	0.361	0.9889	0.9991	0.9999	0.9999	0.9999	0.9999	0.9999
	39	4	0.482	0.9763	0.9977	0.9997	0.9999	0.9999	0.9999	0.9999
	27	4	0.602	0.9551	0.9950	0.9993	0.9999	0.9999	0.9999	0.9999
	15	4	0.903	0.8705	0.9788	0.9965	0.9993	0.9998	0.9999	0.9999
	11	4	1.204	0.7287	0.9366	0.9866	0.9970	0.9993	0.9998	0.9999
	9	4	1.505	0.5732	0.8659	0.9645	0.9908	0.9975	0.9993	0.9997
	7	4	1.806	0.5795	0.8572	0.9587	0.9884	0.9966	0.9989	0.9996

Table 4: Values of the OC function for values of λ/λ_0 **at** $(n = 9, c = 4, t/\lambda_0 = 0.903)$

λ/λ_0	2	2.5	3	3.5	4
L(p)	0.9856	0.9984	0.9997	0.9999	0.9999

Table 5: Values of the OC function for values of λ/λ_0 **at** $(n = 7, c = 4, t/\lambda_0 = 1.806)$

λ/λ_0	2	2.5	3	3.5	4	4.5	5
L(p)	0.5795	0.8572	0.9587	0.9884	0.9966	0.9989	0.9996

Table 6: Minimum of λ/λ_0 for the acceptability of a lot with producer's risk of 0.05

p^*	c	t/λ_0							
		0.241	0.361	0.482	0.602	0.903	1.204	1.505	1.806
0.75	0	4.4202	4.6647	5.0961	5.4557	6.3239	6.7439	7.9027	9.2078
	1	2.6177	2.6953	2.9073	3.1296	3.4508	3.7272	3.7349	4.2172
	2	2.0644	2.1350	2.2313	2.3367	2.6114	2.7601	3.2916	3.2154
	3	1.8393	1.8984	1.9580	2.0957	2.3727	2.5925	2.7851	3.2154
	4	1.7188	1.7616	1.8008	1.8853	1.9619	2.1962	2.3978	2.7845
	5	1.6234	1.6725	1.7327	1.7773	1.8712	2.1316	2.1554	2.5061
	6	1.5622	1.6067	1.6543	1.6999	1.8052	1.9422	2.1922	2.2987
	7	1.5160	1.5582	1.5945	1.6415	1.7548	1.8168	2.0356	2.1467
	8	1.4796	1.5078	1.5479	1.5962	1.6361	1.8256	1.9156	2.0420
	9	1.4459	1.4784	1.5107	1.5293	1.6141	1.7264	1.8159	1.9256
	10	1.4182	1.4546	1.4804	1.5026	1.5949	1.6536	1.7332	1.8456
0.90	0	5.5043	5.8420	6.3102	6.4500	7.6763	8.4810	10.5436	10.1158
	1	2.9925	3.1559	3.2234	3.3993	3.7805	4.4528	4.6590	5.5908
	2	2.3788	2.4760	2.5309	2.7041	2.8758	3.1049	3.3046	3.9655
	3	2.0771	2.1603	2.2278	2.3005	2.4810	2.7966	3.1275	3.2283
	4	1.9092	1.9837	2.0451	2.0857	2.2539	2.4017	2.6903	2.7896
	5	1.8031	1.8494	1.9003	1.9837	2.1086	2.3241	2.3978	2.5061
	6	1.7165	1.7739	1.8216	1.8785	2.0026	2.1140	2.1922	2.6307
	7	1.6558	1.6997	1.7458	1.7932	1.9211	2.0932	2.2509	2.4552
	8	1.6064	1.6464	1.6942	1.7292	1.8635	1.9541	2.1100	2.2987
	9	1.5685	1.6095	1.6466	1.6791	1.8133	1.9541	1.9979	2.1791
	10	1.5355	1.5748	1.6167	1.6377	1.7172	1.8530	1.8996	2.0798
0.95	0	6.1698	6.4672	6.8688	7.3394	8.6576	10.1799	10.6012	12.7215
	1	3.2479	3.3789	3.5622	3.7318	4.1507	4.4294	4.6590	5.5908
	2	2.5480	2.6414	2.7313	2.9068	3.1138	3.4832	3.8811	3.9558
	3	2.2262	2.3104	2.4118	2.4561	2.6408	3.0678	3.1275	3.2348
	4	2.0322	2.0980	2.1983	2.2504	2.3868	2.6242	2.6903	3.2283
	5	1.9003	1.9664	2.0179	2.0759	2.2092	2.4962	2.6688	2.8774
	6	1.8144	1.8693	1.9220	1.9924	2.0919	2.2655	2.4336	2.6307
	7	1.7428	1.7964	1.8492	1.9015	2.0026	2.2189	2.2509	2.4552
	8	1.6853	1.7283	1.7741	1.8205	1.9304	2.0729	2.2832	2.2987
	9	1.6416	1.6859	1.7281	1.7605	1.8721	1.9541	2.1580	2.2269
	10	1.6012	1.6414	1.6779	1.7354	1.8256	1.9541	2.0530	2.2796
0.99	0	7.3513	7.8632	8.1828	8.6528	9.5893	11.4865	12.6403	12.7215
	1	3.7743	3.9580	4.1543	4.3619	4.7903	5.5126	5.5958	6.6232
	2	2.8869	3.0280	3.1469	3.2614	3.5104	3.8505	4.3540	4.6433
	3	2.4840	2.6006	2.6998	2.7758	3.0848	3.3080	3.5050	3.7417
	4	2.2557	2.3496	2.4185	2.5203	2.7413	3.0228	3.2723	3.2283
	5	2.1068	2.1774	2.2695	2.3402	2.5059	2.6634	2.9051	3.2026
	6	1.9953	2.0701	2.1332	2.1804	2.3501	2.5615	2.6635	2.9203
	7	1.9227	1.9558	2.0314	2.0857	2.2155	2.3494	2.6372	2.7011
	8	1.8377	1.8936	1.9542	1.9924	2.1256	2.2896	2.4426	2.5942
	9	1.7811	1.8370	1.8958	1.9414	2.0488	2.1522	2.3121	2.5897
	10	1.7328	1.7899	1.8419	1.8636	1.9827	2.1266	2.3142	2.4587

$$l(\theta) = n\ln 2 + n\ln\alpha + n\ln\gamma + n\gamma\ln\lambda + n\ln(2-q) + (\gamma-1)\sum_{i=1}^{n}\ln x_i + \frac{q-3}{q-1}\sum_{i=1}^{n}\ln z_i(x) + (\alpha-1)\sum_{i=1}^{n}\ln\{1-z_i(x)^k\}.$$

Differentiating $l(\theta)$ with respect to α, γ, λ,and q,we have

$$\frac{\partial l(\theta)}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\{1-z_i(x)^k\}.$$

$$\frac{\partial l(\theta)}{\partial\gamma} = \frac{n}{\gamma} + n\ln(\lambda) + \sum_{i=1}^{n}\ln(x_i) + (q-3)\sum_{i=1}^{n}\ln(\lambda x_i)(\lambda x_i)^{\gamma}\frac{1}{z_i(x)} - (\alpha-1)(q-1)k\sum_{i=1}^{n}\ln(\lambda x_i)(\lambda x_i)^{\gamma}\frac{z_i(x)^{k-1}}{1-z_i(x)^k}.$$

$$\frac{\partial l(\theta)}{\partial\lambda} = \frac{n\gamma}{\lambda} + \left[(q-1)\gamma\lambda^{\gamma-1}\right]\left\{\frac{q-3}{q-2}\sum_{i=1}^{n}\frac{x_i^{\gamma}}{z_i(x)} - (\alpha-1)k\sum_{i=1}^{n}x_i{}^{\gamma}\frac{z_i(x)^{k-1}}{1-z_i(x)^k}\right\}.$$

$$\frac{\partial l(\theta)}{\partial q} = -\frac{n}{2-q} + \frac{2}{(q-1)^2}\sum_{i=1}^{n}\ln z_i(x) + \frac{q-3}{q-1}\sum_{i=1}^{n}\frac{(\lambda x_i)^{\gamma}}{z_i(x)} - (\alpha-1)\sum_{i=1}^{n}\frac{1}{1-z_i(x)^k}\frac{d}{dq}(z_i(x))^k,$$

where

$$\frac{d}{dq}(z_i(x))^k = z_i(x)^k\{k\frac{(\lambda x_i)^{\gamma}}{z_i(x)} + \frac{k+2}{q-1}\ln z_i(x)\}.$$

Now, setting the non-linear system of equations, $\frac{\partial l(\theta)}{\partial\alpha} = 0$, $\frac{\partial l(\theta)}{\partial\gamma} = 0$, $\frac{\partial l(\theta)}{\partial\lambda} = 0$, $\frac{\partial l(\theta)}{\partial q} = 0$ and solving them simultaneously we obtain the maximum likelihood estimate, $\hat{\theta} = (\hat{\alpha}, \hat{\lambda}, \hat{\gamma}, \hat{q})^T$. One can utilise iterative techniques like the Newton-Raphson type algorithm to calculate the estimate when solving non-linear equations numerically.

5. Numerical illustration

5.1. Simulation study

In this section, we do simulation tests to assess how well the MLEs of the TLqW distribution parameters perform over the long term. Numerous finite sample sizes are taken into account and to be more specific, we create samples from the TLqW distribution with n = 50, 75, 100 and 110 for the parameter values $\alpha = 1.275$, $\lambda = 1.5$, $\gamma = 7.8$ and $q = 1.7$. Also, the iteration is conducted 1000 times. The mean values of the biases, root mean squared errors (RMSEs), 95% (asymptotic) coverage probabilities (CPs), and average lengths (ALs) of the 95% (asymptotic) CIs corresponding to each of the parameter estimates for every replication are calculated with respect to the corresponding sample sizes. From Table 7 it can be seen that the RMSEs and ALs corresponding to each estimate decrease as the sample size increases.

Table 7: Simulation results

Sample Size	Parameter	MLE	Bias	RMSE	CP	AL
n=50	α	1.415	0.140	0.141	1.000	0.425
	λ	1.469	-0.031	0.131	0.931	0.437
	γ	9.557	1.758	10.14	0.971	11.469
	q	0.024	-0.023	0.197	0.906	0.534
n=75	α	1.415	0.140	0.140	1.000	0.347
	λ	1.477	-0.023	0.100	0.937	0.351
	γ	8.727	0.927	3.177	0.967	7.500
	q	1.695	-0.005	0.111	0.917	0.371
n=100	α	1.415	0.139	0.140	0.999	0.300
	λ	1.481	-0.019	0.084	0.936	0.299
	γ	8.562	0.762	1.838	0.969	6.146
	q	1.703	0.003	0.083	0.925	0.302
n=110	α	1.415	0.139	0.139	0.878	0.286
	λ	1.482	-0.017	0.077	0.944	0.282
	γ	8.539	0.738	1.674	0.964	5.779
	q	1.706	0.006	0.076	0.916	0.283

5.2. Data illustration for failure time data

In the reliability tests described in this section, lifetime data from engineering equipment are used to show one use of the TLqW distribution. The example uses data from a set measuring how long it took 500 MW generators to fail for the first time (see Jia *et al.* (2020)). The data are (thousands of hours) 0.058, 0.070, 0.090, 0.105, 0.113, 0.121, 0.153, 0.159, 0.224, 0.421, 0.570, 0.596, 0.618, 0.834, 1.019, 1.104, 1.497, 2.027, 2.234, 2.372, 2.433, 2.505, 2.690, 2.877, 2.879, 3.166, 3.455, 3.551, 4.378, 4.872, 5.085, 5.272, 5.341, 8.952, 9.188 and 11.399. The use of the TLqW illustrates the ability of this distribution in dealing with the non-monotonic hazard rate function, which includes a set of problems with relevant applications in the reliability context; for more information, see Jiang *et al.* (2003). Commonly used distributions like Weibull are barely suitable to fit the mentioned failure data. The

Table 8: Goodness of fit for different distributions on the failure time data

Model	Estimates(SE)	lnL	K-S	p value	AIC
Weibull	$\hat{\lambda}$=2.3118(0.256) $\hat{\gamma}$= 0.8156(0.058)	-68.6906	0.1219	0.1880	141.3812
MWE	$\hat{\lambda}$=0.2130(0.133) $\hat{\theta}$= 10.0923(0.003) $\hat{\gamma}$= 0.6920(0.001)	-68.2628	0.1046	0.2900	142.5276
ENH	$\hat{\lambda}$=0.1430(1.934) $\hat{\eta}$= 1.6347 (0.248) $\hat{\gamma}$= 0.6415(0.181)	-68.3560	0.1021	0.3330	142.712
TPW	$\hat{\lambda}$=0.4754(0.424) $\hat{\alpha}$= 1.3378(0.859) $\hat{\gamma}$= 0.1337(0.046)	-68.4044	0.2483	0.0192	142.8089
TLqW	$\hat{\lambda}$=0.1816(0.029) $\hat{\alpha}$= 0.0794(0.019) $\hat{\gamma}$= 6.4944(0.154) $\hat{q}$=1.8799(0.013)	-46.9633	0.0852	0.9361	101.9265

$P-P$ plot of the failure time data is given in Figure 3. The estimated standard error values

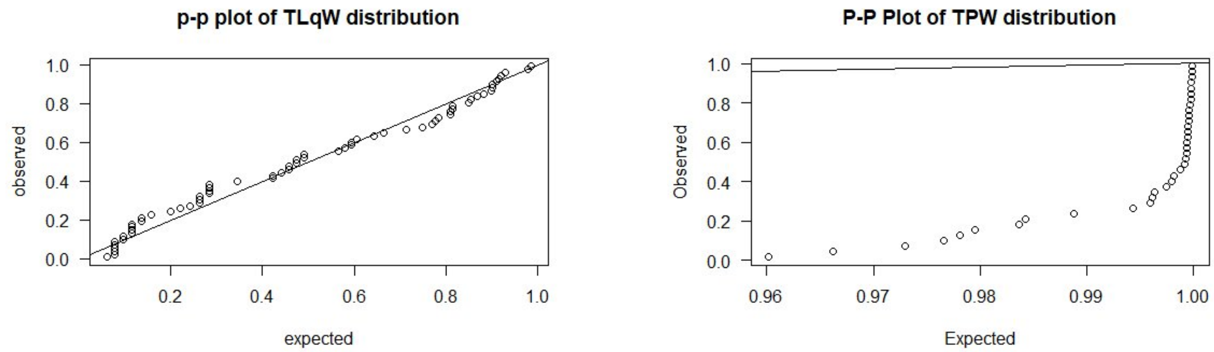

Figure 3: P-P **plot of failure time data**

are given in parentheses. It can be easily seen from the Table 8 that the TLqW distribution is a good alternative to the other lifetime models, namely the Weibull, modified Weibull extension (MWE), exponentiated Nadarajah-Haghighi (ENH), and TPW distributions.

5.3. Data illustration for fibre strength data

We use the original uncensored observations of the 1.5 cm glass fibre strengths made by employees of the UK National Physical Laboratory (see Merovci *et al.* (2016)). The fibre strength data are 0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28, 1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50, 1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82, 1.84, 1.84, 2.00, 2.01 and 2.24.

Table 9: Goodness of fit for different distributions on fibre strength data

Model	Estimates(SE)	lnL	K-S	p value	AIC
qW	$\hat{\lambda}$=0.0357(0.028) $\hat{\gamma}$= 1.2934(0.830) $\hat{q}$= 1.2934(0.142)	-296.15	0.1113	0.4053	598.31
TLqW	$\hat{\lambda}$=0.0360(0.007) $\hat{\gamma}$= 0.8304(0.206) $\hat{\alpha}$= 3.0546(0.024) $\hat{q}$=1.1747(0.012)	-294.74	0.1062	0.4654	594.49

Figure 4 gives the $P-P$ plot of the fibre strength data. It can be easily seen from the Table 9 that TLqW distribution gives better fit than q-Weibull (qW) distribution.

5.4. Reliability test comparison with Marshall-Olkin extended exponential distribution

Comparing Reliability Test Plans for Marshall-Olkin Extended Exponential distribution (see Rao *et al.* (2011)) with TLqW distribution, the minimal sample size is 49 using binomial approximation, whereas for α=2, acceptance number c=9, for the stated ratio t/λ_0=0.482 and confidence level p^*=0.75, whereas for TLqW distribution it is 35. The

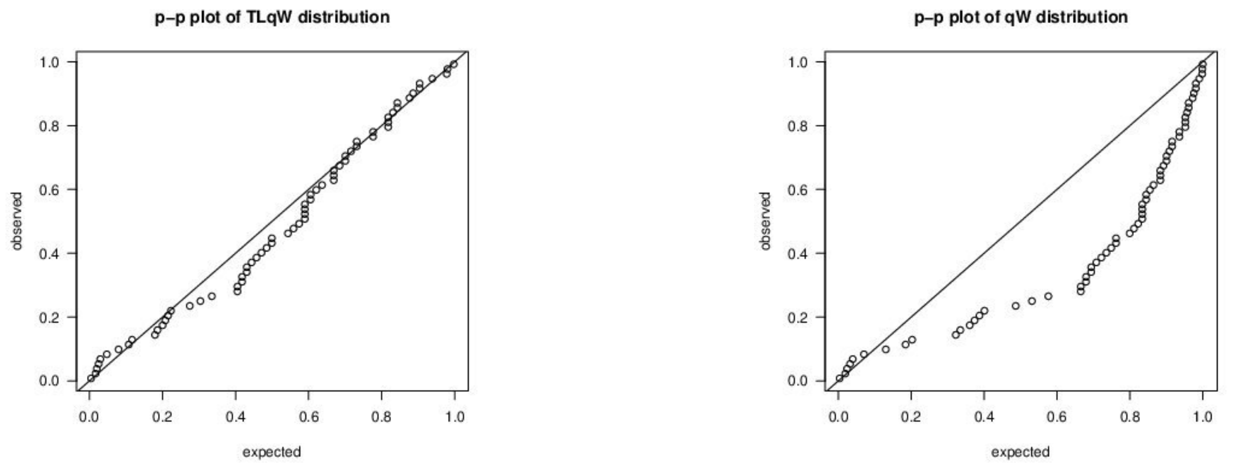

Figure 4: P-P plot of fibre strength data

scaled termination time is uniformly less than that for the current reliability test plans if we take into account each value of c and each value of t/λ_0. The new test plan is now more advantageous due to this modification, which also aids in selecting the best possible decisions.

5.5. Real life application of the new test plan

Take into account the following software release failure times, which are ordered and expressed in hours from the moment the software begins to run until a failure occurs (see, Wood(1996)). The observations 254, 788, 1054, 1393, 2216, 2880, 3593, 4281, and 5180 make up an ordered sample of this data with a size of $n = 9$.

Let's assume that the desired average lifetime is 1000 hours and that the testing time is 903 hours. This results in a ratio of $t/\lambda_0 = 0.903$, with a corresponding sample size of $n = 9$ and an acceptance number of $c = 4$, which is determined from Table 1 for $p^* = 0.75$. As a result, the sampling strategy for the sample data presented above is $(n = 9, c = 4, t/\lambda_0 = 0.903)$. We must choose whether to accept or reject the product in light of the observations. Only products with fewer than or equal to four failures prior to 903 hours are accepted. The sampling plan, however, only ensures the confidence level if the given life times follow the TLqW distribution. We compared the sample quantiles and the corresponding population quantiles and discovered a reasonable agreement, proving that the given sample is produced by lifetimes following the TLqW distribution. As a result, it would seem appropriate to adopt the sampling plan's decision rule. There are just two failures in the sample of 9 units, occurring 254 and 788 hours before $t = 903$ hours. Consequently, we approve the product.

6. Conclusion

The TLqW distribution is introduced in this paper as a generalization of the Weibull distribution. Class L is where the new distribution fits in. Additionally, the generation of random variates using the new model is straightforward. The Weibull distribution is shown to be a competitor of the new model, and the model's adaptability is demonstrated by fitting it to two sets of real-world data. Additionally, we determine the minimal sample size required for a lot to be accepted or rejected using percentiles. The test strategy was established using some helpful tables that were provided. Therefore, we draw the conclusion that the Topp Leone q-Weibull distribution is the most appropriate model among those taken into consideration, as well as a model that is particularly capable of explaining lifetime scenarios. We anticipate that the new model will grab researchers' attention as a serious threat to the Weibull distribution.

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