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Linear Trend-Free Group Divisible Design

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Abstract

In this paper, we extend the construction method of Srivastava for a linear trend-free balanced incomplete block design of size k=2 into a linear trend-free group divisible design. Another construction method for linear trend-free group divisible design has also been developed.

Key words: Linear trend free; Association scheme; Block design; Group divisible.

AMS Subject Classifications: 62K10

1. Introduction

In specific experiments where several treatments are compared in blocks and within blocks, the treatments are applied to the experimental units sequentially over time or space; there is a possibility that a systematic effect or trend effect influences the observations in addition to the block and the treatment effects. In such a situation, a common polynomial trend in one or more dimensions is assumed to exist over the plots in each block of a classical experimental design. One may think of a suitable design that is orthogonal to trend effects, in the sense that the analysis of the design could be done in the usual manner as if no trend effects were present. Bradley and Yeh (1980) have called such designs as Trend Free Block (TFB) designs. The idea is that starting from a block design, a good design is chosen by permuting the treatments to plot positions within blocks. For example, Latin square and Youden square designs with blocks formed by their column are trend-free designs. TFB design has been extensively studied in the literature by Yeh and Bradley (1983), Chai and Majumdarn(1993), Lal *et al.* (2005), Gupta *et al.* (2020), Srivastava R. (accessed on 21.11.2023) gave on the construction of TFB designs.

2. Notation and preliminary results

We assume that within blocks there is a common polynomial trend of order p on the k periods that can be expressed by the orthogonal polynomials $\phi_\alpha(l), 1 \leq \alpha \leq p$, on $l = 1, 2, \ldots, k$, where $\phi_\alpha(l)$ is a polynomial of degree α. The polynomials $\phi_1(l), \ldots, \phi_p(l)$

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satisfy

$$\sum_{l=1}^{k} \phi_\alpha(l) = 0, \sum_{l=1}^{k} \phi_\alpha(l)\phi_{\alpha'}(l) = \delta_{\alpha\alpha'}$$

where $\delta_{\alpha\alpha'}$ denotes the Kronecker delta, $\alpha, \alpha' = 1, 2, \ldots, p$.

If the trend is linear then, $p = 1$.

Let a design d will be represented by a $k \times b$ array of symbols $1, \ldots, \nu$, with columns denoting blocks and row periods. Thus, if the entry in cell (l, j) of d is i, it means that under d, treatment i, has to be applied in period l of block j. Let $D(\nu, b, k)$ be all connected designs in b blocks, k periods based on ν treatments.

Let $d \in D(\nu, b, k)$ and S_{dil} denote the number of times treatment i appears in row (period) l. It has been shown by Chai and Majumdar(1993) that a design is linear trend-free block (LTFB) design iff

$$\sum_{l=1}^{k} S_{dil}\phi_1(l) = 0, \ i = 1, \ldots, \nu \tag{1}$$

where $\phi_1(l)$ is the orthogonal polynomials of degree 1, $l = 1, 2, \ldots, k$ and S_{dil} denotes the number of times treatment i appears in row (period) l.

Condition (1) holds for binary as well as non-binary designs, and also irrespective of whether k is large, equal or smaller than ν, see Lin and Dean (1991). The polynomials $\phi_1(l)$ satisfy the condition

$$\phi_1(l) = -\phi_1(k - l + 1) \tag{2}$$

In addition,

$$\phi_1\left(\frac{k+1}{2}\right) = 0, \text{when } k \text{ is odd.}$$

3. Construction of linear trend-free group divisible (LTFGD) designs

3.1. Extension of Srivastava construction

Srivastava proposed a construction method of linear trend-free (LTF) balanced incomplete block design (BIBD) with parameters $v^* = 2q + 1, b^* = v^*(v^* - 1)/2, r^* = v^* - 1, k^* = 2, \lambda^* = 1$ for q positive integer. Then, such designs can be converted into LTF group divisible (GD) designs by augmenting some more treatments and blocks.

Theorem 1: The existence of an LTFBIBD with parameter $v^* = 2q + 1, b^* = v^*(v^* - 1)/2, r^* = v^* - 1, k^* = 2, \lambda^* = 1$ implies that an LTFGD design with parameters $v = v^*m, b = b^*m, r = r^*, k = 2, \lambda_1 = 1, \lambda_2 = 0, m, n = v^*$.

Proof: Let D be an LTFBIB design. Consider a group divisible association scheme (GDAS) on m different groups each of $n = v^*$ different treatments.

By using all treatments of every group of the GDAS as the treatment of the design, then b^*m blocks are constructed, resulting in a new incomplete block design, d with $v = v^*m, b = b^*m$.

Obviously, $r = r^*, k = k^*, m, n = v^*$.

As all the treatments in a group of the GDAS are treated as treatments of the given BIBD, D, with $\lambda^* = 1$ any two treatments in the same group of GDAS occur once in a block of d, *i.e.*, $\lambda_1 = 1$.

By the construction method of LTFBIBD, no two treatments from different groups can occur together in any block of d. It follows that $\lambda_2 = 0$.

From (2), $\phi_1(1) = -\phi_1(2)$

By the construction method of LTFBIBD, $\sum_{l=1}^{2} S_{dil}\phi_1(l) = 0,\ i = 1, \ldots, v^*$.

Now,

$$\sum_{l=1}^{2} S_{dil}\phi_1(l) = \sum_{i=1}^{v^*m}[S_{dil}\phi_1(1) + S_{dil}\phi_1(2)]$$

As each period of the LTFBIB design are replicated m times by the construction method of LTFGD design.

$$\sum_{l=1}^{2} S_{dil}\phi_1(l) = m\sum_{i=1}^{v^*m}[S_{dil}\phi_1(1) + S_{dil}\phi_1(2)] = 0$$

Hence, proof of the theorem is complete. □

Starting from an LTFBIB design with the parameters $v^* = 5, b^* = 10, r^* = 4, k^* = 2, \lambda^* = 1$ when every treatment occupies all the period (*viz.* 1st and 2nd) the same number of times, *i.e.*, twice, a LTFGD is constructed as an example of the theorem 1.

Example 1: Given a group divisible association scheme (m=2, n=5) as follows

1st group: 0, 1, 2, 3, 4;

2nd group: 5, 6, 7, 8, 9,

using the LTFBIB design with the parameters $v^* = 5, b^* = 10, r^* = 4, k^* = 2, \lambda^* = 1$,

θ	-1	1
B_1	a	b
B_2	a	c
B_3	d	a
B_4	e	a
B_5	b	c
B_6	d	e
B_7	b	d
B_8	e	b
B_9	c	d
B_{10}	c	e

considering $0, 1, 2, 3, 4$ and again $5, 6, 7, 8, 9$, the elements in the 1st group and in the 2nd group respectively as treatments of the LTFGD design, 20 blocks of LTFGD design with the parameters $v = 10, b = 20, r = 4, k = 2, \lambda_1 = 1, \lambda_2 = 0, m = 2, n = 5$, and the first row represent the orthogonal trend component of degree one without normalization,

θ	-1	1
B_1	0	1
B_2	0	2
B_3	3	0
B_4	4	0
B_5	1	2
B_6	3	4
B_7	1	3
B_8	4	1
B_9	2	3
B_{10}	2	4
B_{11}	5	6
B_{12}	5	7
B_{13}	8	5
B_{14}	9	5
B_{15}	6	7
B_{16}	8	9
B_{17}	6	8
B_{18}	9	6
B_{19}	7	8
B_{20}	7	9

3.2. LTFGD designs for $k \geq 2$

Consider a GDAS with m groups each of n elements where the i^{th} group is given by

$$G_i = \{(i-1)n + 1, (i-1)n + 2, \cdots, in\}$$

Consider m latin square arrays of the same order n (whether they are the same or not, but the order should be the same).

Treating all the n elements of the i^{th} group as the elements of the i^{th} latin square and considering each column of the resulting i^{th} latin square array with elements from G_i, as block for each group, n blocks are constructed as given by

$$B_j^{(i)} = l_j^{(i)} \tag{3}$$

where $l_j^{(i)}$ is the j^{th} column of the i^{th} resulting latin square array L_i, say, with elements $(i-1)n+1, (i-1)n+2, \ldots, in$ from the i^{th} group G_i. Continuing the same process for i, we have mn blocks.

Taking p (positive integer) copies of these mn blocks $B_j^{(i)}$ where $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$, the configuration yields an LTFGD as shown in the following theorem.

Theorem 2: A series of LTFGD design with parameters $v = mn, b = mnp, r = np, k = n; m, n, \lambda_1 = r, \lambda_2 = 0$ for p positive integer can always be constructed.

Proof: As the GDAS under consideration is on m different groups, each of n different elements, so $v = mn$.

By the construction method of blocks given in the relation (3), from each resulting latin square array L_i, n blocks $l_j^{(i)}$, are constructed. Counting the p copies of n blocks from the resulting latin square array L_i for all $i; i = 1, 2, \dots, m$, the configuration has mnp blocks. Further, any treatment of the i^{th} group G_i gets replicated once in each of the columns of the resulting latin square array L_i and gets replicated n times in those n blocks $B_j^{(i)}$ given by the relation (3); $j = 1, 2, \dots, n$. By the process of taking p copies of each block, $r = np$.

Since each column of these m latin square designs has n distinct treatments, then $k = n$.

The construction method of blocks given in the relation (3), it can be seen that any two treatments from the i^{th} group of the GDAS occurs together exactly once in each column of the latin square array, under consideration, *i.e.*, the i^{th} latin square array, as any element in a latin square array occurs exactly once in each column of the latin square array. So, from those n blocks constructed based on the i^{th} latin square array, any two treatments from the i^{th} group of the GDAS occurs together in n blocks which have been constructed based on that i^{th} latin square array. Treating of p copies of each of the constructed blocks by the construction method given in the relation (3) gives as $\lambda_1 = np = r$.

From the construction method of blocks given in the relation (3), it is known that no two treatments from different groups occur together in any block. Thus, $\lambda_2 = 0$.

Since every treatment of the i^{th} group appears n times in each position l.

Then,

$$\begin{aligned} S_{dil} &= \text{number of times treatment } i \text{ appears in position } l \\ &= n \end{aligned}$$

By, $\phi_1(l) = -\phi_1(k - l + 1)$; where $\phi_1(l)$ is the orthogonal polynomial of degree 1 and $\phi_1[(k + 1)/2] = 0$; when k is odd,

We get, $\phi_1(l) = -\phi_1(k); \phi_1(2) = -\phi_1(k - 1)$ and so on.

Now, for $k =$ even

$$\begin{aligned} \sum_{i=1}^{k} S_{dil}\phi_1(l) &= n \sum_{i=1}^{k} \phi_1(l) \\ &= n \left[\phi_1(1) + \phi_1(2) + \cdots + \phi_1\left(\frac{k}{2} - 1\right) + \phi_1\left(\frac{k}{2}\right) + \phi_1\left(\frac{k}{2} + 1\right) \cdots + \phi_1(k - 1) \right. \\ &\quad \left. + \phi_1(k) \right] \\ &= n \left[\phi_1(1) + \phi_1(2) + \cdots + \phi_1\left(\frac{k}{2}\right) - \phi_1\left(\frac{k}{2}\right) - \cdots - \phi_1(2) - \phi_1(1) \right] \\ &= n \times 0 \\ &= 0 \end{aligned}$$

Again, for $k =$ odd

$$\begin{aligned}\sum_{i=1}^{k} S_{dil}\phi_1(l) &= n\sum_{i=1}^{k}\phi_1(l)\\ &= n\bigg[\phi_1(1)+\phi_1(2)+\cdots+\phi_1\left(\frac{k+1}{2}-1\right)+\phi_1\left(\frac{k+1}{2}\right)\\ &\qquad +\phi_1\left(\frac{k+1}{2}+1\right)+\cdots+\phi_1(k-1)+\phi_1(k)\bigg]\\ &= n\bigg[\phi_1(1)+\phi_1(2)+\cdots+\phi_1\left(\frac{k+1}{2}-1\right)+\phi_1\left(\frac{k+1}{2}\right)\\ &\qquad -\phi_1\left(\frac{k+1}{2}-1\right)-\cdots-\phi_1(2)-\phi_1(1)\bigg]\\ &= n\times 0\\ &= 0\end{aligned}$$

Hence, proof of the theorem is complete. □

An example of Theorem 2 is shown as an illustration below,

Consider GDAS($m = 2, n = 3$) such that $G_1 = (1, 2, 3); G_2 = (4, 5, 6)$ and also consider 2 latin square arrays of order 3.

$$L_1 = \begin{pmatrix} a & b & c\\ b & c & a\\ c & a & b\end{pmatrix}, L_2 = \begin{pmatrix} \beta & \gamma & \alpha\\ \alpha & \beta & \gamma\\ \gamma & \alpha & \beta\end{pmatrix}$$

From these Latin squares L_1 and L_2, by the construction method given in the relation (3), using the elements $(1, 2, 3)$ and $(4, 5, 6)$, respectively given below

$$L_1^* = (l_1^1, l_2^1, l_3^1) = \begin{pmatrix} 1 & 2 & 3\\ 2 & 3 & 1\\ 3 & 1 & 2\end{pmatrix}; L_2^* = (l_1^2, l_2^2, l_3^2) = \begin{pmatrix} 5 & 6 & 4\\ 4 & 5 & 6\\ 6 & 4 & 5\end{pmatrix}$$

Considering each column of L_1^* and L_2^* as blocks for each group

$$B_1^{(1)} = l_1^1 = \begin{pmatrix} 1\\ 2\\ 3\end{pmatrix}; B_2^{(1)} = l_2^1 = \begin{pmatrix} 2\\ 3\\ 1\end{pmatrix}; B_3^{(1)} = l_3^1 = \begin{pmatrix} 3\\ 1\\ 2\end{pmatrix};$$

$$B_1^{(2)} = l_1^2 = \begin{pmatrix} 4\\ 5\\ 6\end{pmatrix}; B_2^{(2)} = l_2^2 = \begin{pmatrix} 5\\ 6\\ 4\end{pmatrix}; B_3^{(2)} = l_3^2 = \begin{pmatrix} 6\\ 4\\ 5\end{pmatrix}.$$

Taking 2 copies of these 6 blocks, the configuration yields an LTFGD design, as shown in the example given below.

Example 2: Following is a plan of LTFGD design with the parameters $v = 6, b = 12, r = 6, k = 3, m = 2, n = 3, \lambda_1 = 6, \lambda_2 = 0$ and 1^{st} row represents orthogonal trend component of degree one without normalization.

θ	-1	0	1
B_1	1	2	3
B_2	2	3	1
B_3	3	1	2
B_4	4	5	6
B_5	5	6	4
B_6	6	4	5
B_7	1	2	3
B_8	2	3	1
B_9	3	1	2
B_{10}	4	5	6
B_{11}	5	6	4
B_{12}	6	4	5

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