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Bayesian Modeling of VAR Model with Multiple Covariates

Umme Afifa[1], Varun Agiwal[2] and Jitendra Kumar[1]
[1]*Department of Statistics, Central University of Rajasthan, Bandersindri, Ajmer, India*
[2]*Department of Community Medicine, Jawaharlal Nehru Medical College, Ajmer, India*

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Abstract

This paper aims to provide a comprehensive overview of the Bayesian estimation methodology for the multiple covariate vector autoregressive (MC-VAR) model, in both methodology and application point of view. In that respect, conditional posterior distributions are derived to obtain the Bayesian estimators and influence based on covariate is analyzed by posterior odds ratio. Due to multiple integrations, the Gibbs sampler method is employed for the estimation of the MC-VAR model. Our approach is applied on both simulation and real data series to show the applicability of the proposed model. The real data result is useful for analyzing the relationship of covariates in economic time series.

Key words: Bayesian inference; Covariate; Vector autoregressive model.

AMS Subject Classification: 62K99; 62J05.

1. Introduction

The VAR model is continuously attracting the researchers to depict the behavior of a variable over time (Al-hajj *et al.* (2017), Sharma *et al.* (2018)). In the VAR model, present value of the variable is expressed as a linear function of past values and a random error (Fuller (1985)). For the analysis of multivariate time series data, VAR is frequently used model see Fuller (1985), Juselius (2006), Tsay (2014) and Lutkepohl (2005). VAR is most preferred and equally popular model for understanding the behavior of financial and economical data in various literatures see Wei (1990), Lutkepohl (2005), Al-hajj *et al.* (2017) and Sharma *et al.* (2018). VAR model is/was also used for structural analysis. In structural analysis, causal impacts of the variables are observed when certain hypotheses are imposed and resultant causal impacts are précised in Granger causality and impulse response function (IRF) in Wei (1990), Hamilton (1994), Lutkepohl (2005), etc. In the VAR model, when one includes exogenous variables, the VAR model extended to a covariate vector autoregressive (CVAR) model and allows those variables in the dataset to be modeled jointly over present and past time periods as considered in Hamilton (1994) and Tsay (2015).

The main motive behind the study of time series model with covariate is to make precise inferences about the impact of covariates on the response series under Bayesian framework. There are so many articles to explore the covariate in various univariate and multivariate time series model. Hansen (1995) developed covariate augmented Dickey-Fuller (CADF) unit root test with some stationary covariates for autoregressive parameter. This CADF test further extended to a point optimal covariate (POC) unit root test by Elliott and Jansson (2003).

Corresponding Author: Jitendra Kumar
E-mail: vjitendrav@gmail.com

Costantini and Lupi (2013) developed panel data model with stationary covariate which is the extension of Hansen (1995) model. Chaturvedi *et al.* (2017) discussed the Bayesian unit root hypothesis for covariate autoregressive model. Chang *et al.* (2017) developed bootstrap unit root tests with covariate method to the CADF test to deal the nuisance parameter dependency and provided a valid basis for inference based on the CADF test. Anggraeni *et al.* (2017) discussed performance of autoregressive integrated moving average with explanatory variable (ARIMAX) with VAR model using Indonesia economic data sets. Based on MAPE results, observed that performance of ARIMAX model is better than VAR model. Kumar *et al.* (2018) discussed Bayesian estimation and testing procedure for panel autoregressive time series model with covariate. Recently, Ji Linying *et al.* (2019) implemented VAR model with non-ignorable missingness in dependent variables and covariates under Bayesian framework. They introduced a Bayesian model which simultaneously represents the time dependency in multivariate and multiple subject time series data via VAR model.

The purpose of present paper is to make inference of multiple covariate-vector autoregressive (MC-VAR) model under Bayesian framework. We use Monte Carlo simulation method to estimate the parameters using conditional posterior distributions and then testing the impact of stationary covariate using posterior odds ratio. A simulation study has been carried out to validate the theoretical results. An empirical study of GDP series with export and import series as covariates is carried out to evaluate the performance of proposed model and obtained the Bayes estimators.

2. Model Description

In this section, we begin with vector autoregressive (VAR) model that captures the complex dynamics behaviour of multiple time series and their interactions and provides multiple series in a systematic manner. The basic form of VAR model represents a vector of N-dimensional time series measured at a particular time period. Let $\{Y_t, t = 1,2,\ldots,T\}$ be a VAR process expresses as a linear combination of past observations at lag p. Then, model is defined as

$$Y_t = \mu + A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \eta_t. \tag{1}$$

Generally, influence regarding the observation is based not only the study variables but also other associated variables. If these associated variables are also included in the process, then efficiency of the process may be increased. So, we include K stationary covariates (W_t, $t = 1,2,\ldots,T$) in the model that having dependence with its own past observations up to lag q then it can be written as

$$Y_t = \mu + A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \sum_{l=1}^{K} B_{l1} W_{lt-1} + \ldots + \sum_{l=1}^{K} B_{lq} W_{lt-q} + \eta_t,$$

$$Y_t = \mu + \sum_{i=1}^{p} A_i Y_{t-i} + \sum_{l=1}^{K}\sum_{j=1}^{q} B_{lj} W_{lt-j} + \eta_t,$$

$$Y_t = X_t^{'}\theta + Z_t^{'}\delta + \eta_t. \tag{2}$$

Here,

$$Y_t = \begin{pmatrix} y_{1t} & y_{2t} & \cdots & y_{Nt} \end{pmatrix}', \quad Y = \begin{pmatrix} Y_1 & Y_2 & \cdots & Y_T \end{pmatrix}', \quad Y = \begin{pmatrix} y_{11} & y_{21} & \cdots & y_{N1} \\ y_{12} & y_{22} & \cdots & y_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1T} & y_{2T} & \cdots & y_{NT} \end{pmatrix},$$

$$\eta_t = \begin{pmatrix} \eta_{1t} & \eta_{2t} & \cdots & \eta_{Nt} \end{pmatrix}', \quad \eta = \begin{pmatrix} \eta_1 & \eta_2 & \cdots & \eta_T \end{pmatrix}', \quad \eta = \begin{pmatrix} \eta_{11} & \eta_{21} & \cdots & \eta_{N1} \\ \eta_{12} & \eta_{22} & \cdots & \eta_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{1T} & \eta_{2T} & \cdots & \eta_{NT} \end{pmatrix},$$

$$\theta = \begin{pmatrix} \mu & A_1 & \cdots & A_P \end{pmatrix}', \quad A_i = \begin{pmatrix} A_{1i} & A_{2i} & \cdots & A_{Ni} \end{pmatrix}, \quad A_{ij} = \begin{pmatrix} \alpha_{i1}^{(j)} & \alpha_{i2}^{(j)} & \cdots & \alpha_{iN}^{(j)} \end{pmatrix}'$$

$$\gamma_l = \begin{pmatrix} B_{l1} & B_{l2} & \cdots & B_{lq} \end{pmatrix}, \quad B_{li} = \begin{pmatrix} B_{1i}^l & B_{2i}^l & \cdots & B_{Ni}^l \end{pmatrix}, \quad B_{ij}^l = \begin{pmatrix} \beta_{i1}^{(lj)} & \beta_{i2}^{(lj)} & \cdots & \beta_{iN}^{(lj)} \end{pmatrix},$$

$$\delta = \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_K \end{pmatrix}, \quad X_t' = \begin{pmatrix} 1 & Y_{t-1} & \cdots & Y_{t-p} \end{pmatrix}, \quad X = \begin{pmatrix} X_1 & X_2 & \cdots & X_T \end{pmatrix}',$$

$$Z = \begin{pmatrix} Z_1 & Z_2 & \cdots & Z_T \end{pmatrix}', \quad Z_t' = \begin{pmatrix} Z_{1t}' & Z_{2t}' & \cdots & Z_{Kt}' \end{pmatrix}; \quad Z_{lt}' = \begin{pmatrix} W_{lt-1} & W_{lt-2} & \cdots & W_{lt-q} \end{pmatrix}.$$

The final model in terms of matrix notation as

$$Y = X\theta + Z\delta + \eta \tag{3}$$

where N is the numbers of variables under study, K is the number of covariates, Y_t and η_t are 1 x N, A_{ij} is N x N. The disturbances η_t are unobservable random variable with $E(\eta_t) = 0$ and $VAR(\eta_t) = \Sigma$. The model is multiple covariate-vector autoregressive of order p time series model.

3. Bayesian Inference

The following prior distributions are considered for Bayesian analysis. We consider a basic prior distribution that enables analytical derivation of the posterior distribution and, thus, fast computations. The matrix variate normal conditional prior distribution is considered for θ and δ. An inverse Wishart marginal prior distribution is assumed for Σ. Let us assume the following prior distributions for the parameters used in the models

$$\theta|\Sigma \sim MN(\theta_0, \Sigma, V_1), \qquad \theta_0 \in \Re;\ 0 \le \Sigma, V_1 \le \infty, \tag{4}$$

$$\delta|\Sigma \sim MN(\delta_0, \Sigma, V_2), \qquad \delta_0 \in \Re;\ 0 \le \Sigma, V_2 \le \infty, \tag{5}$$

$$\Sigma \sim IW_N(S, \nu), \qquad 0 \le S \le \infty; \nu > N - 1. \tag{6}$$

Here MN and IW denote matrix variate normal distribution and inverse Wishart distribution, respectively. The joint prior probability of all parameters $(\Theta = (\theta, \delta, \Sigma))$ for MC-VAR model is determined using the equations (4) to (6)

$$P(\Theta)=\frac{(2\pi)^{-\frac{N(1+Np+NKq)}{2}}|V_1|^{-\frac{(1+NP)}{2}}|V_2|^{-\frac{NKq}{2}}|S|^{\frac{\nu}{2}}}{|\Sigma|^{\frac{\nu+3N+1}{2}}2^{\frac{\nu N}{2}}\Gamma\left(\frac{\nu}{2}\right)} \tag{7}$$
$$\exp\left[-\frac{1}{2}tr\left\{\Sigma^{-1}\left((\theta-\theta_0)'V_1^{-1}(\theta-\theta_0)+(\delta-\delta_0)'V_2^{-1}(\delta-\delta_0)+S\right)\right\}\right].$$

The likelihood function of the model is

$$L(Y\,|\,\Theta)=\frac{1}{(2\pi)^{\frac{NT}{2}}|\Sigma|^{\frac{T}{2}}}\exp\left[-\frac{1}{2}tr\left\{\Sigma^{-1}(Y-X\theta-Z\delta)'(Y-X\theta-Z\delta)\right\}\right]. \tag{8}$$

The posterior distribution is expressed as the product of likelihood function given in equation (8) and joint prior distribution given in equation (7)

$$P(\Theta\,|\,Y)=KL(Y\,|\,\Theta)P(\Theta)$$
$$=K\frac{(2\pi)^{-\frac{N(1+Np+NKq+NT)}{2}}|V_1|^{-\frac{(1+NP)}{2}}|V_2|^{-\frac{NKq}{2}}|S|^{\frac{\nu}{2}}}{|\Sigma|^{\frac{T+\nu+3N+1}{2}}2^{\frac{\nu N}{2}}\Gamma\left(\frac{\nu}{2}\right)}\exp\left[-\frac{1}{2}tr\left\{\Sigma^{-1}\left((Y-X\theta-Z\delta)'(Y-X\theta-Z\delta)\right.\right.\right. \tag{9}$$
$$\left.\left.\left.+(\theta-\theta_0)'V_1^{-1}(\theta-\theta_0)+(\delta-\delta_0)'V_2^{-1}(\delta-\delta_0)+S\right)\right\}\right]$$

where K is the normalizing constant which is given by $K^{-1}=\int_{\Theta}L(Y\,|\,\Theta)P(\Theta)d\Theta$.

3.1. Bayesian estimation

For Bayesian estimation, the estimator of the parameter is derived by using loss function and the posterior distribution. We consider two loss functions, one is symmetric known as quadratic loss or squared error loss and other one is asymmetric, entropy loss function. The Bayes estimators of any parametric function, say $\phi(\Theta)$ under squared error loss function (SELF), entropy loss function (ELF) and precautionary loss function (PLF) are defined by

$$\phi_{SELF}(\Theta\,|\,Y)=E_{\pi}(\phi(\Theta)\,|\,Y)=K\int_{\Theta}\phi(\Theta)P(\Theta\,|\,Y)d\Theta, \tag{10}$$

$$\phi_{ELF}(\Theta\,|\,Y)=\left[E_{\pi}\left(\phi^{-1}(\Theta)\,|\,Y\right)\right]^{-1}=\left(K\int_{\Theta}\phi^{-1}(\Theta)P(\Theta\,|\,Y)d\Theta\right)^{-1}, \tag{11}$$

$$\phi_{PLF}(\Theta\,|\,Y)=\sqrt{\left[E_{\pi}\left(\phi^{2}(\Theta)\,|\,Y\right)\right]}=\sqrt{K\int_{\Theta}\phi^{2}(\Theta)P(\Theta\,|\,Y)d\Theta}. \tag{12}$$

It is to be noticed here that a major difficulty in the implementation of Bayes procedure is the evaluation of the ratio of two integrals as described in equations (10) to (12) for which closed expression is not easy to obtain analytically. Therefore, we use Gibbs sampler algorithm to obtain the posterior samples from posterior distribution. For this, expression of full conditional posterior distribution are obtained

$$\pi(\theta|Y,\delta,\Sigma) \sim MN\left(BA^{-1},\Sigma,A^{-1}\right), \tag{13}$$

$$\pi(\delta|Y,\theta,\Sigma) \sim MN\left(DC^{-1},\Sigma,C^{-1}\right), \tag{14}$$

$$\pi(\Sigma|Y,\theta,\delta) \sim IW\left(S^{*},v^{*}\right), \tag{15}$$

where

$$A = X^{'}X + V_1^{-1}, \qquad B = X^{'}(Y - Z\delta) + V_1^{-1}\theta_0,$$

$$C = Z^{'}Z + V_2^{-1}, \qquad D = Z^{'}(Y - X\theta) + V_2^{-1}\delta_0,$$

$$S^{*} = (Y - X\theta - Z\delta)^{'}(Y - X\theta - Z\delta) + (\theta-\theta_0)^{'}V_1^{-1}(\theta-\theta_0) + (\delta-\delta_0)^{'}V_2^{-1}(\delta-\delta_0) + S,$$

$$v^{*} = v + 3N + T + 1.$$

Using the generated samples from the above conditional posterior distributions, Bayes estimates of the parameters are evaluated under different loss functions. Bayes estimate under SELF, ELF and PLF is the posterior mean, $\left[E\left(\Theta^{-1}\,|\,Y\right)\right]^{-1}$ and $\sqrt{E\left(\theta^{2}\,|\,Y\right)}$ respectively.

3.2. Bayesian testing procedure

Under Bayesian perspective, posterior odds ratio (POR)/ Bayes Factor (BF) is used in decision making for hypothesis testing problem/model selection procedure. POR (β_{01}) is product of prior odds ratio with Bayes factor (BF_{01}) of the null (H_0) and alternative (H_1) hypothesis. Here, the null hypothesis considers that covariates do not impact the study series, *i.e.*, H_0: $\delta = 0$ against the alternative hypothesis assumes that there is a significant relationship exists between study variable and covariates, *i.e.*, H_1: $\delta \neq 0$. The model under null and alternative hypothesis is

$$\text{Under } H_0 : Y = X\theta + \eta$$

$$\text{Under } H_1 : Y = X\theta + Z\delta + \eta$$

Then, β_{01} is expressed as

$$\beta_{01} = \frac{P(H_0\,|\,y)}{P(H_1\,|\,y)} = O(H_0)\frac{P(y\,|\,H_0)}{P(y\,|\,H_1)} = \frac{p_0}{1-p_0}\,\frac{P(y\,|\,H_0)}{P(y\,|\,H_1)}$$

where $O(H_o)$ is the prior odds ratio, *i.e.*, it is the ratio of prior probability under null is p_0 and alternative is $(1 - p_0)$.

The decision is taken to reject H_0 if POR is less than one, otherwise accept. So, the posterior probability under null and alternative hypothesis is computed as

$$P(y \mid H_0) = \frac{(2\pi)^{-\frac{NT}{2}} |V_1|^{-\frac{(1+NP)}{2}} |S|^{\frac{\nu}{2}} 2^{\frac{NT}{2}} \Gamma\left(\frac{T+\nu}{2}\right)}{|A_0^{-1}|^{-\frac{(1+NP)}{2}} |S_0|^{\frac{T+\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} \tag{16}$$

$$P(y \mid H_1) = \frac{(2\pi)^{-\frac{NT}{2}} |V_1|^{-\frac{(1+NP)}{2}} |V_2|^{-\frac{NKq}{2}} |S|^{\frac{\nu}{2}} 2^{\frac{NT}{2}} \Gamma\left(\frac{T+\nu}{2}\right)}{|A_1^{-1}|^{-\frac{(1+NP)}{2}} |C_1^{-1}|^{-\frac{NKq}{2}} |S_1|^{\frac{T+\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} \tag{17}$$

where

$$A_0 = A_1 = X'X + V_1^{-1}, \quad B_0 = X'Y + V_1^{-1}\theta_0,$$

$$C_1 = Z'Z + V_2^{-1} - Z'XA_1^{-1}X'Z, \quad D_1 = Z'Y + V_2^{-1}\delta_0 - Z'XA_1^{-1}\left(X'Y + V_1^{-1}\theta_0\right),$$

$$S_0 = Y'Y + \theta_0' V_1^{-1}\theta_0 + S - B_0' A_0^{-1} B_0,$$

$$S_1 = Y'Y + \theta_0' V_1^{-1}\theta_0 + \delta_0' V_2^{-1}\delta_0 + S - \left(X'Y + V_1^{-1}\theta_0\right)' A_1^{-1}\left(X'Y + V_1^{-1}\theta_0\right) - D_1' C_1^{-1} C_1.$$

Then, the POR is constructed as

$$\beta_{01} = \frac{p_0}{1-p_0} \frac{P(y \mid H_0)}{P(y \mid H_1)} = \frac{p_0}{1-p_0} \frac{|C_1^{-1}|^{-\frac{NKq}{2}} |S_1|^{\frac{T+\nu}{2}}}{|V_2|^{-\frac{NKq}{2}} |S_0|^{\frac{T+\nu}{2}}} \tag{18}$$

4. Simulation Study

This section discusses the appropriateness of the testing of hypothesis and record the performance of the estimators in the proposed model using the simulation study. For simulation purpose, a bivariate VAR(2) model with single covariate is generated from equation (2) with starting value of observed series is $Y = (4\ 6)$ and covariate series is $W = (2\ 3)$. The results are obtained based on R-language version 3.6.2. We have considered different sizes of the time series $T = c\,(200, 300)$. For series generation, fixed arbitrarily values are defined for the model parameters in the equation (19).

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0.2 & 0.3 \\ 0.25 & 0.15 \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} + \begin{pmatrix} 0.15 & 0.15 \\ 0.2 & 0.1 \end{pmatrix} \begin{pmatrix} y_{1t-2} \\ y_{2t-2} \end{pmatrix} + \begin{pmatrix} 0.3 & 0.15 \\ 0.2 & 0.1 \end{pmatrix} \begin{pmatrix} w_{1t-1} \\ w_{2t-1} \end{pmatrix} + \begin{pmatrix} 0.09 & 0.01 \\ 0.01 & 0.25 \end{pmatrix} \begin{pmatrix} \eta_{1t} \\ \eta_{2t} \end{pmatrix} \tag{19}$$

where $\eta_{.t}$ is a normally distributed random variable. To get more appropriate results, process is repeated 1000 times and each time 5000 posterior samples are generated using the Gibbs sampling procedure. For the different sizes of the series, average estimates (AE) and its standard deviation (SD) are summarized in Tables 1-2 (given in Appendix). For comparison between the different estimators, average absolute bias (AB) and average mean square error (MSE) of the estimators of the parameters are recorded in Tables 3-4 (given in Appendix).

Tables 1-2 conclude that average estimates are near to the true value of the parameters and standard deviation is also small that shows less variability in the estimation of the parameters. From Tables 3-4, we observed that size of the ser1ies increases, MSE and AB decreases from near to bottom of the series. In comparison of different loss functions, ELF performance better as compared to other estimators because MSE and AB is minimum. The next target for this study is to determine the significant affect of the covariate in the time series. For that, hypothesis testing for the presence or absence of covariate is carried out and records the POR results in Table 5 for different sizes of series and different number of covariates. From Table 5, we reject the null hypothesis as the POR values are less than one for different sizes of simulated series. This concludes that including the covariates in the model, better inference is drawn from the given series. Here, we also notice that number of covariates depends upon the series size since it does not much explain the small series with higher covariate as seen in T = 100 and K = 3.

Table 5: Posterior odds ratio with T and K

T	K=1	K=2	K=3
100	1.47E-05	3.74E-02	2.17E+00
200	1.54E-23	2.82E-21	3.70E-18
300	2.25E-47	2.63E-36	1.69E-31
400	4.07E-66	5.22E-45	3.68E-42
500	1.06E-103	2.50E-65	4.01E-57

5. Real Data Analysis

A macroeconomic data set is taken to illustrate the performance of proposed model. We use yearly series on gross domestic product (GDP), export and import for the period 1962 to 2018 from IMF's International Financial Statistics as well as The World Bank data source (http://datahelp.imf.org/ and https://data.worldbank.org/). For analysis purpose, most developing countries India and China is considered as vector form where GDP is the study variable and export and import are two covariates. The reason behind the selection of these two countries is that most of the Indian market is depended upon the import material of China product so this impacts the GDP of both countries. First, we determine the best suitable order of each variable using the in-built function in R-language and display in Table 6. Based on Table 6, we observe that GDP series have VAR order two (p = 2) model whereas import and export series obtain VAR order one (q = 1) model.

Table 6: Order selection based on various selection criterion

Series	Order (lag)	1	2	3
GDP	AIC	5.0812	4.6713	4.7246
	HQ	5.1670	4.8142	4.9248
	SC	5.3043	5.0430	5.2451
Import	AIC	1.9540	2.0192	2.0602
	HQ	2.0397	2.1621	2.2603
	SC	2.1770	2.3909	2.5806
Export	AIC	1.3716	1.4176	1.4832
	HQ	1.4574	1.5605	1.6834
	SC	1.5947	1.7893	2.0037

Once, we get the order of the series, estimation of model parameters are carried out using the proposed methodology and then obtain the consequence of covariate(s) in the response series. Here, we analyze the inference of the proposed model based on one and two covariates, *i.e.*, show the suitable impact of GDP versus import or export or both series. As per simulation study, we recorded that the best estimated values of the parameters of the proposed model is obtained through ELF estimators. So, we only estimate the MC-VAR model parameters under ELF and recorded in equations (20)-(22) for single and bivariate covariates.

$$\begin{pmatrix} GDP_{India,t} \\ GDP_{China,t} \end{pmatrix} = \begin{pmatrix} 4.8678 \\ 4.5303 \end{pmatrix} + \begin{pmatrix} -0.1150 & -0.0963 \\ -0.1810 & -0.0275 \end{pmatrix} \begin{pmatrix} GDP_{India,t-1} \\ GDP_{China,t-1} \end{pmatrix} + \begin{pmatrix} -0.2342 & 0.5426 \\ 0.6116 & -0.4277 \end{pmatrix} \begin{pmatrix} GDP_{India,t-2} \\ GDP_{China,t-2} \end{pmatrix} + \begin{pmatrix} -0.5573 & -0.2204 \\ 0.2431 & 0.2526 \end{pmatrix} \begin{pmatrix} IMPORT_{India,t-1} \\ IMPORT_{China,t-1} \end{pmatrix} \; with\, \hat{\Sigma} = \begin{pmatrix} 3.1242 & 0.3569 \\ 0.3569 & 4.8835 \end{pmatrix} \quad (20)$$

$$\begin{pmatrix} GDP_{India,t} \\ GDP_{China,t} \end{pmatrix} = \begin{pmatrix} 5.0978 \\ 4.8146 \end{pmatrix} + \begin{pmatrix} -0.0098 & 0.5431 \\ -0.1086 & 0.0042 \end{pmatrix} \begin{pmatrix} GDP_{India,t-1} \\ GDP_{China,t-1} \end{pmatrix} + \begin{pmatrix} -0.0156 & 0.5457 \\ 0.5810 & -0.4243 \end{pmatrix} \begin{pmatrix} GDP_{India,t-2} \\ GDP_{China,t-2} \end{pmatrix} + \begin{pmatrix} -0.2777 & 0.1859 \\ -0.3034 & 0.2396 \end{pmatrix} \begin{pmatrix} EXPORT_{India,t-1} \\ EXPORT_{China,t-1} \end{pmatrix} \; with\, \hat{\Sigma} = \begin{pmatrix} 3.114 & 0.7960 \\ 0.7960 & 4.8539 \end{pmatrix} \quad (21)$$

$$\begin{pmatrix} GDP_{India,t} \\ GDP_{China,t} \end{pmatrix} = \begin{pmatrix} 4.7290 \\ 4.5312 \end{pmatrix} + \begin{pmatrix} -0.1298 & -0.1083 \\ -0.1928 & -0.0206 \end{pmatrix} \begin{pmatrix} GDP_{India,t-1} \\ GDP_{China,t-1} \end{pmatrix} + \begin{pmatrix} -3.1044 & 0.5108 \\ 0.6121 & -0.4286 \end{pmatrix} \begin{pmatrix} GDP_{India,t-2} \\ GDP_{China,t-2} \end{pmatrix} + \begin{pmatrix} 0.2385 & 0.1588 \\ 0.2838 & 0.2024 \end{pmatrix} \begin{pmatrix} EXPORT_{India,t-1} \\ EXPORT_{China,t-1} \end{pmatrix} + \begin{pmatrix} 0.2948 & 0.1376 \\ 0.3809 & 0.1749 \end{pmatrix} \begin{pmatrix} IMPORT_{India,t-1} \\ IMPORT_{China,t-1} \end{pmatrix} \; with\, \hat{\Sigma} = \begin{pmatrix} 3.0678 & 0.6221 \\ 0.6221 & 5.0160 \end{pmatrix} \quad (22)$$

The calculated POR for the proposed model under study are recorded in Table 7 that shows the suitable covariate is necessary to analyze the growth of the GDP series. We see that individual covariate has an impact on GDP series as compared when both covariates are jointly analyzed because it does not reject the null hypothesis.

Table 7: POR value based on real data series

Covariate	Export	Import	Both Export and Import
POR	0.9266	0.4488	1241.5250

6. Conclusion

In this paper, we develop a Bayesian approach for analyzing vector autoregressive (VAR) model with multiple covariates. The model is estimated by deriving the conditional posterior distribution and Bayesian estimators are obtained under different loss functions. We also test the association of covariates in the VAR model using the derived posterior odds ratio. Based on our simulation and empirical results, indicates that Bayesian estimators appropriate estimates the parameter values and import and export variables are related to GDP series individually.

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APPENDIX

Table 1: AE and SD of C-Var(2) model at *T* = 200

Estimator	SELF		ELF		PLF	
Parameter (True Value)	AE	SD	AE	SD	AE	SD
μ_1 (1)	0.6718	0.1224	0.6624	0.1358	0.6929	0.1198
μ_2 (2)	1.6965	0.0896	1.5995	0.1067	1.6833	0.0874
$\alpha_{11}^{(1)}$ (0.2)	0.2017	0.0573	0.1835	0.0701	0.2085	0.0554
$\alpha_{12}^{(1)}$ (0.3)	0.3203	0.0424	0.3146	0.0433	0.3230	0.0420
$\alpha_{11}^{(2)}$ (0.15)	0.1468	0.0492	0.1340	0.0634	0.1542	0.0462
$\alpha_{12}^{(2)}$ (0.15)	0.1646	0.0458	0.1495	0.0562	0.1706	0.0425
$\alpha_{21}^{(1)}$ (0.25)	0.2464	0.0965	0.2635	0.0854	0.2629	0.0886
$\alpha_{22}^{(1)}$ (0.15)	0.1523	0.0695	0.1671	0.0932	0.1591	0.0615
$\alpha_{21}^{(2)}$ (0.2)	0.2101	0.0872	0.2100	0.0821	0.2249	0.0794
$\alpha_{22}^{(2)}$ (0.1)	0.1198	0.0830	0.0971	0.0692	0.0973	0.0639
$\beta_{11}^{(11)}$ (0.3)	0.3062	0.0217	0.3050	0.0218	0.3068	0.0217
$\beta_{12}^{(11)}$ (0.15)	0.1524	0.0216	0.1499	0.0220	0.1536	0.0214
$\beta_{21}^{(11)}$ (0.2)	0.2117	0.0391	0.2066	0.0406	0.2141	0.0386
$\beta_{22}^{(11)}$ (0.1)	0.1070	0.0373	0.0947	0.0735	0.1119	0.0355
Σ_{11} (0.09)	0.0739	0.0073	0.0725	0.0072	0.0747	0.0075
Σ_{12} (0.01)	0.0085	0.0670	0.0085	0.0652	0.0086	0.0681
Σ_{21} (0.01)	0.0085	0.0670	0.0085	0.0652	0.0086	0.0681
Σ_{22} (0.25)	0.1972	0.0195	0.1936	0.0191	0.1994	0.0198

Table 2: AE and SD of MC-Var(2) model at *T* = 300

Estimator	SELF		ELF		PLF	
Parameter (True Value)	AE	SD	AE	SD	AE	SD
μ_1 (1)	0.7122	0.1338	0.7194	0.1530	0.7304	0.1309
μ_2 (2)	1.8086	0.0600	1.7557	0.0760	1.7909	0.0620
$\alpha_{11}^{(1)}$ (0.2)	0.1974	0.0476	0.1843	0.0600	0.2026	0.0462
$\alpha_{12}^{(1)}$ (0.3)	0.3168	0.0387	0.3126	0.0392	0.3189	0.0384
$\alpha_{11}^{(2)}$ (0.15)	0.1529	0.0410	0.1524	0.3054	0.1581	0.0393
$\alpha_{12}^{(2)}$ (0.15)	0.1625	0.0404	0.1506	0.0563	0.1668	0.0391
$\alpha_{21}^{(1)}$ (0.25)	0.2483	0.0773	0.2374	0.0606	0.2596	0.0731
$\alpha_{22}^{(1)}$ (0.15)	0.1507	0.0596	0.1561	0.0756	0.1533	0.0557
$\alpha_{21}^{(2)}$ (0.2)	0.2126	0.0690	0.1893	0.1131	0.2226	0.0653
$\alpha_{22}^{(2)}$ (0.1)	0.1162	0.0650	0.1032	0.0451	0.1136	0.0511
$\beta_{11}^{(11)}$ (0.3)	0.3009	0.0182	0.3000	0.0182	0.3013	0.0181
$\beta_{12}^{(11)}$ (0.15)	0.1536	0.0171	0.1518	0.0173	0.1544	0.0170
$\beta_{21}^{(11)}$ (0.2)	0.2022	0.0289	0.1986	0.0295	0.2040	0.0287
$\beta_{22}^{(11)}$ (0.1)	0.1089	0.0272	0.1010	0.0359	0.1121	0.0263
Σ_{11} (0.09)	0.0810	0.0065	0.0800	0.0063	0.0815	0.0065
Σ_{12} (0.01)	0.0098	0.0634	0.0099	0.0607	0.0099	0.0651
Σ_{21} (0.01)	0.0098	0.0634	0.0099	0.0607	0.0099	0.0651
Σ_{22} (0.25)	0.2128	0.0176	0.2103	0.0173	0.2142	0.0178

Table 3: MSE and AB of MC-Var(2) model at *T* = 200

Estimator	SELF		ELF		PLF	
Parameter (True value)	MSE	AB	MSE	ABS	MSE	ABS
μ_1 (1)	0.3728	0.5515	0.3405	0.5093	0.3870	0.5674
μ_2 (2)	0.2655	0.2466	0.2683	0.2848	0.2530	0.2722
$\alpha_{11}^{(1)}$ (0.2)	0.2121	0.2193	0.2180	0.2193	0.2104	0.2203
$\alpha_{12}^{(1)}$ (0.3)	0.2246	0.2752	0.2248	0.2715	0.2246	0.2770
$\alpha_{11}^{(2)}$ (0.15)	0.2308	0.2252	0.2445	0.2311	0.2282	0.2227
$\alpha_{12}^{(2)}$ (0.15)	0.2427	0.2236	0.2481	0.2297	0.2408	0.2219
$\alpha_{21}^{(1)}$ (0.25)	0.2066	0.2432	0.2841	0.3032	0.2037	0.2455
$\alpha_{22}^{(1)}$ (0.15)	0.2525	0.2398	0.2642	0.2541	0.2481	0.2375
$\alpha_{21}^{(2)}$ (0.2)	0.2170	0.2202	0.2561	0.2443	0.2119	0.2215
$\alpha_{22}^{(2)}$ (0.1)	0.2789	0.2551	0.2540	0.2393	0.2650	0.2377
$\beta_{11}^{(11)}$ (0.3)	0.1965	0.2558	0.1965	0.2549	0.1965	0.2562
$\beta_{12}^{(11)}$ (0.15)	0.2464	0.2149	0.2472	0.2155	0.2460	0.2146
$\beta_{21}^{(11)}$ (0.2)	0.2022	0.2141	0.2031	0.2131	0.2018	0.2146
$\beta_{22}^{(11)}$ (0.1)	0.2653	0.2342	0.2753	0.2493	0.2631	0.2311
Σ_{11} (0.09)	0.2869	0.2632	0.2876	0.2642	0.2865	0.2626
Σ_{12} (0.01)	0.2462	0.2440	0.2475	0.2462	0.2454	0.2427
Σ_{21} (0.01)	0.2927	0.2734	0.2942	0.2755	0.2918	0.2721
Σ_{22} (0.25)	0.2023	0.1969	0.2029	0.1963	0.2019	0.1974

Table 4: MSE and AB of MC-Var(2) model at *T* = 300

Estimator	SELF		ELF		PLF	
Parameter (True Value)	MSE	ABS	MSE	ABS	MSE	ABS
μ_1 (1)	0.3874	0.5755	0.4016	0.5442	0.3612	0.5897
μ_2 (2)	0.2441	0.2555	0.2493	0.2821	0.2331	0.2768
$\alpha_{11}^{(1)}$ (0.2)	0.2244	0.2179	0.2231	0.2204	0.2280	0.2181
$\alpha_{12}^{(1)}$ (0.3)	0.2118	0.2686	0.2118	0.2657	0.2118	0.2701
$\alpha_{11}^{(2)}$ (0.15)	0.2342	0.2178	0.2325	0.2368	0.3262	0.2162
$\alpha_{12}^{(2)}$ (0.15)	0.2351	0.2157	0.2337	0.2198	0.2401	0.2149
$\alpha_{21}^{(1)}$ (0.25)	0.2190	0.2421	0.2170	0.2340	0.2201	0.2467
$\alpha_{22}^{(1)}$ (0.15)	0.2352	0.2253	0.2319	0.2775	0.2257	0.2276
$\alpha_{21}^{(2)}$ (0.2)	0.2223	0.2269	0.2200	0.2353	0.2381	0.2274
$\alpha_{22}^{(2)}$ (0.1)	0.2546	0.2431	0.2258	0.2331	0.2425	0.2279
$\beta_{11}^{(11)}$ (0.3)	0.2093	0.2561	0.2092	0.2555	0.2093	0.2564
$\beta_{12}^{(11)}$ (0.15)	0.2380	0.2136	0.2377	0.2139	0.2385	0.2134
$\beta_{21}^{(11)}$ (0.2)	0.2237	0.2139	0.2233	0.2134	0.2246	0.2142
$\beta_{22}^{(11)}$ (0.1)	0.2534	0.2328	0.2520	0.2386	0.2572	0.2308
Σ_{11} (0.09)	0.2661	0.2476	0.2658	0.2483	0.2665	0.2472
Σ_{12} (0.01)	0.2682	0.2558	0.2677	0.2572	0.2691	0.2550
Σ_{21} (0.01)	0.2728	0.2584	0.2723	0.2599	0.2737	0.2576
Σ_{22} (0.25)	0.2197	0.2147	0.2195	0.2140	0.2202	0.2151