Statistics and Applications {ISSN 2454-7395 (online)}
Volume 20, No. 2, 2022 (New Series), pp 203–218

# Inference on $P(X < Y)$ for Morgenstern Type Bivariate Exponential Distribution Based on Record Values

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Received: 26 June 2021; Revised: 24 September 2021; Accepted: 18 November 2021

**Abstract**

In this paper, we consider the problem of estimation of $R = P(X < Y)$, when $X$ and $Y$ are dependent. The maximum likelihood estimates and Bayes estimates of $R$ are obtained based on record values when $(X, Y)$ follows Morgenstern type bivariate exponential distribution. The percentile bootstrap and HPD confidence intervals for R are also obtained. Monte Carlo simulations are carried out to study the accuracy of the proposed estimators.

*Key words:* Morgenstern type bivariate exponential distribution; Record values; Maximum likelihood estimation; Bayes estimation.

**AMS Subject Classifications:** 62N05, 62F15

## 1. Introduction

Record value data arise in a wide variety of practical situations. Examples include destructive stress testing, mateorological analysis, hydrology, seismology, sporting and athletic events and oil and mining surveys. Interest in records has increased steadily over the years since Chandler (1952) formulation. Let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed (iid) random variables having an absolutely continuous cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. An observation $X_j$ is called an upper record if $X_j > X_i$, for every $i < j$ (see Arnold *et al.* 1998, p.8). An analogous definition deals with lower record values. Let $(X_1, Y_1)$, $(X_2, Y_2)$,... be a sequence of iid random variables with common continuous joint cdf $F(x, y), (x, y) \in R \times R$. Let $F_X(x)$ and $F_Y(y)$ be the marginal cdfs of $X$ and $Y$ respectively. Let $R_n, n \geq 1$ be the sequence of upper record values arising from the sequence of X's. Then the Y-variate associated with the X-value, which qualified as the nth record will be called the concomitant of the nth record and will be denoted by $R_{[n]}$. Suppose in an experiment, individuals are measured based on an inexpensive test, and only those individuals whose measurement breaks the previous records are retained for the measurement based on an expensive test; then the resulting data involves record values and concomitants of record values. For a detailed discussion on the distribution theory of concomitants of record values see, Arnold *et al.* (1998), Ahsanullah and Nevzorov (2000), Barakat *et al.* (2013) and Ahsanullah and Shakil (2013). Chacko and Thomas (2006,2008) considered the problem of estimation of parameters of Morgenstern

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type bivariate logistic distribution and bivariate normal distribution based on concomitants of record values.

The joint pdf of first $n$ upper record values and its concomitants $(\mathbf{R_{(n)}}, \mathbf{R_{[n]}}) = ((R_{(1)}, R_{[1]}), (R_{(2)}, R_{[2]}), \ldots, (R_{(n)}, R_{[n]}))$ is given by

$$f_{(\mathbf{R_{(n)}},\mathbf{R_{[n]}})}(\mathbf{r_{(n)}}, \mathbf{r_{[n]}}) = \prod_{i=1}^{n} f(r_{[i]}|r_{(i)}) f_{1,2,\ldots,n}(r_{(1)}, r_{(2)}, \ldots, r_{(n)}), \tag{1}$$

where $f_{1,2,\ldots,n}(r_{(1)}, r_{(2)}, \ldots, r_{(n)})$ is the joint pdf of first $n$ upper record values and is given by

$$f_{1,2,\ldots,n}(r_{(1)}, r_{(2)}, \ldots, r_{(n)}) = f(r_{(n)}) \prod_{i=1}^{n-1} \frac{f(r_{(i)})}{1 - F(r_{(i)})}. \tag{2}$$

Now a days the inference on $R = P(X < Y)$ is studied in many branches of sciences and social sciences such as psychology, medicine, pedagogy, pharmaceutics and engineering. In the context of reliability the stress-strength model describes the life of a component which has a random strength $Y$ and is subjected to a random stress $X$. The component fails at the instant that the stress applied to it exceeds the strength and the component will function satisfactorily whenever $X < Y$. Thus $R = P(X < Y)$ is a measure of component reliability. It has found applications in many life testing problems and engineering. The application of $R$ in engineering includes deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures etc. For example, if $X$ and $Y$ are future observations on the stability of an engineering design, then $R$ would be predictive probability that $X$ is less than $Y$. Similarly, if $X$ and $Y$ represents life times of two electronic devices, then $R$ is the probability that one fails before the other. For more details on applications of $R$ in engineering see, Nadarajah and Kotz (2006).

The estimation of $R$ has been extensively investigated in the literature when $X$ and $Y$ are independent random variables belonging to the same bivariate family of distributions. However, there is a relative little work when $X$ and $Y$ are dependent random variables. The problem of estimating $R$ when the $X$ and $Y$ are dependent was considered by Abu-Salih and Shamseldin (1988), Awad *et al.* (1981), Jana and Roy (1994) and Cramer (2001). Estimation of $R$ when $(X, Y)$ follows bivariate normal distribution has been discussed by Enis and Geisser (1971) and Mukherjee and Saran (1985). Jana(1994) and Hanagal (1995) discussed the estimation of $R$ when $(X, Y)$ follows Marshall-Olkin bivariate exponential distribution. Hanagal (1997) discussed the estimation of $R$ when $(X, Y)$ has a bivariate Pareto distribution. Chacko and Mathew (2019) considered the estimation of $R = P(X < Y)$ for bivariate normal distribution based on ranked set sample. Chacko and Mathew (2020) considered the estimation of $R = P(X < Y)$ for bivariate normal distribution based on record values. In this paper, we focus on estimation of $R = P(X < Y)$ based on upper record values and its concomitants, corresponding to a bivariate random variable $(X, Y)$ which follows a Morgenstern Type Bivariate Exponential distribution (MTBED) with pdf given by (see, Kotz *et al.*, 2000, P.353)

$$f(x,y) = \begin{cases} \theta_1\theta_2 exp(-\theta_1 x - \theta_2 y)[1 + \alpha(1 - 2exp(-\theta_1 x))(1 - 2exp(-\theta_2 y))], \\ \qquad x > 0, y > 0; -1 \le \alpha \le 1; \theta_1 > 0, \theta_2 > 0 \\ 0, \qquad \text{otherwise} \end{cases} \tag{3}$$

It may be noted that if $(X, Y)$ has a MTBED as defined in (3) then the marginal distributions of both $X$ and $Y$ have exponential distributions. The correlation between $X$ and $Y$ is $\alpha/4$. As $\alpha$ lies between -1 and 1, MTBED accomodates correlation in the range of (-1/4,1/4). Exponential distributions are the most popular and the most applied life time models in many areas, including life testing and reliability studies. Let $T_1$ and $T_2$ be two dependent components of a system with lifetimes $X$ and $Y$ respectively. Then $R = P(X < Y)$ is the probability that the first component $T_1$ fails before second component $T_2$. If (X,Y) follows a bivariate exponential distribution and the data available are in the form of upper record values and its concomitants then the methods describe in this paper can easily be used to estimate $R = P(X < Y)$.

The organization of the paper is as follows. In section 2, we consider maximum likelihood estimation of $R$ and also obtain the bootstrap confidence interval (CI) based on the maximum likelihood estimator (MLE). In section 3, we consider the Bayes estimation of $R$ using importance sampling method under both symmetric and assymetric loss functions. Section 4 is devoted to some simulation studies and in section 5, we give concluding remarks.

## 2. Maximum Likelihood Estimation

Let (X,Y) follows MTBED with pdf defined in (3), then $R = P(X < Y)$ is given by

$$\begin{aligned} R &= P(X<Y) \\ &= \frac{\theta_1}{\theta_1+\theta_2}[1+\alpha\frac{\theta_1(\theta_1-\theta_2)}{(2\theta_1+\theta_2)(2\theta_2+\theta_1)}]. \end{aligned} \tag{4}$$

If we denote $\theta = (\theta_1, \theta_2, \alpha)$ then we can write $R$ as

$$R = R(\theta).$$

In this section, we obtain the MLE of $R$ for MTBED using record values and its concomitants. Let $(R_{(i)}, R_{[i]}), i = 1, 2, \ldots, n$ be the upper record values and its concomitants arising from MTBED. Then from (1), the likelihood function is given by

$$\begin{aligned} L(\theta) &= (\theta_1\theta_2)^n \prod_{i=1}^{n} exp(-\theta_1 r_{(i)} - \theta_2 r_{[i]})[1+\alpha(1-2exp(-\theta_1 r_{(i)})) \\ &\quad \times(1-2exp(-\theta_2 r_{[i]}))] \prod_{i=1}^{n-1} \frac{1}{exp(-\theta_1 r_{(i)})}. \end{aligned}$$

Then the log-likelihood function is given by

$$\begin{aligned} \log L(\theta) &= n\log\theta_1 + n\log\theta_2 - \theta_1 r_{(n)} - \theta_2\sum_{i=1}^{n} r_{[i]} \\ &\quad + \sum_{i=1}^{n}\log[1+\alpha(1-2exp(-\theta_1 r_{(i)}))(1-2exp(-\theta_2 r_{[i]}))]. \end{aligned}$$

Thus we have

$$\frac{\partial \log L}{\partial\theta_1} = \frac{n}{\theta_1} - r_{(n)} + \sum_{i=1}^{n}\frac{2\alpha r_{(i)}(1-2exp(-\theta_2 r_{[i]}))exp(-\theta_1 r_{(i)})}{[1+\alpha(1-2exp(-\theta_1 r_{(i)}))(1-2exp(-\theta_2 r_{[i]}))]},$$

$$\frac{\partial \log L}{\partial \theta_2} = \frac{n}{\theta_2} - \sum_{i=1}^{n} r_{[i]} + \sum_{i=1}^{n} \frac{2\alpha r_{[i]}(1-2exp(-\theta_1 r_{(i)}))exp(-\theta_2 r_{[i]})}{[1+\alpha(1-2exp(-\theta_1 r_{(i)}))(1-2exp(-\theta_2 r_{[i]}))]}$$

and

$$\frac{\partial \log L}{\partial \alpha} = \sum_{i=1}^{n} \frac{(1-2exp(-\theta_1 r_{(i)}))(1-2exp(-\theta_2 r_{[i]}))}{[1+\alpha(1-2exp(-\theta_1 r_{(i)}))(1-2exp(-\theta_2 r_{[i]}))]}.$$

The MLEs of $\theta_1$, $\theta_2$ and $\alpha$ can be obtained as the solutions of the following non-linear equations

$$\frac{n}{\theta_1} - r_{(n)} + \sum_{i=1}^{n} \frac{2\alpha r_{(i)}(1-2exp(-\theta_2 r_{[i]}))exp(-\theta_1 r_{(i)})}{[1+\alpha(1-2exp(-\theta_1 r_{(i)}))(1-2exp(-\theta_2 r_{[i]}))]} = 0,$$

$$\frac{n}{\theta_2} - \sum_{i=1}^{n} r_{[i]} + \sum_{i=1}^{n} \frac{2\alpha r_{[i]}(1-2exp(-\theta_1 r_{(i)}))exp(-\theta_2 r_{[i]})}{[1+\alpha(1-2exp(-\theta_1 r_{(i)}))(1-2exp(-\theta_2 r_{[i]}))]} = 0$$

and

$$\sum_{i=1}^{n} \frac{(1-2exp(-\theta_1 r_{(i)}))(1-2exp(-\theta_2 r_{[i]}))}{[1+\alpha(1-2exp(-\theta_1 r_{(i)}))(1-2exp(-\theta_2 r_{[i]}))]} = 0.$$

If $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\alpha})$ is the MLE of $\theta$ obtained by solving the above nonlinear equations, then the MLE of $R$ is given by

$$\hat{R}_{ML} = \frac{\hat{\theta}_1}{\hat{\theta}_1 + \hat{\theta}_2}\left[1 + \hat{\alpha}\frac{\hat{\theta}_1(\hat{\theta}_1 - \hat{\theta}_2)}{(2\hat{\theta}_1 + \hat{\theta}_2)(2\hat{\theta}_2 + \hat{\theta}_1)}\right]. \quad (5)$$

### 2.1. Asymptotic confidence interval

In this subsection, the asymptotic confidence interval of $R$ is obtained. Towards this, we consider the observed information matrix of $\theta$. Let

$$I(\theta) = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix},$$

where

$$\begin{aligned} I_{11} &= \frac{\partial^2 \log L}{\partial \theta_1^2} = \frac{-n}{\theta_1^2} - \sum_{i=1}^{n} 2\alpha r_{(i)}(1-2exp(-\theta_2 r_{[i]})) \\ &\quad \times \left(\frac{1-\alpha r_{(i)} exp(-\theta_1 r_{(i)})(1-2exp(-\theta_2 r_{[i]})}{[1+\alpha(1-2exp(-\theta_1 r_{(i)})(1-2exp(-\theta_2 r_{[i]})]^2}\right), \end{aligned}$$

$$I_{12} = \frac{\partial^2 \log L}{\partial \theta_1 \partial \theta_2} = \sum_{i=1}^{n} \frac{2\alpha r_{(i)} exp(-\theta_1 r_{(i)})}{[1+\alpha(1-2exp(-\theta_1 r_{(i)})(1-2exp(-\theta_2 r_{[i]})]^2},$$

$$I_{13} = \frac{\partial^2 \log L}{\partial\theta_1\partial\alpha} = \sum_{i=1}^{n} \frac{2r_{(i)}exp(-\theta_1 r_{(i)})(1-2exp(-\theta_2 r_{[i]})}{[1+\alpha(1-2exp(-\theta_1 r_{(i)})(1-2exp(-\theta_2 r_{[i]})]^2},$$

$$I_{21} = \frac{\partial^2 \log L}{\partial\theta_2\partial\theta_1} = \sum_{i=1}^{n} \frac{2\alpha r_{[i]}exp(-\theta_2 r_{[i]})}{[1+\alpha(1-2exp(-\theta_1 r_{(i)})(1-2exp(-\theta_2 r_{[i]})]^2},$$

$$\begin{aligned} I_{22} &= \frac{\partial^2 \log L}{\partial\theta_2^2} = \frac{-n}{\theta_2^2} + \sum_{i=1}^{n} 2\alpha r_{[i]}(1-2exp(-\theta_1 r_{(i)}) \\ &\times \left(\frac{1-\alpha r_{[i]}exp(-\theta_2 r_{[i]})(1-2exp(-\theta_1 r_{(i)})}{[1+\alpha(1-2exp(-\theta_1 r_{(i)})(1-2exp(-\theta_2 r_{[i]})]^2}\right), \end{aligned}$$

$$I_{23} = \frac{\partial^2 \log L}{\partial\theta_2\partial\alpha} = \sum_{i=1}^{n} \frac{2r_{[i]}exp(-\theta_2 r_{[i]})(1-2exp(-\theta_1 r_{(i)}))}{[1+\alpha(1-2exp(-\theta_1 r_{(i)})(1-2exp(-\theta_2 r_{[i]})]^2},$$

$$I_{31} = \frac{\partial^2 \log L}{\partial\alpha\partial\theta_1} = \sum_{i=1}^{n} \frac{(1-2exp(-\theta_2 r_{[i]}))}{[1+\alpha(1-2exp(-\theta_1 r_{(i)})(1-2exp(-\theta_2 r_{[i]})]^2},$$

$$I_{32} = \frac{\partial^2 \log L}{\partial\alpha\partial\theta_2} = \sum_{i=1}^{n} \frac{(1-2exp(-\theta_1 r_{(i)}))}{[1+\alpha(1-2exp(-\theta_1 r_{(i)})(1-2exp(-\theta_2 r_{[i]})]^2}$$

and

$$I_{33} = \frac{\partial^2 \log L}{\partial\alpha^2} = -\sum_{i=1}^{n} \frac{(1-2exp(-\theta_1 r_{(i)})^2(1-2exp(-\theta_2 r_{[i]})^2}{[1+\alpha(1-2exp(-\theta_1 r_{(i)})(1-2exp(-\theta_2 r_{[i]})]^2}.$$

Let $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\alpha})$ be the MLE of $\theta$. Then the observed information matrix is given by $I(\hat{\theta})$. Thus by using delta method, we obtain the asymptotic distribution of $\hat{R}$. For that we have

$$\begin{aligned} \hat{Var}(\hat{R}_{ML}) &= \hat{Var}(R(\hat{\theta})) \\ &\approx h(\hat{\theta})[I(\hat{\theta})]^{-1}h(\hat{\theta})^{\top}. \end{aligned}$$

where

$$h(\hat{\theta}) = \left(\frac{\partial R}{\partial\theta_1}, \frac{\partial R}{\partial\theta_2}, \frac{\partial R}{\partial\alpha}\right)\bigg|_{\theta=\hat{\theta}}$$

with

$$\begin{aligned} \frac{\partial R}{\partial\theta_1} &= \frac{\theta_2}{(\theta_1+\theta_2)^2} + \alpha\frac{\theta_1\theta_2(4\theta_2^3+7\theta_1\theta_2^2-2\theta_1^3)}{(2\theta_1^3+2\theta_2^3+7\theta_1\theta_2^2+7\theta_1^2\theta_2)^2}, \\ \frac{\partial R}{\partial\theta_2} &= \frac{-\theta_1}{(\theta_1+\theta_2)^2} + \alpha\frac{\theta_1^2(2\theta_1^3-4\theta_2^3-7\theta_1\theta_2^2)}{(2\theta_1^3+2\theta_2^3+7\theta_1\theta_2^2+7\theta_1^2\theta_2)^2} \end{aligned}$$

and

$$\frac{\partial R}{\partial\alpha} = \frac{\theta_1^2\theta_2}{(2\theta_1^3+2\theta_2^3+7\theta_1\theta_2^2+7\theta_1^2\theta_2)}.$$

Thus $\frac{\hat{R}-R}{\sqrt{v\hat{a}r(\hat{R})}}$ is aymptotically distributed as $N(0,1)$. Thus a $(1-\nu)100\%$ confidence interval for $R$ based on the MLE is $(\hat{R} - z_{\nu/2}\sqrt{\hat{Var}(\hat{R})}, \hat{R} + z_{\nu/2}\sqrt{\hat{Var}(\hat{R})})$, where $z_{\nu/2}$ is the $(1-\nu/2)100th$ percentile of N(0,1).

### 2.2. Bootstrap confidence interval

In this subsection, we consider percentile bootstrap CI for $R$ based on MLEs. For that we do the following.

1. Compute the MLEs $\hat{\theta}_1^{(0)}$, $\hat{\theta}_2^{(0)}$ and $\hat{\alpha}^{(0)}$ of $\theta_1, \theta_2$ and $\alpha$ using original record values and its concomitants and set k=1.
2. Generate a bootstrap sample using $\hat{\theta}_1^{(0)}$, $\hat{\theta}_2^{(0)}$ and $\hat{\alpha}^{(0)}$ from MTBED and obtain the MLEs $\hat{\theta}_1^{(k)}$, $\hat{\theta}_2^{(k)}$ and $\hat{\alpha}^{(k)}$ using the bootstrap sample.
3. Obtain the MLE $\hat{R}_k = R(\hat{\theta}_1^{(k)}, \hat{\theta}_2^{(k)}, \hat{\alpha}^{(k)})$.
4. Set $k = k + 1$.
5. Repeat steps (2)to(4) B times to obtain the MLEs $\hat{R}_1, \hat{R}_2, \cdots, \hat{R}_B$, for sufficiently large $B$ .
6. Arrange $\hat{R}_1, \hat{R}_2, \cdots, \hat{R}_B$ in ascending order as $\hat{R}_{(1)} \leq \hat{R}_{(2)}, \ldots, \leq \hat{R}_{(B)}$. Then the $100(1-\nu)$ percentile bootstrap CI for R is given by $\left(\hat{R}_{([B(\nu/2)])}, \hat{R}_{([B(1-\nu/2)])}\right)$, $[.]$ is the greatest integer function.

## 3. Bayesian Estimation

In this section, we consider Bayesian estimation of R for MTBED under symmetric as well as asymmetric loss functions. For symmetric loss function we consider squared error loss (SEL) function and for asymmetric loss function we consider both LINEX loss (LL) and the general entropy loss (EL) function. The Bayes estimate of any parameter $\mu$ under SEL function is the posterior mean of $\mu$. The Bayes estimate of any parameter $\mu$ under LL is given by

$$\hat{d}_{LB}(\mu) = \frac{-1}{h} \log\{E_\mu(e^{-h\mu}|\underline{x})\}, h \neq 0, \tag{6}$$

provided $E_\mu$ exists. The Bayes estimate of any parameter $\mu$ under EL function is given by

$$\hat{d}_{EB}(\mu) = (E_\mu(\mu^{-q}|\underline{x}))^{\frac{-1}{q}}, q \neq 0, \tag{7}$$

provided $E_\mu$ exists.
Let $(R_{(i)}, R_{[i]}), i = 1, 2, \ldots, n$ be the vector of record value and its concomitants arising from MTBED $(\theta_1, \theta_2, \alpha)$. Then from (1) the likelihood function is given by

$$\begin{aligned} L(\theta) &= (\theta_1\theta_2)^n \prod_{i=1}^{n} exp(-\theta_1 r_{(i)} - \theta_2 r_{[i]})[1 + \alpha(1 - 2exp(-\theta_1 r_{(i)})) \\ &\quad \times(1 - 2exp(-\theta_2 r_{[i]}))] \prod_{i=1}^{n-1} \frac{1}{exp(-\theta_1 r_{(i)})}. \end{aligned}$$

Assume that the prior distributions of $\theta_1 \sim$ Gamma$(a, b)$, $\theta_2 \sim$ Gamma$(c, d)$ and $\alpha \sim U[-1, 1]$. Thus the prior density functions of $\theta_1, \theta_2$ and $\alpha$ are respectively given by

$$\pi_1(\theta_1|a, b) = \frac{b^a}{\Gamma(a)} \theta_1^{a-1} e^{-b\theta_1}; a > 0, b > 0, \tag{8}$$

$$\pi_2(\theta_2|c,d) = \frac{d^c}{\Gamma(c)}\theta_2^{c-1}e^{-d\theta_2}; c > 0, d > 0 \tag{9}$$

and

$$\pi_3(\alpha) = \frac{1}{2}, -1 \leq \alpha \leq 1. \tag{10}$$

Then the joint prior distribution of $\theta$ is given by

$$\pi(\theta) = \frac{1}{2}\frac{b^a}{\Gamma(a)}\frac{d^c}{\Gamma(c)}\theta_1^{a-1}\theta_2^{c-1}e^{-b\theta_1}e^{-d\theta_2} \tag{11}$$

Then the joint posterior density of $\theta$ is given by

$$\pi^*(\theta) = \frac{L(\theta)\pi(\theta)}{\int L(\theta)\pi(\theta)d\theta}. \tag{12}$$

Therefore the Bayes estimate of $R(\theta)$ under SEL, LL and EL are respectively given by

$$\hat{R}_S = \frac{\int R(\theta)L(\theta)\pi(\theta)d\theta}{\int L(\theta)\pi(\theta)d\theta}, \tag{13}$$

$$\hat{R}_L = \frac{-1}{h}\log\frac{\int e^{-hR(\theta)}L(\theta)\pi(\theta)d\theta}{\int L(\theta)\pi(\theta)d\theta} \tag{14}$$

and

$$\hat{R}_E = \left[\frac{\int R(\theta)^{-q}L(\theta)\pi(\theta)d\theta}{\int L(\theta)\pi(\theta)d\theta}\right]^{\frac{-1}{q}}. \tag{15}$$

It is not possible to compute (13)-(15) explicitly. The popular approach to perform the integrals (13) to (15) is the Markov Chain Monte Carlo (MCMC) method which replace the expectation values of the parameters with the average values over Monte Carlo (posterior) samples obtained through the Markov Chain. A drawback of the MCMC method is that the time series of the Monte Carlo samples obtained through the Markov Chain are usually correlated. The importance sampling method introduces an importance sampling density which should be handled easily and can generate Monte Carlo data randomly. The Monte Carlo data generated randomly by the importance sampling method can be autocorrelation-free. The autocorrelation-free nature of the importance sampling could be considered to be an advantage over the MCMC method. Thus we consider importance sampling method to find the Bayes estimates for $R$.

### 3.1. Importance sampling method

In this subsection, we consider the importance sampling method to generate samples from the posterior distributions and then find the Bayes estimate of $R$. The numerator in the posterior distribution given in (12) can be written as

$$L(\theta)\pi(\theta) \propto Q(\theta)f_1(\theta_1)f_2(\theta_2)f_3(\alpha),$$

where

$$Q(\theta) = \prod_{i=1}^{n}[1 + \alpha(1 - 2exp(-\theta_1 r_{(i)}))(1 - 2exp(-\theta_2 r_{[i]}))], \tag{16}$$

$$f_1(\theta_1) \propto \theta_1^{n+a-1} exp[-\theta_1(r_{(n)} + b)] \tag{17}$$

$$f_2(\theta_2) \propto \theta_2^{m+c-1} exp\left[-\theta_2\left(\sum_{i=1}^{n} r_{[i]} + d\right)\right] \tag{18}$$

and

$$f_3(\alpha) = \frac{1}{2}. \tag{19}$$

Thus from (17) we can see that distribution of $\theta_1$ follows Gamma distribution with parameters $(n + a)$ and $(r_{(n)} + b)$. Again from (18) one can see that distribution of $\theta_2$ follows gamma distribution with parameters $(m+c)$ and $(\sum_{i=1}^{n} r_{[i]}+d)$. From (19) we can see that $\alpha \sim U(-1, 1)$. Let $\theta^{(t)} = (\theta_1^{(t)}, \theta_2^{(t)}, \alpha^{(t)})$, $t = 1, 2, \ldots, N$ be the observations generated from (17),(18) and (19) respectively. Then by importance sampling method the Bayes estimators under SEL, LL and EL given by (13)-(15) can be respectively written as

$$\hat{R}_S = \frac{\sum_{t=1}^{N} R(\theta^{(t)})Q(\theta^{(t)})}{\sum_{t=1}^{N} Q(\theta^{(t)})}, \tag{20}$$

$$\hat{R}_L = \frac{-1}{h} \log\left[\frac{\sum_{t=1}^{N} exp(-hR(\theta^{(t)})Q(\theta^{(t)})}{\sum_{t=1}^{N} Q(\theta^{(t)})}\right] \tag{21}$$

and

$$\hat{R}_E = \left[\frac{\sum_{t=1}^{N} (R(\theta^{(t)}))^{-q}Q(\theta^{(t)})}{\sum_{t=1}^{N} Q(\theta^{(t)})}\right]^{-1/q}. \tag{22}$$

### 3.2. HPD interval

In this subsection, we construct HPD intervals for $R$ as described in Chen and Shao (1999). In this method a Monte Carlo approach is used to approximate the pth quantile of $R$ and then obtain an estimate of Bayesian credible or HPD interval. Define $R_t = R(\theta^{(t)})$, where $\theta^{(t)} = (\theta_1^{(t)}, \theta_2^{(t)}, \alpha^{(t)})$ for $t = 1, 2, \ldots, M$ are posterior samples generated respectively from (17), (18) and (19) for $\theta_1, \theta_2$ and $\alpha$. Let $R_{(t)}$ be the ordered values of $R_t$. Define

$$w_t = \frac{Q(\theta^{(t)})}{\sum_{t=1}^{M} Q(\theta^{(t)})}.$$

Then the pth quantile of $R$ can be estimated as

$$\hat{R}^{(p)} = \begin{cases} R_1 & \text{if} \quad p = 0 \\ R_{(i)} & if \quad \sum_{j=1}^{i-1} w_{(j)} < p < \sum_{j=1}^{i} w_{(j)}, \end{cases}$$

where $w_{(j)}$ is the weight associated with jth ordered value $R_{(j)}$. Then the $100(1-\nu)\%$, $0 < \nu < 1$, confidence interval for $R$ is given by $(\hat{R}^{(j/M)}, \hat{R}^{(j+[(1-\nu)M])/M)})$, $j = 1, 2, \ldots, M$, where [.] is the greatest integer function. Then the required HPD interval for R is the interval with smallest width.

## 4. Simulation Study

In this section, we carry out a simulation study for illustrating the estimation procedures developed in the previous sections. First we obtain the MLE of $R$ using (5). We have obtained the bias and MSE of MLEs for different combinations of $\theta_1$, $\theta_2$ and $\alpha$ and are given in Table 1. The bootstrap CI for $R$ are also obtained. The average interval length (AIL) and coverage probability (CP) are also obtained and are included in Table 1. We consider four sets of true parameter values, $(\theta_1, \theta_2)$= (5,1), (3,2), (2,4) and (0.5,5). Since prior distribution of $\theta_1$ follows gamma distribution with mean $\frac{a}{b}$, we take the hyperparameters for $\theta_1$ =5, 3, 2, and 0.5 as $(a, b)$=(5,1), (3,1), (2,1) and (0.5,1) respectively. Similarly we take the hyperparamerters of $\theta_2$=1,2,4, and 5 as $(c, d)$ = (1,1), (2,1), (4,1) and (5,1). We have obtained the Bayes estimators for $R$ of MTBED under SEL, LL and EL functions using importance sampling method and are given in Table 2. For importance sampling method we use the following algorithm.

1. Generate $n$ upper record values and its concomiants from MTBED distribution with parameters $\theta_1$, $\theta_2$ and $\alpha$.
2. Calculate the Bayes estimators of $R$ as described below.
   (a) Set t=1
   (b) Generate $\theta_1^{(t)}$ from Gamma distribution with parameters $n + a$ and $r_{(n)} + b$.
   (c) Generate $\theta_2^{(t)}$ from Gamma distribution with parameters $m + c$ and $\sum_{i=1}^{n} r_{[i]} + d$.
   (d) Generate $\alpha^{(t)}$ from $Uniform(-1, 1)$ distribution.
   (e) Calculate $\hat{R}(\theta^{(t)})$ using (5) and $Q(\theta^{(t)})$ using (16).
   (f) Set t=t+1.
   (g) Repeat steps (b) to (f) 50,000 times.
   (h) Calculate the Bayes estimators for $R$ using (20)-(22)
3. Repeat steps 1 and 2 for 500 times to obtain the estimators $\hat{R}_1, \hat{R}_2, \cdots, \hat{R}_{500}$.
4. Calculate the average bias= $\frac{1}{500}\sum_i^{500}(\hat{R}_i - R)$ and $MSE = \frac{1}{500}\sum_i^{500}(\hat{R}_i - \bar{R}) + bias^2$ of the estimators.

We repeat the simulation study for different values of $\alpha$ and $n$. From the tables we can see that the bias and MSE of all estimators decrease when the number of records increase. We can also see that among different estimators Bayes estimator under SEL have minimum bias and MSE. From Table 1 we can see that the AILs of HPD intervals are smaller than that of bootstrap CIs and the CPs of HPD intervals are higher than that of bootstrap CIs.

**Table 1: The AIL and CP for bootstrap CIs and HPD intervals**

| $\alpha$ | n | $\theta_1$ | $\theta_2$ | R | Bootstrap | | HPD | |
|---|---|---|---|---|---|---|---|---|
| | | | | | AIL | CP | AIL | CP |
| -0.75 | 6 | 5 | 1 | 0.67100 | 0.22830 | 0.85 | 0.11755 | 0.94 |
| | | 3 | 2 | 0.57589 | 0.21336 | 0.85 | 0.13603 | 0.92 |
| | | 2 | 4 | 0.34583 | 0.18995 | 0.87 | 0.12971 | 0.93 |
| | | 0.5 | 5 | 0.09334 | 0.18229 | 0.86 | 0.12178 | 0.92 |
| | 8 | 5 | 1 | 0.67100 | 0.17457 | 0.87 | 0.14590 | 0.95 |
| | | 3 | 2 | 0.57589 | 0.16680 | 0.87 | 0.11195 | 0.94 |
| | | 2 | 4 | 0.34583 | 0.15912 | 0.88 | 0.12494 | 0.95 |
| | | 0.5 | 5 | 0.09334 | 0.15215 | 0.88 | 0.11086 | 0.94 |
| | 10 | 5 | 1 | 0.67100 | 0.14993 | 0.88 | 0.11234 | 0.96 |
| | | 3 | 2 | 0.57589 | 0.13375 | 0.87 | 0.08454 | 0.96 |
| | | 2 | 4 | 0.34583 | 0.16985 | 0.89 | 0.12985 | 0.95 |
| | | 0.5 | 5 | 0.09334 | 0.21065 | 0.85 | 0.14707 | 0.95 |
| -0.5 | 6 | 5 | 1 | 0.72511 | 0.23566 | 0.85 | 0.13176 | 0.94 |
| | | 3 | 2 | 0.58393 | 0.17582 | 0.84 | 0.11582 | 0.93 |
| | | 2 | 4 | 0.34167 | 0.17737 | 0.84 | 0.13245 | 0.93 |
| | | 0.5 | 5 | 0.09253 | 0.16639 | 0.86 | 0.11629 | 0.94 |
| | 8 | 5 | 1 | 0.72511 | 0.18512 | 0.86 | 0.12621 | 0.95 |
| | | 3 | 2 | 0.58393 | 0.15397 | 0.86 | 0.13670 | 0.95 |
| | | 2 | 4 | 0.34167 | 0.14288 | 0.88 | 0.12190 | 0.94 |
| | | 0.5 | 5 | 0.09253 | 0.15278 | 0.87 | 0.13116 | 0.95 |
| | 10 | 5 | 1 | 0.72511 | 0.14377 | 0.88 | 0.11098 | 0.94 |
| | | 3 | 2 | 0.58393 | 0.16415 | 0.87 | 0.18154 | 0.95 |
| | | 2 | 4 | 0.34167 | 0.16366 | 0.88 | 0.08478 | 0.96 |
| | | 0.5 | 5 | 0.09253 | 0.23831 | 0.85 | 0.12197 | 0.95 |
| -0.25 | 6 | 5 | 1 | 0.77922 | 0.22563 | 0.85 | 0.13355 | 0.93 |
| | | 3 | 2 | 0.59196 | 0.14293 | 0.84 | 0.11844 | 0.94 |
| | | 2 | 4 | 0.33750 | 0.14189 | 0.85 | 0.12333 | 0.93 |
| | | 0.5 | 5 | 0.09172 | 0.16277 | 0.86 | 0.16214 | 0.94 |
| | 8 | 5 | 1 | 0.77922 | 0.15815 | 0.86 | 0.12882 | 0.95 |
| | | 3 | 2 | 0.59196 | 0.14140 | 0.87 | 0.11425 | 0.95 |
| | | 2 | 4 | 0.33750 | 0.13703 | 0.85 | 0.12929 | 0.95 |
| | | 0.5 | 5 | 0.09172 | 0.14633 | 0.88 | 0.11552 | 0.94 |
| | 10 | 5 | 1 | 0.77922 | 0.12808 | 0.87 | 0.12552 | 0.95 |
| | | 3 | 2 | 0.59196 | 0.13565 | 0.89 | 0.12982 | 0.96 |
| | | 2 | 4 | 0.33750 | 0.12893 | 0.88 | 0.11873 | 0.95 |
| | | 0.5 | 5 | 0.09172 | 1.12536 | 0.89 | 0.11320 | 0.96 |

**Table 1: Continued**

| $\alpha$ | n | $\theta_1$ | $\theta_2$ | R | Bootstrap | | HPD | |
|---|---|---|---|---|---|---|---|---|
| | | | | | AIL | CP | AIL | CP |
| 0.25 | 6 | 5 | 1 | 0.88745 | 0.15200 | 0.84 | 0.09624 | 0.93 |
| | | 3 | 2 | 0.60804 | 0.17320 | 0.84 | 0.11650 | 0.93 |
| | | 2 | 4 | 0.32917 | 0.18756 | 0.85 | 0.08888 | 0.92 |
| | | 0.5 | 5 | 0.09010 | 0.16752 | 0.85 | 0.09039 | 0.92 |
| | 8 | 5 | 1 | 0.88745 | 0.13493 | 0.85 | 0.09131 | 0.95 |
| | | 3 | 2 | 0.60804 | 0.16607 | 0.85 | 0.11750 | 0.94 |
| | | 2 | 4 | 0.32917 | 0.16677 | 0.86 | 0.08159 | 0.95 |
| | | 0.5 | 5 | 0.09010 | 0.15563 | 0.86 | 0.08744 | 0.95 |
| | 10 | 5 | 1 | 0.88745 | 0.12890 | 0.86 | 0.08205 | 0.93 |
| | | 3 | 2 | 0.60804 | 0.13831 | 0.87 | 0.10826 | 0.95 |
| | | 2 | 4 | 0.32917 | 0.12724 | 0.88 | 0.07175 | 0.95 |
| | | 0.5 | 5 | 0.09010 | 0.13398 | 0.88 | 0.07624 | 0.96 |
| 0.5 | 6 | 5 | 1 | 0.94156 | 0.16974 | 0.85 | 0.11738 | 0.94 |
| | | 3 | 2 | 0.61607 | 0.15253 | 0.86 | 0.12961 | 0.93 |
| | | 2 | 4 | 0.32500 | 0.14018 | 0.83 | 0.11854 | 0.94 |
| | | 0.5 | 5 | 0.08929 | 0.14557 | 0.84 | 0.12974 | 0.95 |
| | 8 | 5 | 1 | 0.94156 | 0.13592 | 0.85 | 0.11546 | 0.94 |
| | | 3 | 2 | 0.61607 | 0.13528 | 0.85 | 0.10860 | 0.95 |
| | | 2 | 4 | 0.32500 | 0.12432 | 0.86 | 0.10255 | 0.96 |
| | | 0.5 | 5 | 0.08929 | 0.12819 | 0.88 | 0.11897 | 0.95 |
| | 10 | 5 | 1 | 0.94156 | 0.12177 | 0.88 | 0.12423 | 0.94 |
| | | 3 | 2 | 0.61607 | 0.12387 | 0.86 | 0.12557 | 0.96 |
| | | 2 | 4 | 0.32500 | 0.11362 | 0.88 | 0.12771 | 0.96 |
| | | 0.5 | 5 | 0.08929 | 0.11695 | 0.89 | 0.08317 | 0.95 |
| 0.75 | 6 | 5 | 1 | 0.99567 | 0.18762 | 0.84 | 0.11731 | 0.93 |
| | | 3 | 2 | 0.62411 | 0.17517 | 0.85 | 0.12341 | 0.92 |
| | | 2 | 4 | 0.32083 | 0.16081 | 0.83 | 0.09304 | 0.94 |
| | | 0.5 | 5 | 0.08847 | 0.16947 | 0.84 | 0.09253 | 0.94 |
| | 8 | 5 | 1 | 0.99567 | 0.14872 | 0.85 | 0.12235 | 0.93 |
| | | 3 | 2 | 0.62411 | 0.13736 | 0.87 | 0.13994 | 0.95 |
| | | 2 | 4 | 0.32083 | 0.12521 | 0.87 | 0.09638 | 0.95 |
| | | 0.5 | 5 | 0.08847 | 0.13757 | 0.88 | 0.09070 | 0.96 |
| | 10 | 5 | 1 | 0.99567 | 0.12422 | 0.86 | 0.10111 | 0.95 |
| | | 3 | 2 | 0.62411 | 0.12523 | 0.88 | 0.11063 | 0.96 |
| | | 2 | 4 | 0.32083 | 0.12777 | 0.87 | 0.08842 | 0.96 |
| | | 0.5 | 5 | 0.08847 | 0.12388 | 0.89 | 0.09357 | 0.96 |

**Table 2: The bias and MSE for MLE and Bayes estimators for $R$**

| $\alpha$ | n | $\theta_1$ | $\theta_2$ | MLE | | SE | | LL | | EL | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | Bias | MSE | Bias | MSE | Bias | MSE | Bias | MSE |
| -0.75 | 6 | 5 | 1 | 0.15841 | 0.12508 | 0.12893 | 0.02014 | 0.13763 | 0.02264 | 0.14868 | 0.02623 |
| | | 3 | 2 | -0.12726 | 0.12173 | 0.02384 | 0.00752 | 0.00917 | 0.00907 | 0.01595 | 0.00935 |
| | | 2 | 4 | 0.12681 | 0.15718 | -0.01435 | 0.00854 | 0.00669 | 0.00825 | 0.01677 | 0.00864 |
| | | 0.5 | 5 | 0.19078 | 0.16283 | 0.00522 | 0.00162 | 0.02135 | 0.00240 | 0.02290 | 0.00256 |
| | 8 | 5 | 1 | -0.14307 | 0.11006 | 0.12063 | 0.01724 | 0.12942 | 0.01956 | 0.14035 | 0.02282 |
| | | 3 | 2 | 0.11088 | 0.11727 | -0.00596 | 0.00775 | 0.00451 | 0.00815 | 0.01270 | 0.00837 |
| | | 2 | 4 | 0.11998 | 0.09947 | -0.01343 | 0.00837 | 0.00377 | 0.00653 | 0.01565 | 0.00675 |
| | | 0.5 | 5 | 0.16437 | 0.14880 | 0.00417 | 0.00158 | 0.01192 | 0.00195 | 0.01295 | 0.00216 |
| | 10 | 5 | 1 | 0.12832 | 0.10176 | 0.10952 | 0.01594 | 0.11912 | 0.01818 | 0.13063 | 0.02139 |
| | | 3 | 2 | -0.10925 | 0.10386 | -0.00592 | 0.00612 | 0.00388 | 0.00769 | 0.00961 | 0.00783 |
| | | 2 | 4 | 0.11148 | 0.09322 | 0.01237 | 0.00576 | 0.00108 | 0.00549 | 0.00530 | 0.00564 |
| | | 0.5 | 5 | 0.13270 | 0.08616 | 0.00172 | 0.00149 | 0.00901 | 0.00178 | 0.00985 | 0.00185 |
| -0.5 | 6 | 5 | 1 | -0.14634 | 0.13198 | 0.07667 | 0.00944 | 0.08515 | 0.00995 | 0.09591 | 0.01322 |
| | | 3 | 2 | 0.17554 | 0.18434 | 0.01319 | 0.00853 | 0.01647 | 0.00841 | 0.02230 | 0.00877 |
| | | 2 | 4 | 0.19828 | 0.15297 | -0.00146 | 0.00621 | 0.00271 | 0.00981 | 0.00674 | 0.00915 |
| | | 0.5 | 5 | -0.12645 | 0.16493 | 0.00700 | 0.00122 | 0.01692 | 0.00178 | 0.01831 | 0.00190 |
| | 8 | 5 | 1 | 0.13805 | 0.12617 | 0.06490 | 0.00858 | 0.07459 | 0.00898 | 0.08654 | 0.01228 |
| | | 3 | 2 | 0.15785 | 0.13086 | -0.00762 | 0.00762 | 0.00343 | 0.00719 | 0.01002 | 0.00732 |
| | | 2 | 4 | 0.11615 | 0.14922 | -0.00123 | 0.00580 | -0.00863 | 0.00826 | 0.00412 | 0.00845 |
| | | 0.5 | 5 | 0.10410 | 0.14556 | 0.00433 | 0.00095 | 0.01231 | 0.00163 | 0.01315 | 0.00128 |
| | 10 | 5 | 1 | -0.12854 | 0.11433 | 0.06238 | 0.00673 | 0.07095 | 0.00798 | 0.08148 | 0.00989 |
| | | 3 | 2 | 0.12208 | 0.12936 | -0.00613 | 0.00641 | 0.00849 | 0.00665 | 0.00884 | 0.00678 |
| | | 2 | 4 | 0.12199 | 0.13811 | 0.00108 | 0.00476 | -0.00417 | 0.00712 | 0.00353 | 0.00730 |
| | | 0.5 | 5 | 0.08247 | 0.11382 | -0.00365 | 0.00079 | 0.00789 | 0.00109 | 0.00884 | 0.00104 |

**Table 2: Continued**

| $\alpha$ | n | $\theta_1$ | $\theta_2$ | MLE | | SE | | LL | | EL | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | Bias | MSE | Bias | MSE | Bias | MSE | Bias | MSE |
| -0.25 | 6 | 5 | 1 | 0.12725 | 0.15281 | 0.01976 | 0.00428 | 0.02879 | 0.00486 | 0.04018 | 0.00604 |
| | | 3 | 2 | -0.14357 | 0.12200 | 0.02778 | 0.00977 | -0.01681 | 0.00880 | -0.01049 | 0.00871 |
| | | 2 | 4 | 0.12894 | 0.14564 | -0.00505 | 0.00807 | 0.01759 | 0.00801 | 0.02039 | 0.00841 |
| | | 0.5 | 5 | 0.12438 | 0.15906 | 0.00797 | 0.00146 | 0.01257 | 0.00193 | 0.01381 | 0.00204 |
| | 8 | 5 | 1 | 0.11938 | 0.12006 | 0.01673 | 0.00374 | 0.02615 | 0.00417 | 0.03892 | 0.00518 |
| | | 3 | 2 | -0.14834 | 0.10475 | -0.01844 | 0.00855 | 0.01445 | 0.00796 | 0.01226 | 0.00819 |
| | | 2 | 4 | -0.11049 | 0.13814 | -0.00708 | 0.00782 | 0.01541 | 0.00785 | 0.02008 | 0.00818 |
| | | 0.5 | 5 | 0.11796 | 0.13364 | 0.00640 | 0.00118 | 0.01118 | 0.00149 | 0.01213 | 0.00155 |
| | 10 | 5 | 1 | 0.10124 | 0.10853 | 0.01262 | 0.00255 | 0.02425 | 0.00304 | 0.03439 | 0.00396 |
| | | 3 | 2 | 0.11841 | 0.11926 | -0.01207 | 0.00800 | -0.01166 | 0.00735 | -0.00628 | 0.00730 |
| | | 2 | 4 | 0.09921 | 0.12727 | 0.00337 | 0.00563 | 0.01496 | 0.00534 | 0.00435 | 0.00546 |
| | | 0.5 | 5 | -0.09222 | 0.11548 | -0.00393 | 0.00071 | 0.00190 | 0.00077 | 0.00565 | 0.00079 |
| 0.25 | 6 | 5 | 1 | 0.16839 | 0.12261 | -0.07787 | 0.00996 | -0.08808 | 0.01129 | -0.09631 | 0.01265 |
| | | 3 | 2 | -0.13160 | 0.11563 | -0.02081 | 0.01138 | -0.02764 | 0.01163 | -0.03948 | 0.01314 |
| | | 2 | 4 | 0.12543 | 0.10017 | 0.01213 | 0.00565 | 0.01863 | 0.00848 | 0.01393 | 0.00552 |
| | | 0.5 | 5 | 0.12936 | 0.10947 | 0.01771 | 0.00260 | 0.01652 | 0.00477 | 0.00358 | 0.00189 |
| | 8 | 5 | 1 | -0.15803 | 0.11962 | -0.07764 | 0.00917 | -0.08717 | 0.01043 | -0.09485 | 0.01169 |
| | | 3 | 2 | -0.12533 | 0.10829 | -0.01730 | 0.00680 | -0.02298 | 0.00697 | -0.03243 | 0.00783 |
| | | 2 | 4 | 0.11079 | 0.09383 | 0.01208 | 0.00795 | 0.00802 | 0.00771 | 0.00669 | 0.00786 |
| | | 0.5 | 5 | 0.12187 | 0.09976 | 0.01282 | 0.00180 | 0.01196 | 0.00145 | 0.00282 | 0.00114 |
| | 10 | 5 | 1 | 0.13589 | 0.11125 | -0.06289 | 0.00709 | -0.07466 | 0.00835 | -0.08383 | 0.00965 |
| | | 3 | 2 | 0.11101 | 0.10076 | -0.01277 | 0.00537 | -0.01760 | 0.00547 | 0.02536 | 0.00601 |
| | | 2 | 4 | -0.10887 | 0.08118 | 0.00638 | 0.00149 | 0.00141 | 0.00470 | -0.01720 | 0.00509 |
| | | 0.5 | 5 | -0.11165 | 0.08092 | 0.01127 | 0.00091 | 0.01059 | 0.00088 | 0.00209 | 0.00069 |

**Table 2: Continued**

| $\alpha$ | n | $\theta_1$ | $\theta_2$ | MLE | | SE | | LL | | EL | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | Bias | MSE | Bias | MSE | Bias | MSE | Bias | MSE |
| 0.5 | 6 | 5 | 1 | 0.15162 | 0.11095 | 0.11730 | 0.01587 | -0.12781 | 0.01819 | -0.13604 | 0.02023 |
| | | 3 | 2 | -0.13110 | 0.09081 | -0.04190 | 0.00861 | -0.04643 | 0.00895 | 0.05423 | 0.00998 |
| | | 2 | 4 | 0.11021 | 0.09824 | 0.02081 | 0.00658 | 0.01599 | 0.00629 | -0.08451 | 0.00617 |
| | | 0.5 | 5 | 0.15137 | 0.10037 | 0.01325 | 0.00631 | 0.02208 | 0.00219 | 0.01924 | 0.00158 |
| | 8 | 5 | 1 | -0.13262 | 0.09341 | -0.11504 | 0.01456 | 0.12539 | 0.01776 | -0.12349 | 0.01969 |
| | | 3 | 2 | 0.12683 | 0.08046 | -0.02679 | 0.00650 | -0.03326 | 0.00686 | 0.04428 | 0.00810 |
| | | 2 | 4 | -0.10137 | 0.07384 | 0.01085 | 0.00570 | 0.00701 | 0.00552 | -0.07148 | 0.00564 |
| | | 0.5 | 5 | 0.13209 | 0.08589 | 0.00947 | 0.00538 | 0.00871 | 0.00133 | 0.00729 | 0.00110 |
| | 10 | 5 | 1 | 0.12558 | 0.08917 | -0.11237 | 0.01364 | -0.12415 | 0.01608 | 0.11328 | 0.02029 |
| | | 3 | 2 | -0.11186 | 0.06539 | -0.01501 | 0.00628 | -0.02041 | 0.00641 | -0.02901 | 0.00709 |
| | | 2 | 4 | -0.10113 | 0.06314 | -0.00301 | 0.00439 | -0.00617 | 0.00433 | -0.01820 | 0.00465 |
| | | 0.5 | 5 | 0.11845 | 0.07321 | 0.00211 | 0.00089 | 0.00453 | 0.00086 | 0.00315 | 0.00076 |
| 0.75 | 6 | 5 | 1 | 0.17269 | 0.10671 | -0.18041 | 0.03485 | -0.19013 | 0.03819 | -0.19789 | 0.05107 |
| | | 3 | 2 | 0.14495 | 0.11425 | -0.02737 | 0.00675 | -0.05236 | 0.00700 | 0.04044 | 0.00786 |
| | | 2 | 4 | 0.13727 | 0.10991 | 0.02759 | 0.00617 | 0.03377 | 0.00680 | 0.03682 | 0.00923 |
| | | 0.5 | 5 | 0.13290 | 0.11231 | 0.01795 | 0.00222 | 0.01814 | 0.00214 | 0.02786 | 0.00728 |
| | 8 | 5 | 1 | 0.12592 | 0.09042 | -0.15980 | 0.02955 | 0.16805 | 0.02916 | 0.18938 | 0.04232 |
| | | 3 | 2 | -0.12520 | 0.07051 | -0.02599 | 0.00568 | -0.04199 | 0.00612 | 0.05215 | 0.00734 |
| | | 2 | 4 | 0.12947 | 0.09929 | 0.01740 | 0.00562 | 0.02961 | 0.00582 | -0.02351 | 0.00736 |
| | | 0.5 | 5 | 0.11302 | 0.08131 | 0.01433 | 0.00182 | 0.01335 | 0.00174 | 0.01885 | 0.00234 |
| | 10 | 5 | 1 | -0.12679 | 0.07175 | -0.11297 | 0.02318 | -0.13501 | 0.02640 | -0.17271 | 0.03915 |
| | | 3 | 2 | -0.11129 | 0.06732 | -0.01954 | 0.00459 | -0.03385 | 0.00583 | -0.04074 | 0.00651 |
| | | 2 | 4 | 0.12162 | 0.06734 | 0.00728 | 0.00419 | 0.00826 | 0.00508 | 0.00897 | 0.00518 |
| | | 0.5 | 5 | 0.10832 | 0.06674 | 0.00415 | 0.00060 | 0.00363 | 0.00088 | 0.00858 | 0.00073 |

## 5. Illustration Using Simulated Data

In this section, we illustrate the estimation procedures developed in the previous sections using a simulated data. For that we have generated 10 upper record values and its concomitants from MTBED with parameters $\theta_1 = 2$, $\theta_2 = 1$ and $\alpha = 0.5$. The generated record values and its concomitants are given below.

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $r_{(i)}$ | 0.201 | 0.383 | 0.868 | 1.433 | 1.589 | 1.7034 | 2.258 | 3.123 | 3.657 | 4.166 |
| $r_{[i]}$ | 0.245 | 0.066 | 0.563 | 3.379 | 0.685 | 0.411 | 1.111 | 3.526 | 2.721 | 5.317 |

Based on the simulated data we have obtained the MLE of $R = P(X < Y)$ and also the bootstrap CL of $R$ based on the MLE. For the Bayesian estimation we took hyperparameters as $a = 2$, $b = 1$, $c = 2$ and $d = 2$. The HPD interval of $R$ under SEL is also obtained. The estimated values are given below.

| MLE (Bootstrap CI) | Bayes estimates | | |
|---|---|---|---|
| | SEL (HPD Interval) | LL | EL |
| 0.6375 (0.4124,0.7124) | 0.6587 (0.4841, 0.7124) | 0.6457 | 0.6387 |

## 6. Conclusion

In this work, we considered the problem of estimation of $R = P(X < Y)$ for Morgenstern type bivariate exponential distribution using record values and its concomitants. The maximum likelihood and Bayesian estimators were obtained for $R$. For obtaining the Bayes estimates, importance sampling method was applied. Based on the simulation study we concluded that among different estimators, Bayes estimators under squared error loss function perform better in terms of bias and MSE. AILs of HPD intervals are smaller and the associated CPs are higher than that of bootstrap confidence intervals.

## Acknowledgements

The authors would like to thank the editor and the referees for the constructive comments.

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