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On Three-Level A-Optimal Designs for Test-Control Discrete Choice Experiments

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Abstract

Choice experiments are conducted when it is important to study the importance of different factors based on the perceived utility of choice options. We study the optimality of discrete choice experiments under a newly introduced inference problem of test-control discrete choice experiments; it is akin to the test-control inference problem in factorial experiments. For each factor, we have one control level and this control level is then compared with all the test levels of the same factor. For three-level choice designs with multiple factors, we first obtain a lower bound to the A-values for estimating the two test-control contrasts for each factor. We then provide some A-optimal designs for a small number of factors obtained through a complete search. For practical use with a somewhat large number of factors, we then provide some highly efficient designs.

Key words: Choice set; Test-control contrast effects; Hadamard matrix; Multinomial logit model; Linear paired comparison model.

AMS Subject Classifications: 62K05, 05B05

1. Introduction

Discrete choice experiments are used for quantifying the influence of the attributes which characterize the choice options. They are useful in many applied sciences, for example, psychology, marketing research, etc., where options (or, products) have to be judged with respect to a subjective criterion like preference or taste. For a latest application, see Ong *et al.* (2020). In choice experiments, respondents are shown a collection of choice sets and each of these choice sets consists of several options. Respondents are then asked to select one preferred option from each of the choice sets. We consider choice experiments with N choice sets each having two options (referred to as choice pairs hereafter); so, N choice pairs are shown to respondents and they are asked to pick one of the two options that they prefer from each of the N pairs. Each option is described by the same k factors, with each factor having two or more levels. We consider each factor at three levels. A choice design d then is a collection of these N choice pairs. Excellent reviews of the choice designs are provided in Street and Burgess (2012) and Großmann and Schwabe (2015), and a recent paper (Das and

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Singh, 2020) provides a unified theory on optimal choice experiments connecting different approaches to choice experiments.

Discrete choice experiments (DCEs) have usually been studied under the multinomial logit model (El-Helbawy and Bradley, 1978; Street and Burgess, 2007). Under the multinomial logit model, D-optimal designs have been studied for several situations, and orthogonal contrasts of main effects and two-factor interactions for the k factors are usually of interest. The multinomial logit model is non-linear, hence, the information matrix is a function of unknown parameters. The locally optimal designs are therefore obtained, and under the indifference assumption (that all treatment combinations have equal utility) of choice experiments, these locally optimal designs have been just termed as optimal designs. DCEs with only two options in each choice set can be equivalently studied under the traditional linear paired comparison model (McFadden, 1974; Huber and Zwerina, 1996; Großmann and Schwabe, 2015). The relationship between the two approaches (MNL models and linear paired comparison models) for studying DCEs has been studied in Das and Singh (2020). They also obtained the information matrices under different inference problems including briefly introducing the test-control inference problem in DCEs. So far, the inference problems that have been studied in choice experiments focus on comparing all levels of each factor with equal importance (Street and Burgess, 2007; Großmann and Schwabe, 2015; Chai *et al.*, 2017). We focus on the test-control inference problem for paired choice DCEs with each factor at three levels. The same setup with a traditional inference problem (of equal focus on all pairwise comparisons) was studied in Chai *et al.* (2017). The difference between the current paper and Chai *et al.* (2017) lies only in the studied inference problem, which ultimately leads to obtaining different optimal designs. We defer most of the technical details until the next section.

The primary goal in a test-control inference problem is to compare the test levels to a (pre-specified) control level. Here, we are not interested in making all pairwise comparisons, we are only interested in a subset of those comparisons. To the best of our knowledge, no one has worked on finding optimal choice designs when the interest might lie in making test-control comparisons. We are also not aware of any practical choice experiment which was conducted with this intention, however, it is not too hard to imagine that such an inference problem will find its use with practitioners. This is useful when manufacturers/service providers or policymakers want to study the effect of new test levels as against the existing control levels. Test-control inference problem has been studied by several authors; see, for example, Hedayat *et al.* (1988) and Majumdar (1996) for block designs, and Gupta (1995) and Gupta (1998) for multiple factors.

D-optimality is invariant to reparameterizations, and thus, D-optimal designs remain optimal even when the inference problem is changed (Großmann and Schwabe, 2015). On the other hand, A-optimal designs change with the inference problem which is one of the reasons behind us studying the A-optimal designs under the inference problem of test-control experiments. For linear models, it has been shown that when the inference problem is test-control, one often benefits by using the A-optimal designs specially designed for catering to this problem (see Banerjee and Mukerjee, 2008, for example). By studying optimal designs for the test-control inference problem for DCEs, we intend to do the same for DCEs (results in Table 2 and the final paragraph). A-optimal designs are the designs that minimize the sum of variances of the treatment contrasts of interest. For example, if the information matrix for treatment contrasts of interest is M_d, then the design d^* which minimizes $trace(M_d^{-1})$ among all designs is called an A-optimal design. We provide constructions of A-optimal and A-efficient designs for estimating the test-control contrasts under the indifference assumption of the multinomial logit model. We also provide designs having high A-efficiencies.

2. Background

We only present here the details relevant to the current problem, and for more details, Das and Singh (2020) is suggested to be consulted. With each of the k factors at three levels, there are a total of 3^k options. Let the systematic component of the utility for options be denoted by a 3^k-tuple vector τ. Without loss of generality, let the options be arranged lexicographically. For example, for $k = 2$, the systematic component of the utility vector is $\tau = (\tau_{00}, \tau_{01}, \tau_{02}, \tau_{10}, \tau_{11}, \tau_{12}, \tau_{20}, \tau_{21}, \tau_{22})$. The coding that we use is more commonly known as effects coding, see Großmann and Schwabe (2015), for example. In effects coding, for one factor at three levels, level 0 is coded as (1 0), level 1 is coded as (0 1) and level 2 is coded as $(-1\ -1)$; here, level 2 is the control level, and levels 0 and 1 are test levels.

The nth choice pair is denoted by $T_n = (t_{(n1)}, t_{(n2)})$, with $t_{(nj)}$ is the jth option in the nth choice pair, $n = 1, \ldots, N$, $j = 1, 2$. Corresponding to the jth option in N choice sets, $A_j = (t^T_{(1j)}\ t^T_{(2j)} \cdots\ t^T_{(Nj)})^T$ is a $N \times k$ matrix representing the levels of the k attributes. Let a $N \times 2k$ matrix X_j denote the effects coded matrix corresponding to A_j implying that 0, 1 and 2 in A_j is replaced by the vectors $(1, 0)$, $(0, 1)$ and $(-1, -1)$, respectively, in X_j. Then, the effects coded difference matrix for the first and second option is

$$X = X_1 - X_2. \tag{1}$$

Let B be a $2k \times 3^k$ matrix such that the ith column of B corresponds to the effects coding for the ith option, $i = 1, \ldots, 3^k$. It is assumed that the 3^k options are arranged lexicographically. For example, if $k = 3$, the 3rd column in B would correspond to the effects coding corresponding to the option (002) which is $(1\ 0\ 1\ 0\ -1\ -1)$, or that, the 7th column would correspond to option (020), that is, $(1\ 0\ -1\ -1\ 1\ 0)$. The matrix B has been called B_E in Das and Singh (2020). For simplicity, we drop the subscript E in the current work. This should not be confused with B used in Street and Burgess (2007), since, the matrix B has traditionally corresponded to the orthonormal coding.

The inference problem studied in the current paper is $B\tau$ which corresponds to the situations where the primary interest lies in making test-control comparisons which means that some new levels (called test levels) of factors are compared with an existing control level for the same factor. From Das and Singh (2020), the average information matrix for the inference problem $B\tau$ is $\mathcal{I}(B\tau) = \frac{1}{4N} M_d$ where

$$M_d = (BB^T)^{-1} X^T X (BB^T)^{-1}. \tag{2}$$

Note that the word average here comes from using N in the definition of the information matrix implying that the information considered here is per choice pair. Given the structure of B, it is easy to see that $(BB^T)^{-1} = (\frac{1}{3^k}) diag(V_1^{-1}, \ldots, V_k^{-1})$ where $V_i^{-1} = (3I_2 - J_2)$ for all $i = 1, \ldots, k$. For three-level factors, a choice design d is connected if all the test-control contrasts are estimable, and this happens if and only if M_d has rank $2k$. In what follows, the class of all connected paired choice designs with k three-level factors and N choice pairs is denoted by $\mathcal{D}_{k,N}$. As stated before, we use the standard A-optimality criteria. The A-value of a design d in $\mathcal{D}_{k,N}$ is $4Ntrace(M_d^{-1})$. A design that minimizes the A-value among all designs in $\mathcal{D}_{k,N}$ is said to be A-optimal.

3. Lower Bounds to the A-value

To find the lower bound to the A-value under the inference problem $B\tau$, we adopt the same strategy as in Chai *et al.* (2017). We first find the lower bound of the A-value for designs with only one factor, and then use the same to find a naïve bound to the A-value when k factors are taken into consideration. Let the matrix X in (1) be partitioned as $(X_{(1)}|X_{(2)}|\cdots|X_{(k)})$, where $X_{(p)}$ is a $N\times 2$ matrix corresponding to the pth factor. Notice that rows in $X_{(p)}$ determine the corresponding options in a design for the pth factor. In $X_{(p)}$, rows $(+2,\ +1)$, $(-2,\ -1)$, $(+1,\ +2)$, $(-1,\ -2)$, $(+1,\ -1)$ and $(-1,\ +1)$ correspond to choice pairs $(0,\ 2)$, $(2,\ 0)$, $(1,\ 2)$, $(2,\ 1)$, $(0,\ 1)$ and $(1,\ 0)$ respectively. Similarly, row $(0,\ 0)$ in $X_{(p)}$ could correspond to any of the three choice pairs $(0,\ 0)$, $(1,\ 1)$ or $(2,\ 2)$.

Similarly, the matrix $M_d = (M_{dpq})$ for a design d can also be partitioned into 2×2 blocks such that the block corresponding to the pth and qth factor is $M_{dpq} = \frac{1}{3^{2k}}(3I_2 - J_2)X_{(p)}^T X_{(q)}(3I_2 - J_2)$; $p = 1,\ldots,k; q = 1,\ldots,k$. It can be shown that we always benefit by not considering the pairs corresponding to the type $(0,0)$ in $X_{(p)}$ (for an explanation, see Chai *et al.* (2017)). Also, note that $(0,0)$ in $X_{(p)}$ implies that the corresponding value for a factor in both the options are same. Let y be the number of rows of $X_{(p)}$ that are equal to either $(2,\ 1)$ or $(-2,-1)$ and z be the number of rows of $X_{(p)}$ that are equal to either $(1,\ 2)$ or $(-1,-2)$. Then the remaining $N-y-z$ $(=x$, say) rows of $X_{(p)}$ are necessarily equal to either $(1,\ -1)$ or $(-1,\ 1)$. It can then be shown that for the pth factor,

$$M_{dpp} = \frac{1}{3^{2(k-1)}}\begin{bmatrix} N-z & y+z-N \\ y+z-N & N-y \end{bmatrix} = \frac{1}{3^{2(k-1)}}C_{dpp}, \tag{3}$$

where

$$C_{dpp} = \begin{bmatrix} N-z & y+z-N \\ y+z-N & N-y \end{bmatrix}.$$

We need to obtain a lower bound to $trace(M_{dpp}^{-1}) = 3^{2(k-1)}trace(C_{dpp}^{-1})$, which is equivalent to obtaining a lower bound to $trace(C_{dpp}^{-1})$. The

$$trace(C_{dpp}^{-1}) = (2N-y-z)/h_N(y,z) = g_N(y,z),$$

where

$$h_N(y,z) = det(C_{dpp}) = yz + N(y+z) - (y+z)^2.$$

Note that both $h_N(y,z)$ and $g_N(y,z)$ are symmetric in y and z. We now find the values y and z for which $g_N(y,z)$ is minimized for $1\le y+z\le N$, $y\ne N$, $z\ne N$. These conditions are required for every p so that the design d is connected, that is, $rank(M_d) = 2k$. We need these conditions because our eventual goal is to find a lower bound to the A-value for a design with k factors. Let $\lfloor x\rfloor$ denote the greatest integer contained in x. Let $L_a = min_{d\in\mathcal{D}_{1,N}} trace(C_{dpp}^{-1}) = min_{1\le y+z\le N, y\ne N, z\ne N}\, g_N(y,z) = g_N(a^*,b^*)$.

Lemma 1: For the pth $(p = 1, \ldots, k)$ factor in design $d \in \mathcal{D}_{1,N}$ with $N > 3$,

$$trace(C_{dpp}^{-1}) = g_N(y, z) \geq g_N(a^*, b^*)$$

where $g_N(a^*, b^*) = min\{g_N(a_1, b_1), g_N(a_2, b_2), g_N(a_3, b_3)\}$ with

(i) $a_1 = b_1 = t$,

(ii) $a_2 = b_2 = t + 1$,

(iii) $a_3 = t, b_3 = t + 1$

and $t = \lfloor N(3 - \sqrt{3})/3 \rfloor$.

Proof: The proof follows on similar lines as the proof of corresponding lemma in Chai *et al.* (2017). Treating y and z as continuous variables and adopting the usual derivative approach to minimize $g_N(y, z)$, we get $\partial g_N(y, z)/\partial y = (2N(y + z - N) - yz + y(2N - y - z))/h_N^2(y, z)$. Similarly, $\partial g_N(y, z)/\partial z = (2N(y + z - N) - yz + z(2N - y - z))/h_N^2(y, z)$.

Now, $\partial g_N(y, z)/\partial y = \partial g_N(y, z)/\partial z = 0$ implies that $(y - z)(2N - y - z) = 0$. In other words, y=z, since $2N - y - z > 0$.

Now, for $y = z$, it follows that $\partial g_N(y, z)/\partial y = 0$ implies that $3y^2 - 6Ny + 2N^2$=0 or $y = N(3 \pm \sqrt{3})/3$. However, since $y < N$, the only feasible solution of y is $N(3 - \sqrt{3})/3 = t_1$.

Similarly, checking the matrix of second derivatives, we see that the minimum of $g_N(y, z)$ is attained at $y = z = t_1$. Since t_1 is non-integer, $g_N(y, z) = L_a$ at one of the integer points nearest to (t_1, t_1). □

Using Lemma 1, we have computed the values of a^* and b^* for $4 \leq N \leq 64$ and summarize it in Table 1. Also, note that since $g_N(y, z)$ is symmetric in y and z, interchanging the values of a^* and b^* yield the same values of $g_N(a^*, b^*)$ and therefore from Table 1, we could either say that $y = a^*$ and $z = b^*$ or that $z = a^*$ and $y = b^*$. The optimal value for x can then be computed as $N - y - z$. Recall that we consider choice designs with y number of rows of $X_{(p)}$ equal to either $(2,\ 1)$ or $(-2, -1)$, z number of rows equal to either $(1,\ 2)$ or $(-1, -2)$, and the remaining x rows equal to either $(1,\ -1)$ or $(-1,\ 1)$.

Results in Table 1 are not surprising since we know that for block designs or for factorial experiments, when test-control inference problem is of interest then unequal replication of levels, often with control treatment being repeated more number of times than other treatments, is common. Let $L_A = min_{d \in \mathcal{D}_{k,N}} trace(M_d^{-1})$. We now give a lower bound of L_A for paired choice designs with k factors in N choice pairs.

Theorem 1: For a paired choice design $d \in \mathcal{D}_{k,N}$, $trace(M_d^{-1}) \geq L_A \geq k3^{2(k-1)}L_a = k3^{2(k-1)}g_N(a^*, b^*)$ where a^* and b^* are as in Lemma 1.

Proof: Similar to the proof of Theorem 2.1 in Chai *et al.* (2017), first we apply the inequality $trace(M_d^{-1}) \geq \sum_{p=1}^{k} trace(M_{dpp}^{-1})$ which, using Schur complement and the inverse of partitioned matrices, follows easily for $k = 2$. For example, for $k = 2$, let the 2×2 partitioned

Table 1: Values of a^* and b^* for $4 \leq N \leq 64$ for Lemma 1

N	a^*	b^*	N	a^*	b^*	N	a^*	b^*	N	a^*	b^*	N	a^*	b^*	N	a^*	b^*
4	2	2	14	6	6	24	10	10	34	14	15	44	18	19	54	23	23
5	2	2	15	6	6	25	10	11	35	15	15	45	19	19	55	23	23
6	2	3	16	7	7	26	11	11	36	15	15	46	19	20	56	24	24
7	3	3	17	7	7	27	11	12	37	15	16	47	20	20	57	24	24
8	3	4	18	7	8	28	12	12	38	16	16	48	20	20	58	24	25
9	4	4	19	8	8	29	12	12	39	16	17	49	21	21	59	25	25
10	4	4	20	8	9	30	13	13	40	17	17	50	21	21	60	25	26
11	4	5	21	9	9	31	13	13	41	17	17	51	21	22	61	26	26
12	5	5	22	9	9	32	13	14	42	18	18	52	22	22	62	26	26
13	5	6	23	10	10	33	14	14	43	18	18	53	22	23	63	26	27
															64	27	27

matrix M be $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$. Then

$$M^{-1} = \begin{bmatrix} (M_{11} - M_{12}M_{22}^{-1}M_{21})^{-1} & -M_{11}^{-1}M_{12}(M_{22} - M_{21}M_{11}^{-1}M_{12})^{-1} \\ -M_{22}^{-1}M_{21}(M_{11} - M_{12}M_{22}^{-1}M_{21})^{-1} & (M_{22} - M_{21}M_{11}^{-1}M_{12})^{-1} \end{bmatrix}.$$

Since $M_{12}M_{22}^{-1}M_{21}$ is non-negative definite, $(M_{11} - M_{12}M_{22}^{-1}M_{21}) \leq M_{11}$ and therefore $(M_{11} - M_{12}M_{22}^{-1}M_{21})^{-1} \geq M_{11}^{-1}$. Similarly, $(M_{22} - M_{21}M_{11}^{-1}M_{12})^{-1} \geq M_{22}^{-1}$. Therefore, $trace(M^{-1}) = trace(M_{11} - M_{12}M_{22}^{-1}M_{21})^{-1} + trace(M_{22} - M_{21}M_{11}^{-1}M_{12})^{-1} \geq trace(M_{11}^{-1}) + trace(M_{22}^{-1})$. Now, using the method of induction, one can see that the inequality holds for a general k, that is, the $trace(M^{-1}) \geq \sum_{p=1}^{t} trace(M_{pp}^{-1})$. Finally, using Lemma 1, the proof follows. □

In the next section, we provide some A-optimal designs attaining the lower bounds of Theorem 1. In some situations, since we are not able to provide designs attaining the A-lower bounds, A-efficiencies are given.

4. Design Constructions

A design $d \in \mathcal{D}_{k,N}$ would be A-optimal under the test-control inference problem if $trace(M_d^{-1})$ attains the bound obtained in Theorem 1. To attain this bound, the design should not only have the values of a^* and b^*, for each factor, as in Table 1 (or, Lemma 1) but should also satisfy the orthogonality property, that is, the blocks M_{dpq} for $p \neq q = 1, \ldots, k$ should be block matrices with all values equal to 0s. The closer these block matrices are to zero matrices, the higher is the efficiency expected to be. This is a somewhat hard combinatorial problem, less studied, and it is more difficult to deal with the problem as compared to finding designs for other inference problems. For finding optimal designs, an algorithm such as the one recently studied in case of factorial experiments (Chai and Das, 2020) would be more helpful. The A-efficiency of a design $d \in \mathcal{D}(k, N)$ is given by

$$\phi_A = \frac{min_{d_0 \in \mathcal{D}(k,N)} trace(M_{d_0}^{-1})}{trace(M_d^{-1})}.$$

It then follows from Theorem 1 that for $d \in \mathcal{D}_{k,N}$

$$\phi_A \geq \frac{k3^{2(k-1)} g_N(a^*, b^*)}{trace(M_d^{-1})}. \tag{4}$$

As an example, consider $N = 9$ and $k = 2$. Here, the optimal values for a^* and b^* are both equal to 4, and, from the result of complete search, we see that no design with $a^* = b^* = 4$ for $N = 9$ achieves the bound in Theorem 1. One of the designs d_9 with the smallest A-value (= 282.8769) among the designs having $a^* = b^* = 4$ is provided below. In fact, it is surprising to note that designs for $a^* = 3$ and $b^* = 4$ (or vice versa) while satisfying the orthogonality condition have A-value (= 274.1538) which is smaller than d_9. This design is provided below as d_9^+. The bound from Theorem 1 (= $k3^{2(k-1)} g_N(a^*, b^*)$) is 7.5, and, therefore, the bound for A-value is then $4N(7.5) = 270$. Thus, the design d_9 has an A-efficiency of at least 0.9545 whereas d_9^+ has an A-efficiency of at least 0.9848.

Similarly for $N = 7, k = 2$, no design with $a^* = b^* = 3$ achieves the bound in Theorem 1. Design d_7 with the smallest A-value (= 279.7321) among the designs having $a^* = b^* = 3$ is provided below. The designs having either $a^* = 2$, $b^* = 3$ or vice versa for one factor and $a^* = b^* = 3$ for another factor and additionally satisfying the orthogonality condition have A-value (276.1500) which is smaller than d_7. This design is also given below as d_7^+. The bound from Theorem 1 is 9.6, and, therefore the bound for A-value is then $4N(9.6) = 268.8$. Thus, the design d_7 has an A-efficiency of at least 0.9609 whereas d_7^+ has an A-efficiency of at least 0.9734.

$$d_9 = \begin{pmatrix} 00, & 22 \\ 01, & 22 \\ 02, & 11 \\ 02, & 20 \\ 02, & 21 \\ 10, & 22 \\ 11, & 20 \\ 12, & 20 \\ 12, & 21 \end{pmatrix}, \; d_9^+ = \begin{pmatrix} 00, & 22 \\ 01, & 10 \\ 01, & 22 \\ 02, & 11 \\ 02, & 21 \\ 10, & 22 \\ 11, & 20 \\ 12, & 20 \\ 12, & 21 \end{pmatrix} \text{ and } d_7 = \begin{pmatrix} 00, & 22 \\ 01, & 22 \\ 02, & 11 \\ 02, & 20 \\ 10, & 22 \\ 11, & 20 \\ 12, & 21 \end{pmatrix}, \; d_7^+ = \begin{pmatrix} 00, & 22 \\ 01, & 22 \\ 02, & 11 \\ 02, & 20 \\ 10, & 21 \\ 11, & 20 \\ 12, & 21 \end{pmatrix}$$

The designs obtained for $N = 7$ and $N = 9$ suggest that the lower bound in Theorem 1 is not tight. In fact, it suggests that orthogonality is somewhat more important than the designs satisfying the property in Lemma 1 for every factor. For $k = 2$ and $N = 4, 5, 6$ and 8, A-optimal designs have been obtained using complete search and reported below as d_4, d_5, d_6, and d_8, respectively. These designs satisfy the orthogonality property and satisfy the values of a^* and b^* in Lemma 1 thereby attaining the optimal bound in Theorem 1. In fact, the complete search result also shows that d_7^+ and d_9^+ have the smallest A-value and are, therefore, A-optimal. Note that there exists more than one design with the same A-values and only one of them is reported here. Besides, these complete searches are carried out within the class of designs having distinct choice pairs since that is more desirable in practice (Chai *et al.*, 2017).

$$d_4 = \begin{pmatrix} 01, & 22 \\ 02, & 21 \\ 10, & 22 \\ 12, & 20 \end{pmatrix}, d_5 = \begin{pmatrix} 00, & 22 \\ 01, & 10 \\ 02, & 21 \\ 11, & 22 \\ 12, & 20 \end{pmatrix}, d_6 = \begin{pmatrix} 00, & 12 \\ 01, & 20 \\ 02, & 21 \\ 10, & 22 \\ 11, & 22 \\ 12, & 21 \end{pmatrix} \text{ and } d_8 = \begin{pmatrix} 00, & 22 \\ 01, & 22 \\ 02, & 11 \\ 02, & 20 \\ 10, & 21 \\ 11, & 22 \\ 12, & 20 \\ 12, & 21 \end{pmatrix}.$$

A Hadamard matrix H_m is a $m \times m$ matrix with elements ± 1 such that $H_m^T H_m = H_m H_m^T = mI_m$. Using the construction in Chai *et al.* (2017) and designs reported above, called *base designs*, we now find designs with larger numbers of factors k and $N \geq 8$. Consider a base design $d_0 \in \mathcal{D}_{k_0,N_0}$ with the trace $(M_{d_0}^{-1}) = k_0 3^{2(k_0-1)} g_{N_0}(a_0, b_0)$. Using d_0, a paired choice design d_H with parameters $k = mk_0$, $N = mN_0$ is constructed with the corresponding design matrix $X_H = H_m \otimes X$, where X is the design matrix of d_0. This method of construction obtains a final design by performing Kronecker product of the small design with a Hadamard matrix. To find the A-value of design d_H, we first note that

$$X_H^T X_H = H_m^T H_m \otimes X^T X = mI_m \otimes X^T X.$$

Then, from (2), we have

$$\begin{aligned} M_{d_H} &= \frac{1}{3^{2k}} diag(V_1^{-1}, \ldots, V_k^{-1})(mI_m \otimes X^T X) diag(V_1^{-1}, \ldots, V_k^{-1}) \\ &= \frac{1}{3^{2mk_0}} diag(V_1^{-1}, \ldots, V_k^{-1})(mI_m \otimes X^T X) diag(V_1^{-1}, \ldots, V_k^{-1}) \\ &= \frac{m}{3^{2k_0(m-1)}} I_m \otimes M_{d_0}. \end{aligned} \quad (5)$$

Therefore, $trace(M_{d_H}^{-1}) = 3^{2k_0(m-1)} trace(M_{d_0}^{-1})$, and the A-efficiency of d_H is given by

$$\phi_A \geq \frac{k 3^{2(k-1)} g_N(a^*, b^*)}{3^{2k_0(m-1)} trace(M_{d_0}^{-1})} = \frac{k 3^{2(k_0-1)} g_N(a^*, b^*)}{trace(M_{d_0}^{-1})} = \phi_A^*, \quad (6)$$

where d_0 is a base design in $\mathcal{D}_{k_0,N_0}$ and a^* and b^* are as in Theorem 1 for a design with N runs and k factors. In Table 2, for $N \geq 4$ and $k \geq 2$, we provide A-optimal and A-efficient designs with efficiency bounds as in (6), and the corresponding methods of constructions. One of the designs with $k = 4$ and $N = 10$ is A-optimal. We denote designs d_H obtained using the Hadamard matrix H_m and a base design d_{N_0} by $H_m \otimes d_{N_0}$. A design with a smaller k retains its optimality property for given N when factors are deleted from a design with larger k. As is expected, designs that are obtained using d_7^+ and d_9^+ are better (higher efficiency) than the designs obtained using d_7 and d_9. Note that the A-efficiencies of designs reported in Table 2 could actually be higher than the reported lower bounds.

We have obtained optimality bounds for the test-control inference problem for DCEs. From Table 2, we see that the designs obtained for $k \geq 2$ are highly efficient. It is worth observing that corresponding designs obtained in Chai *et al.* (2017) are less efficient as

Table 2: A-optimal/efficient designs with distinct choice pairs ($N \geq 4$ and $k \geq 2$)

k	N	ϕ_A^*	Method	k	N	ϕ_A^*	Method	k	N	ϕ_A^*	Method
2	4	1	d_4	4	14	0.9609	$H_2 \otimes d_7$	8	36	0.9502	$H_4 \otimes d_9$
2	5	1	d_5	4	14	0.9734	$H_2 \otimes d_7^+$	8	36	0.9805	$H_4 \otimes d_9^+$
2	6	1	d_6	4	16	0.987	$H_2 \otimes d_8$	16	32	0.9338	$H_8 \otimes d_4$
2	7	0.9609	d_7	4	18	0.9526	$H_2 \otimes d_9$	16	40	0.9953	$H_8 \otimes d_5$
2	7	0.9734	d_7^+	4	18	0.9829	$H_2 \otimes d_9^+$	16	48	0.9778	$H_8 \otimes d_6$
2	8	1	d_8	8	16	0.9351	$H_4 \otimes d_4$	16	56	0.9609	$H_8 \otimes d_7$
2	9	0.9545	d_9	8	20	0.9973	$H_4 \otimes d_5$	16	56	0.9734	$H_8 \otimes d_7^+$
2	9	0.9848	d_9^+	8	24	0.9778	$H_4 \otimes d_6$	16	64	0.9849	$H_8 \otimes d_8$
4	8	0.9474	$H_2 \otimes d_4$	8	28	0.9609	$H_4 \otimes d_7$	16	72	0.9501	$H_8 \otimes d_9$
4	10	1	$H_2 \otimes d_5$	8	28	0.9734	$H_4 \otimes d_7^+$	16	72	0.9803	$H_8 \otimes d_9^+$
4	12	0.9778	$H_2 \otimes d_6$	8	32	0.9856	$H_4 \otimes d_8$				

compared to the designs reported in Table 2 under the current test-control inference problem. For example, the design for $k = 2, N = 4$ reported in Chai *et al.* (2017) (given below for convenience) is shown to be both A- and D-optimal under the traditional inference problem of pairwise comparisons, but it is only 83% efficient under the current test-control inference problem.

Design for $k = 4$ in Chai *et al.* (2017) is $\begin{pmatrix} 20, & 01 \\ 21, & 10 \\ 12, & 00 \\ 02, & 11 \end{pmatrix}$. Note that this design has with $a = b = 1$, whereas the optimal design for test-control inference problem should have $a^* = b^* = 2$ from Table 1.

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