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Bayesian Inference of Progressive Type - II Censored Data using Mixture of Log Logistic Distributions

S.S. Bhavsar and M.N. Patel

Department of Statistics, School of Sciences, Gujarat University, Ahmedabad, Gujarat, India

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ABSTRACT

Bayesian approach is considered by several authors in mixture models under Type – I, Type – II and other censoring schemes in the area of life testing and survival analysis. In this paper we consider the estimation of parameters of a mixture of log logistic distributions under classical and Bayesian setup. The estimation is done based on progressive Type – II censored sample and the squared error loss function, K- loss function and precautionary loss function are used as loss functions under Bayesian approach. A simulation study is conducted to examine the performance of the proposed estimators based on mean squared error. Bayes estimators under the three types of loss functions are compared using posterior risk too. The results are also compared based on Progressive Type – II censoring and Type – II censoring schemes. Additionally a real life data is considered to determine whether the estimators have similar behavior as seen in simulation study.

Key words: Maximum likelihood; Gamma prior; Log logistic distribution; Posterior risk; Importance sampling.

1. Introduction

The concept of censoring is generally used in life testing experiments and survival analysis since partial or complete destruction of the testing unit becomes quite expensive and time consuming. In literature there are many censoring schemes available, which are mainly based on Type – I or Type – II censoring schemes. When life test is terminated as soon as the pre-determined time is observed, such censoring scheme is known as Type – I censoring, where as in Type - II censoring the test is terminated as soon as pre-determined number of failures observed.

One of the significant shortcoming of conventional censoring schemes is that other than the terminal point of the experiment they do not allow removal of the experimental units at any other points. A censoring scheme in which some of the experimental units are withdrawn during the test and test is continued after the withdrawal is known as progressive censoring scheme. Based on Type – I & Type – II censoring schemes progressive censoring schemes can be formulated as progressive Type – I & progressive Type – II censoring schemes. For detailed study of progressive censoring scheme one may refer Balakrishnan and Aggarwala (2000). The progressive Type – II censoring scheme became very popular among the researchers. Some of the references are Wu *et al.* (2006), Patel and Patel (2007), Gajjar and Patel (2008), Saraçoğlu *et al.* (2010) and Ahmed (2014).

Corresponding author: S. S. Bhavsar
Email: snehbhavsar1@gmail.com

Various types of lifetime models are available in the literature like Exponential, Weibull, Rayleigh, Power function, *etc*. Some of the works available in literature under progressive censoring for above mentioned lifetime models are considered by Fernández (2004), Jung and Chung (2011), Kim and Han (2009), *etc*. In the recent years, the estimation under Bayesian setup for log logistic distribution for progressive censoring is studied by Abbas and Tang (2016), Al-Shomrani *et al.* (2016), Kumar (2018), Yahaya and Ibrahim (2019), *etc.*

In life testing experiments failure of unit may occur due to more than one causes for *e.g.* failure of an electrical component may occur due to fluctuations in voltage or its operating environment or a mechanical shock. Similarly death of a person may occur due to heart attack or old age or any other reason. In such situations a lifetime model based on multiple causes is considered which is known as mixture model. Suppose there are k causes of failure of an experimental unit and p_i be the probability of failure of a unit due to i^{th} cause such that $\sum_{i=1}^{k} p_i = 1$ and $f_i(x)$ be the probability density function or probability mass function of lifetime of the failure units due to i^{th} cause then the mixture model is defined as

$$f(x) = \sum_{i=1}^{k} p_i \, f_i(x) \qquad (1)$$

The mixture model is found useful in engineering, medical, agriculture, and many more fields.

Pearson (1894) introduced a statistical model based on finite mixtures of distributions to analyse crab morphometry data. Based on their causes of failures Mendenhall and Hader (1958) studied a population of failures by dividing them in two sub-populations. For estimating the parameters of a mixture of Rayleigh distribution Saleem and Aslam (2009) have used Bayesian procedure. Bayesian estimation was considered by Kazmi *et al.* (2012) for a mixture of Maxwell distribution under Type-I censoring scheme. Bayesian estimation of mixture of power function distributions using Type – II censored sample was discussed by Bhavsar and Patel (2019). Complete and Type – I censored sample are considered by Saleem *et al.* (2010) for estimation of parameters of the mixture of power function distributions. Very few works are available in the area of estimation of mixture model of the log logistic distribution under Bayesian setup based on progressive censoring. This has motivated us to consider a problem of estimation for mixture of log logistic distribution under progressive Type – II censoring scheme.

In this article, an estimation of the parameters of mixture of two log logistic distributions is carried out using the progressive Type – II censored sample considering the maximum likelihood estimation and Bayesian approach, and their respective mean squared errors and posterior risks are studied. The prior considered for the parameters β_1 and β_2 is gamma prior and uniform prior is considered for proportion parameter p of the mixture model. The squared error loss function, K – loss function and precautionary loss function are considered to obtain the Bayes estimates and a comparison between them based on MSE & posterior risk is done. A simulation study is carried out to obtain some interesting conclusions and a real life data is also considered. The rest of the article is structured as follows. In section 2, a two-component mixture model for log logistic distribution and likelihood function under progressive Type – II censoring is described. In section 3, the parameter estimation is carried out using the maximum likelihood estimation approach and the estimators are derived along with their asymptotic variances. Section 4 covers the estimation carried out under the Bayesian setup considering three different loss functions. MSE and posterior risks are obtained for the Bayes estimators. A simulation study is conducted to compare the performance of the proposed estimators in section 5. In section 6, some discussion on the numerical results are presented. Section 7 gives an idea about the behavior of the estimators for real life data under classical and Bayesian setup. The final conclusion is given in Section 8.

2. Mixture Model

The two - component mixture model for log logistic distribution is defined as follows

$$f(x) = pf_1(x) + (1-p)f_2(x) \tag{2}$$

where $f_i(x) = \frac{\beta_i x^{\beta_i - 1}}{(1+x^{\beta_i})^2},\ x > 0,\ \beta_i > 0, 0 < p < 1;$

is the probability density function of log logistic distribution and corresponding distribution function is

$$F_i(x) = \frac{x^{\beta_i}}{(1+x^{\beta_i})}; \qquad i = 1,2 \tag{3}$$

Here β_1, β_2 are unknown parameters of the log logistic distributions and p is unknown mixing proportion with mixing weight $p : 1-p$.

The life testing experiment under progressive censoring can be conducted as follows. Let us suppose that n experimental units are put on test and as soon as the m^{th} failure is observed the test is terminated. Considering a mixture model an experimental unit may fail due to cause 1 or cause 2. The failed unit can easily be identified whether it is from sub population 1 (which failed due to cause 1) or sub population 2 (which failed due to cause 2). Since this is progressive censoring scheme, as soon as the 1st failure occurs R_1 units are removed from the test which has remaining $(n-1)$ units on the test and the test is continued with $(n-1-R_1)$ units. Similarly on the $(m-1)^{th}$ failure R_m units are withdrawn from the remaining units on the test and test is continued with $(n-m-1-\sum_{i=1}^{m-1} R_i)$ units. The test is finally terminated as soon as the m^{th} failure is observed.

Thus depending upon the cause of failure, we can identify the number of failures m_1 due to cause 1 and m_2 due to cause 2 from the $m(=m_1+m_2)$ observed failures. The remaining $(n-m\ -\sum_{i=1}^{m-1} R_i)$ units are censored which provide no information about the sub population and survive beyond the time $X_{(m)}$, the observed time of the m^{th} failure.

To produce precise inferences a mixture model must be identifiable and in our model we have only shape parameters β_1 and β_2. Suppose x_{1i} and x_{2i} are the i^{th} failure time due to cause 1 and cause 2 respectively. The general form of likelihood function for the two – component mixture distribution under progressive Type – II censoring is given by:

$$L(\beta_1, \beta_2, p|x) \propto \prod_{i=1}^{m_1} p.f_1(x_{1i}) \prod_{i=1}^{m_2} (1-p)\, f_2\,(x_{2i})$$

$$\prod_{i=1}^{m} \left[1 - p{x_i}^{\beta_1} - (1-p){x_i}^{\beta_2}\right]^{R_i} \tag{4}$$

3. Maximum Likelihood Estimation (MLE)

Using Eq. (2) and (4), the likelihood function under progressive Type – II censoring for mixture model is obtained as,

$$L \propto p^{m_1} {\beta_1}^{m_1} \prod_{i=1}^{m_1} \frac{(x_{1i})^{\beta_1 - 1}}{\left(1+{x_{1i}}^{\beta_1}\right)^2} (1-p)^{m_2} {\beta_2}^{m_2} \prod_{i=1}^{m_2} \frac{(x_{2i})^{\beta_2 - 1}}{(1+{x_{2i}}^{\beta_2})^2} \times$$

$$\prod_{i=1}^{m}\left[1-p\frac{x_i^{\beta_1}}{1+x_i^{\beta_1}}-(1-p)\frac{x_i^{\beta_2}}{1+x_i^{\beta_2}}\right]^{R_i} \tag{5}$$

$$\log L \propto m_1 logp + m_1 log\beta_1 + (\beta_1-1)\sum_{i=1}^{m_1} logx_{1i} - 2\sum_{i=1}^{m_1}\log\left(1+x_{1i}^{\beta_1}\right) + m_2\log(1-p) + m_2 log\beta_2 + (\beta_2-1)\sum_{i=1}^{m_2} logx_{2i} - 2\sum_{i=1}^{m_1}\log\left(1+x_{2i}^{\beta_2}\right) + \sum_{i=1}^{m} R_i \log\left[1-p\frac{x_i^{\beta_1}}{1+x_i^{\beta_1}}-(1-p)\frac{x_i^{\beta_2}}{1+x_i^{\beta_2}}\right] \tag{6}$$

The first derivatives of equation Eq. (6) with respect to β_1, β_2 and p are

$$\frac{\partial logL}{\partial\beta_1} = \frac{m_1}{\beta_1} + \sum_{i=1}^{m_1} logx_{1i} - 2\sum_{i=1}^{m_1}\frac{x_{1i}^{\beta_1}\log x_{1i}}{1+x_{1i}^{\beta_1}} + \sum_{i=1}^{m} R_i \frac{1}{\left[1-p\frac{x_i^{\beta_1}}{1+x_i^{\beta_1}}-(1-p)\frac{x_i^{\beta_2}}{1+x_i^{\beta_2}}\right]}\left(-p\frac{x_i^{\beta_1}\, logx_i}{(1+x_i^{\beta_1})^2}\right) \tag{7}$$

$$\frac{\partial logL}{\partial\beta_2} = \frac{m_2}{\beta_2} + \sum_{i=1}^{m_2} logx_{2i} - 2\sum_{i=1}^{m_2}\frac{x_{2i}^{\beta_2}\log x_{2i}}{1+x_{2i}^{\beta_2}} + \sum_{i=1}^{m} R_i \frac{1}{\left[1-p\frac{x_i^{\beta_1}}{1+x_i^{\beta_1}}-(1-p)\frac{x_i^{\beta_2}}{1+x_i^{\beta_2}}\right]}\left(-(1-p)\frac{x_i^{\beta_2}\, logx_i}{(1+x_i^{\beta_2})^2}\right) \tag{8}$$

$$\frac{\partial logL}{\partial p} = \frac{m_1}{p} - \frac{m_2}{(1-p)} - \sum_{i=1}^{m} R_i\left(\frac{1}{\left[1-p\frac{x_i^{\beta_1}}{1+x_i^{\beta_1}}-(1-p)\frac{x_i^{\beta_2}}{1+x_i^{\beta_2}}\right]} + \frac{x_i^{\beta_1}}{(1+x_i^{\beta_1})} - \frac{x_i^{\beta_2}}{(1+x_i^{\beta_2})}\right) \tag{9}$$

On equating the Eq. (7), (8) and (9) with respect to 0, we get the likelihood equations for β_1, β_2 and p as

$$\beta_1 = \frac{m_1}{-\sum_{i=1}^{m_1} logx_{1i} + 2\sum_{i=1}^{m_1}\frac{x_{1i}^{\beta_1}\log x_{1i}}{1+x_{1i}^{\beta_1}} - \sum_{i=1}^{m} R_i \frac{1}{\left[1-p\frac{x_i^{\beta_1}}{1+x_i^{\beta_1}}-(1-p)\frac{x_i^{\beta_2}}{1+x_i^{\beta_2}}\right]}\left(-p\frac{x_i^{\beta_1}\, logx_i}{(1+x_i^{\beta_1})^2}\right)} \tag{10}$$

$$\beta_2 = \frac{m_2}{-\sum_{i=1}^{m_2} logx_{2i} + 2\sum_{i=1}^{m_2}\frac{x_{2i}^{\beta_2}\log x_{2i}}{1+x_{2i}^{\beta_2}} - \sum_{i=1}^{m} R_i \frac{1}{\left[1-p\frac{x_i^{\beta_1}}{1+x_i^{\beta_1}}-(1-p)\frac{x_i^{\beta_2}}{1+x_i^{\beta_2}}\right]}\left(-(1-p)\frac{x_i^{\beta_2}\, logx_i}{(1+x_i^{\beta_2})^2}\right)} \tag{11}$$

$$p = \frac{m_1 - (p-p^2) + \sum_{i=1}^{m} R_i\left(\frac{1}{\left[1-p\frac{x_i^{\beta_1}}{1+x_i^{\beta_1}}-(1-p)\frac{x_i^{\beta_2}}{1+x_i^{\beta_2}}\right]} + \frac{x_i^{\beta_1}}{\left(1+x_i^{\beta_1}\right)} - \frac{x_i^{\beta_2}}{\left(1+x_i^{\beta_2}\right)}\right)}{m} \tag{12}$$

which can be solved by any method of iteration and we get the MLEs $\widehat{\beta_1}$, $\widehat{\beta_2}$ and $\hat{p}$.

To obtain Variance-Covariance matrix of ML estimators, we find second derivatives of $logL$ with respect to the parameters β_1, β_2 and p as

$$\frac{\partial^2 logL}{\partial{\beta_1}^2} = \frac{-m_1}{{\beta_1}^2} - 2\sum_{i=1}^{m_1}\log x_{1i}\left[\frac{(1+x_{1i}^{\beta_1})x_{1i}^{\beta_1}\log x_{1i} - x_{1i}^{\beta_1}x_{1i}^{\beta_1}\log x_{1i}}{\left(1+x_{1i}^{\beta_1}\right)^2}\right] -$$

$$p\ \sum_{i=1}^{m} R_i \log x_i \left[\left(1 - p\frac{x_i^{\beta_1}}{1+x_i^{\beta_1}} - (1-p)\frac{x_i^{\beta_2}}{1+x_i^{\beta_2}}\right)\left(1 + x_i^{\beta_1}\right)^2 x_i^{\beta_1}\, log x_i - \right.$$

$$x_i^{\beta_1}\begin{Bmatrix} 2\left(1 + x_i^{\beta_1}\right)x_i^{\beta_1}\, log x_i - p\left(x_i^{\beta_1}\, log x_i + x_i^{2\beta_1} 2\, log x_i\right) - \\ (1-p)\frac{x_i^{\beta_2}}{1+x_i^{\beta_2}}\ x_i^{\beta_1}\, log x_i\ 2\left(1 + x_i^{\beta_1}\right) \end{Bmatrix} \times$$

$$\left.(1 - p\frac{x_i^{\beta_1}}{1+x_i^{\beta_1}} - (1-p)\frac{x_i^{\beta_2}}{1+x_i^{\beta_2}})^{-2}\left(1 + x_i^{\beta_1}\right)^{-4}\right] \tag{13}$$

$$\frac{\partial^2 logL}{\partial {\beta_2}^2} = \frac{-m_2}{{\beta_2}^2} - 2\sum_{i=1}^{m_2} \log x_{2i}\ \left[\frac{(1+x_{2i}^{\beta_2})x_{2i}^{\beta_2}\log x_{2i} - x_{2i}^{\beta_2} x_{2i}^{\beta_2}\log x_{2i}}{\left(1+x_{2i}^{\beta_2}\right)^2}\right] -$$

$$(1-p)\sum_{i=1}^{m} R_i \log x_i\ \left[\left(1 - p\frac{x_i^{\beta_1}}{1+x_i^{\beta_1}} - (1-p)\frac{x_i^{\beta_2}}{1+x_i^{\beta_2}}\right)\left(1 + x_i^{\beta_2}\right)^2 x_i^{\beta_2}\, log x_i - \right.$$

$$x_i^{\beta_2}\begin{Bmatrix} 2\left(1 + x_i^{\beta_2}\right)x_i^{\beta_2}\, log x_i - (1-p)\left(x_i^{\beta_2}\, log x_i + x_i^{2\beta_2} 2\, log x_i\right) - \\ (1-p)\left(\frac{x_i^{\beta_1}}{1+x_i^{\beta_1}}\ x_i^{\beta_2}\, log x_i\ 2\left(1 + x_i^{\beta_2}\right)\right) \end{Bmatrix} \times$$

$$\left.\left(1 - p\frac{x_i^{\beta_1}}{1+x_i^{\beta_1}} - (1-p)\frac{x_i^{\beta_2}}{1+x_i^{\beta_2}}\right)^{-2}\left(1 + x_i^{\beta_1}\right)^{-4}\right] \tag{14}$$

$$\frac{\partial^2 logL}{\partial\beta_1\partial\beta_2} = -p\,(1-p)\ \sum_{i=1}^{m}\left[\frac{R_i\, x_i^{\beta_1}\, x_i^{\beta_2}\, (\log x_i)^2}{\left(1+x_i^{\beta_1}\right)^2\left(1+x_i^{\beta_2}\right)^2\left(1-p\frac{x_i^{\beta_1}}{1+x_i^{\beta_1}}-(1-p)\frac{x_i^{\beta_2}}{1+x_i^{\beta_2}}\right)^2}\right] \tag{15}$$

$$\frac{\partial^2 logL}{\partial\beta_1\partial p} = \sum_{i=1}^{m}\frac{R_i}{\left(1-p\frac{x_i^{\beta_1}}{1+x_i^{\beta_1}}-(1-p)\frac{x_i^{\beta_2}}{1+x_i^{\beta_2}}\right)^2} - \frac{x_i^{\beta_1}}{1+x_i^{\beta_1}} + \frac{x_i^{\beta_2}}{1+x_i^{\beta_2}} - \frac{x_i^{\beta_1}\log x_i}{\left(1+x_i^{\beta_1}\right)^2} \tag{16}$$

$$\frac{\partial^2 logL}{\partial\beta_2\partial p} = \sum_{i=1}^{m}\frac{R_i}{\left(1-p\frac{x_i^{\beta_1}}{1+x_i^{\beta_1}}-(1-p)\frac{x_i^{\beta_2}}{1+x_i^{\beta_2}}\right)^2} - \frac{x_i^{\beta_1}}{1+x_i^{\beta_1}} + \frac{x_i^{\beta_2}}{1+x_i^{\beta_2}} - \frac{x_i^{\beta_2}\log x_i}{\left(1+x_i^{\beta_2}\right)^2} \tag{17}$$

$$\frac{\partial^2 logL}{\partial p^2} = -\frac{m_1}{p^2} - \frac{m_2}{(1-p)^2}\sum_{i=1}^{m}\frac{R_i}{\left(1-p\frac{x_i^{\beta_1}}{1+x_i^{\beta_1}}-(1-p)\frac{x_i^{\beta_2}}{1+x_i^{\beta_2}}\right)^2} - \frac{x_i^{\beta_1}}{1+x_i^{\beta_1}} + \frac{x_i^{\beta_2}}{1+x_i^{\beta_2}} \tag{18}$$

The Variance – Covariance matrix of MLEs of the parameters is given by,

$$V = \begin{bmatrix} -E\left[\frac{\partial^2 logL}{\partial{\beta_1}^2}\right] & -E\left[\frac{\partial^2 logL}{\partial\beta_2\partial\beta_1}\right] & -E\left[\frac{\partial^2 logL}{\partial p\partial\beta_1}\right] \\ -E\left[\frac{\partial^2 logL}{\partial\beta_2\partial\beta_1}\right] & -E\left[\frac{\partial^2 logL}{\partial{\beta_2}^2}\right] & -E\left[\frac{\partial^2 logL}{\partial p\partial\beta_2}\right] \\ -E\left[\frac{\partial^2 logL}{\partial p\partial\beta_1}\right] & -E\left[\frac{\partial^2 logL}{\partial p\partial\beta_2}\right] & -E\left[\frac{\partial^2 logL}{\partial p^2}\right] \end{bmatrix}^{-1}$$

According to Lawless (2003) the estimate of variance covariance matrix is given as

$$\hat{V} = \begin{bmatrix} -\frac{\partial^2 logL}{\partial \beta_1{}^2} & -\frac{\partial^2 logL}{\partial \beta_2 \partial \beta_1} & -\frac{\partial^2 logL}{\partial p \partial \beta_1} \\ -\frac{\partial^2 logL}{\partial \beta_2 \partial \beta_1} & -\frac{\partial^2 logL}{\partial \beta_2{}^2} & -\frac{\partial^2 logL}{\partial p \partial \beta_2} \\ -\frac{\partial^2 logL}{\partial p \partial \beta_1} & -\frac{\partial^2 logL}{\partial p \partial \beta_2} & -\frac{\partial^2 logL}{\partial p^2} \end{bmatrix}^{-1}_{(\beta_1,\beta_2,p)=(\hat{\beta}_1,\hat{\beta}_2,\hat{p})} \quad (19)$$

The variances of $\widehat{\beta_1}$, $\widehat{\beta_2}$ and $\hat{p}$ are given by diagonal elements of the matrix V.

The mean squared error is calculated for each of the above parameter using the below equation

$$\text{Mean squared error } (\hat{\theta}) = \text{Variance } (\hat{\theta}) + (\hat{\theta} - \theta)^2, \quad \theta = (\beta_1, \beta_2, p) \quad (20)$$

4. Bayes Estimation

The Bayesian approach considers prior information along with the information available from the data to form a posterior distribution which is used for Bayesian inference. Comparatively less sample data is required in Bayesian method than in classical sampling theory, which makes it more preferable in life testing and reliability estimation where sample data is costly and hard to obtain.

Under Bayesian estimation, a joint distribution function $\phi(\theta, \underline{x})$ is obtained using the likelihood function and the specified prior distribution of the unknown parameters. A marginal distribution $m(\underline{x})$ is derived on integrating the joint distribution function over the range of its parameters. The joint posterior distribution $g(\theta|\underline{x})$ is obtained by taking a ratio of joint distribution of β_1, β_2, p and $\underline{x}$ and marginal distribution $m(\underline{x})$. The marginal posterior distribution is derived by integrating joint posterior distribution over the range of its parameters. In this section, Bayes estimates of the parameters are obtained using the marginal posterior distributions of the parameters and their corresponding mean squared errors and posterior risks are also obtained.

Consider the gamma priors for the parameter β_1 and β_2, and uniform prior for the parameter p.

$$\Pi_1(\beta_1) = \frac{a_1{}^{b_1} \beta_1{}^{b_1-1} e^{-a_1\beta_1}}{\Gamma b_1}, \ \beta_1 > 0; \ a_1, b_1 > 0 \quad (21)$$

$$\Pi_2(\beta_2) = \frac{a_2{}^{b_2} \beta_2{}^{b_2-1} e^{-a_2\beta_2}}{\Gamma b_2}, \ \beta_2 > 0; \ a_2, b_2 > 0 \quad (22)$$

$$\Pi_3(p) = 1, \ 0 < p < 1 \quad (23)$$

Using the likelihood function in Eq. (5) and prior distributions in Eq. (21), (22) and (23), the joint distribution of parameters and sample becomes

$$\phi(\beta_1, \beta_2, p, \underline{x}) \propto L\, \Pi_1(\beta_1)\, \Pi_2(\beta_2)\, \Pi_3(p) \quad (24)$$

$$\phi(\beta_1, \beta_2, p, \underline{x}) \propto p^{m_1} \beta_1{}^{m_1} \prod_{i=1}^{m_1} \frac{(x_{1i})^{\beta_1 - 1}}{(1 + x_{1i}{}^{\beta_1})^2} (1-p)^{m_2} \beta_2{}^{m_2} \prod_{i=1}^{m_2} \frac{(x_{2i})^{\beta_2 - 1}}{(1 + x_{2i}{}^{\beta_2})^2}$$

$$\prod_{i=1}^{m}\left[1-p\,\frac{x_i^{\beta_1}}{1+x_i^{\beta_1}}-(1-p)\,\frac{x_i^{\beta_2}}{1+x_i^{\beta_2}}\right]^{R_i}\ \frac{a_1^{b_1}\,\beta_1^{b_1-1}\,e^{-a_1\beta_1}}{\Gamma b_1}\ \frac{a_2^{b_2}\,\beta_2^{b_2-1}\,e^{-a_2\beta_2}}{\Gamma b_2} \tag{25}$$

The joint posterior distribution of β_1, β_2 and p can be obtained as

$$g(\beta_1,\beta_2,p|\underline{x}) = \frac{\phi(\beta_1,\beta_2,p,\underline{x})}{m(\underline{x})} \tag{26}$$

where $m(\underline{x})$ is the marginal distribution of $\underline{x}$ that can be derived from the joint distribution as

$$m(\underline{x}) = \int_{\beta_1}\int_{\beta_2}\int_{p}\phi(\beta_1,\beta_2,p|\underline{x})\,dp\,d\beta_2\,d\beta_1 \tag{27}$$

Using the equations Eq. (25) and (27) the joint posterior distribution can be written as

$$g(\beta_1,\beta_2,p|\underline{x}) \propto p^{m_1}(1-p)^{m_2}\beta_1^{m_1+b_1-1}e^{-\beta_1\left(a_1-\sum_{i=1}^{m_1}logx_{1i}\right)}\beta_2^{m_2+b_2-1}$$
$$e^{-\beta_2\left(a_2-\sum_{i=1}^{m_2}logx_{2i}\right)}\ e^{\sum_{i=1}^{m}R_i\log\left[1-p\frac{x_i^{\beta_1}}{1+x_i^{\beta_1}}-(1-p)\frac{x_i^{\beta_2}}{1+x_i^{\beta_2}}\right]}\ e^{-2\sum_{i=1}^{m_1}\log\left(1+x_{1i}^{\beta_1}\right)}$$
$$e^{-2\sum_{i=1}^{m_2}\log\left(1+x_{2i}^{\beta_2}\right)}e^{-\sum_{i=1}^{m_1}log(x_{1i})}e^{-\sum_{i=1}^{m_2}log(x_{2i})}\ \frac{a_1^{b_1}}{\Gamma b_1}\ \frac{a_2^{b_2}}{\Gamma b_2} \tag{28}$$

The marginal posterior distribution of β_1, β_2 and p can be determined by integrating with respect to the other parameters. The marginal posterior distribution of prior β_1 is given by

$$h_1(\beta_1|\underline{x}) = \int_{\beta_2}\int_{p}p^{m_1}\,(1-p)^{m_2}\,\beta_1^{m_1+b_1-1}\,e^{-\beta_1\left(a_1-\sum_{i=1}^{m_1}logx_{1i}\right)}\,\beta_2^{m_2+b_2-1}$$
$$e^{-\beta_2\left(a_2-\sum_{i=1}^{m_2}logx_{2i}\right)}\ e^{\sum_{i=1}^{m}R_i\log\left[1-p\frac{x_i^{\beta_1}}{1+x_i^{\beta_1}}-(1-p)\frac{x_i^{\beta_2}}{1+x_i^{\beta_2}}\right]}\ e^{-2\sum_{i=1}^{m_1}\log\left(1+x_{1i}^{\beta_1}\right)}$$
$$e^{-2\sum_{i=1}^{m_2}\log\left(1+x_{2i}^{\beta_2}\right)}e^{-\sum_{i=1}^{m_1}log(x_{1i})}e^{-\sum_{i=1}^{m_2}log(x_{2i})}\ \frac{a_1^{b_1}}{\Gamma b_1}\ \frac{a_2^{b_2}}{\Gamma b_2}\ dp\,d\beta_2 \tag{29}$$

Similarly the marginal posterior distribution of prior β_2 and p are given by

$$h_2(\beta_2|\underline{x}) = \int_{\beta_1}\int_{p}p^{m_1}\,(1-p)^{m_2}\,\beta_1^{m_1+b_1-1}\,e^{-\beta_1\left(a_1-\sum_{i=1}^{m_1}logx_{1i}\right)}\,\beta_2^{m_2+b_2-1}$$
$$e^{-\beta_2\left(a_2-\sum_{i=1}^{m_2}logx_{2i}\right)}\ e^{\sum_{i=1}^{m}R_i\log\left[1-p\frac{x_i^{\beta_1}}{1+x_i^{\beta_1}}-(1-p)\frac{x_i^{\beta_2}}{1+x_i^{\beta_2}}\right]}\ e^{-2\sum_{i=1}^{m_1}\log\left(1+x_{1i}^{\beta_1}\right)}$$
$$e^{-2\sum_{i=1}^{m_2}\log\left(1+x_{2i}^{\beta_2}\right)}\ e^{-\sum_{i=1}^{m_1}log(x_{1i})}e^{-\sum_{i=1}^{m_2}log(x_{2i})}\ \frac{a_1^{b_1}}{\Gamma b_1}\ \frac{a_2^{b_2}}{\Gamma b_2}\ dp\,d\beta_1 \tag{30}$$

$$h_3(p|\underline{x}) = \int_{\beta_1}\int_{\beta_2}p^{m_1}\,(1-p)^{m_2}\,\beta_1^{m_1+b_1-1}\,e^{-\beta_1\left(a_1-\sum_{i=1}^{m_1}logx_{1i}\right)}\,\beta_2^{m_2+b_2-1}$$
$$e^{-\beta_2\left(a_2-\sum_{i=1}^{m_2}logx_{2i}\right)}\ e^{\sum_{i=1}^{m}R_i\log\left[1-p\frac{x_i^{\beta_1}}{1+x_i^{\beta_1}}-(1-p)\frac{x_i^{\beta_2}}{1+x_i^{\beta_2}}\right]}\ e^{-2\sum_{i=1}^{m_1}\log\left(1+x_{1i}^{\beta_1}\right)}$$
$$e^{-2\sum_{i=1}^{m_2}\log\left(1+x_{2i}^{\beta_2}\right)}\ e^{-\sum_{i=1}^{m_1}log(x_{1i})}e^{-\sum_{i=1}^{m_2}log(x_{2i})}\ \frac{a_1^{b_1}}{\Gamma b_1}\ \frac{a_2^{b_2}}{\Gamma b_2}\ d\beta_2\,d\beta_1 \tag{31}$$

It is not possible to obtain the above mentioned marginal posterior distributions in closed form, which makes it difficult to obtain Bayes estimators directly using marginal posterior distributions. In literature, there are various methods like numerical integration method, Lindley approximation, importance sampling, MCMC technique *etc.* that are useful in such cases. We have used here the importance sampling method used by Kundu and Pradhan (2009) to obtain Bayes estimates of the parameters in any kind of loss functions such as SELF, KLF, PLF *etc.* This method is discussed in many other articles also, some of them are Sultana *et al.* (2020), Madi and Raqab (2009) and Sultana *et al.* (2018).

Based on theory of Bayes estimation a loss function gauges the difference between the estimate $\hat{\theta}$ and the parameter θ and there is no particular procedures to select any loss functions. A posterior risk is the expected value of loss function and the posterior risks associated with the estimators are compared to evaluate the performances of the Bayes estimators. The loss functions used in this paper are described below:

Squared Error loss function (SELF): The Squared error loss function is given by $l(\hat{\theta},\ \theta) = (\hat{\theta} - \theta)^2$.

The Bayes estimate and the posterior risk are defined as

$$\hat{\theta} = E(\theta|x) \tag{32}$$

and

$$\rho(\hat{\theta}) = E(\hat{\theta} - \theta)^2 \tag{33}$$

K – loss function (KLF): The K – loss function was proposed by Wasan (1970), is defined as

$$l(\hat{\theta},\ \theta) = (\hat{\theta} - \theta)^2 / \hat{\theta}\theta.$$

The Bayes estimate and the posterior risk are defined as

$$\hat{\theta} = \sqrt{E(\theta|x)/E(\theta^{-1}|x)} \tag{34}$$

and

$$\rho(\hat{\theta}) = 2\,\{E(\theta|x)E(\theta^{-1}|x) - 1\} \tag{35}$$

respectively.

Precautionary loss function (PLF): The Precautionary loss function was proposed by Norstrom (1996), is defined as $l\,(\hat{\theta},\ \theta) = (\theta - \hat{\theta})^2 \,/\, \hat{\theta}$.

The Bayes estimate and the posterior risk are defined as

$$\hat{\theta} = \{E(\theta^2|x)\}^{\frac{1}{2}} \tag{36}$$

and

$$\rho(\hat{\theta}) = 2[\{E(\theta^2|x)\}^{\frac{1}{2}} - E(\theta|x)] \tag{37}$$

respectively.

To employ importance sampling method for Bayes estimation we rewrite the joint posterior distribution given in Eq. (28) as

$$g(\beta_1, \beta_2, p|\underline{x}) \propto p^{m_1}\,(1-p)^{m_2}\,{\beta_1}^{m_1+b_1-1}\,e^{-\beta_1 A_1}\,{\beta_2}^{m_2+b_2-1}\,e^{-\beta_2 A_2}$$

$$e^{-2\sum_{i=1}^{m_1}\log(1+{x_{1i}}^{\beta_1})}\;e^{-2\sum_{i=1}^{m_2}\log(1+{x_{2i}}^{\beta_2})}\;e^{\sum_{i=1}^{m} R_i \log\left[1-p\frac{{x_i}^{\beta_1}}{1+{x_i}^{\beta_1}}-(1-p)\frac{{x_i}^{\beta_2}}{1+{x_i}^{\beta_2}}\right]}$$

$$e^{-\sum_{i=1}^{m_1} log(x_{1i})} e^{-\sum_{i=1}^{m_2} log(x_{2i})} \frac{a_1{}^{b_1}}{\Gamma b_1} \frac{a_2{}^{b_2}}{\Gamma b_2} \tag{38}$$

where $A_1 = a_1 - \sum_{i=1}^{m_1} \log x_{1i}$ and $A_2 = a_2 - \sum_{i=1}^{m_2} \log x_{2i}$

The above form can also be written as

$$g(\beta_1, \beta_2, p|\underline{x}) \propto g_1(p \mid m_1, m_2)\, g_2(\beta_1 \mid \underline{x}, m_1)\, g_3(\beta_2 \mid \underline{x}, m_2)\, \Psi(p, \beta_1, \beta_2 \mid \underline{x}, m) \tag{39}$$

where

$g_1(p \mid m_1, m_2)$ is the probability density function of $beta\ (m_1 + 1, m_2 + 1)$ distribution

$g_2(\beta_1 \mid \underline{x}, m_1)$ is the probability density function of $gamma\ (m_1 + b_1, A_1)$ distribution

$g_3(\beta_2 \mid \underline{x}, m_2)$ is the probability density function of $gamma\ (m_2 + b_2, A_2)$ distribution

$\Psi(p, \beta_1, \beta_2 \mid \underline{x}, m) =$

$e^{-2\sum_{i=1}^{m_1} \log(1 + x_{1i}{}^{\beta_1})}\ e^{-2\sum_{i=1}^{m_2} \log(1 + x_{2i}{}^{\beta_2})}\ e^{\sum_{i=1}^{m} R_i \log\left[1 - p\frac{x_i{}^{\beta_1}}{1 + x_i{}^{\beta_1}} - (1-p)\frac{x_i{}^{\beta_2}}{1 + x_i{}^{\beta_2}}\right]}$, a function of β_1, β_2, p and $\underline{x}$.

The hyper parameters used in the prior distributions are determined as follows:

- Find means & variances of the MLEs of parameters β_1, β_2 and p and considered them as prior information of the parameters.
- These estimates are compared with theoretical mean & variance of the prior distribution.
- Solving them we obtain estimates of the hyper parameters.

Algorithm-1

The steps of importance sampling to obtain Bayes estimates are as follows:

Step - 1: Decide the values of β_1, β_2, p and $R_1, R_2, \ldots R_m$ such that $\sum_{i=1}^{m} R_i = (n - m)$.

Step - 2: Generate

$x_{1i}\, ; i = 1, 2, \ldots \ldots \ldots, m_1$ and $x_{2i}; i = 1, 2, \ldots \ldots \ldots, m_2$

Step - 3: Generate

N values of p from $beta\ (m_1 + 1, m_2 + 1)$ as $(p_1, p_2, \ldots, p_N)$

N values of β_1 from $gamma\ (m_1 + b_1, A_1)$ as $(\beta_{11}, \beta_{12}, \ldots, \beta_{1N})$

N values of β_2 from $gamma\ (m_2 + b_2, A_2)$ as $(\beta_{21}, \beta_{22}, \ldots, \beta_{2N})$

Step - 4: Calculate the $E(\theta|\underline{x})$ using the formula:

$$E(\theta|\underline{x}) = \frac{\sum_{i=1}^{N} \theta_i . \Psi(\theta_i|\underline{x}, m)}{\sum_{i=1}^{N} \Psi(\theta_i|\underline{x}, m)} \tag{40}$$

Step - 5: Calculate $E(p|x)$, $E(p^2|x)$, $E\left(\frac{1}{p}|x\right)$, $E(\beta_1|x)$, $E(\beta_1{}^2|x)$, $E\left(\frac{1}{\beta_1}|x\right)$, $E(\beta_2|x)$, $E(\beta_2{}^2|x)$, $E\left(\frac{1}{\beta_2}|x\right)$ using the Eq. (40) to calculate the estimate, PR, and MSE using the squared error loss function, K – loss function and precautionary loss function.

5. Simulation Study

A simulation study is setup to check the performance of ML estimators and Bayes estimators obtained in the earlier sections. We have used the following inputs.

To simulate samples from 2 component mixture of log logistic distributions, we have used the following algorithm with the inputs: S=5000 which is number of simulations, n = 60 and m = 15 (25% censored) with censoring scheme R = (0, 5, 2, 8, 0, 2, 5, 3, 0, 6, 4, 9, 1, 0, 0); m = 25 (42% censored) with censoring scheme R = (0, 2, 5, 3, 2, 3, 0, 0, 0, 1, 4, 1, 1, 1, 2, 0, 0, 5, 2, 0, 1, 0, 0, 2, 0) and m = 35 (58% censored) with censoring scheme R = (0, 0, 2, 0, 1, 0, 0, 3, 0, 1, 1, 1, 1, 1, 2, 0, 0, 2, 2, 0, 1, 0, 0, 2, 0, 0, 2, 0, 0, 1, 0, 0, 0, 2, 0). The values of the prior parameters are considered as β_1= (0.9, 1.9), β_2 = (2.0, 3.5) and p = (0.7, 0.35) based on the method described before the algorithm-1. A Type – II censored sample is also generated using the above mentioned inputs to observe a comparison between performances of estimators using the progressive Type – II censoring and Type – II censoring.
To generate the progressive Type – II censored sample for the mixture model we use the following algorithm.

Algorithm-2

Step - 1: A uniform random number (u) is generated from $U(0,1)$ and if $u \leq p$ (mixture proportion parameter) then select first sub-population $f_1(x)$ having parameter β_1, otherwise second sub-population $f_2(x)$ having parameter β_2.
Step - 2: To generate say r observations from first or second sub-population ($r = m_1 \ or\ m_2$), generate r uniform random numbers $u_1, u_2, \dots, u_r \sim U(0,1)$
Step - 3: Set $\xi_i = \ln(1 - u_i)\,; i = 1, \dots, r$
Step - 4: Let $y_1 = \frac{\xi_1}{r}$ and $y_i = y_{i-1} + \frac{\xi_i}{n - \sum_{j=1}^{i-1} R_j - i + 1}; i = 2, 3, \dots, r$ with $\sum_{i=1}^{r} R_i = n - r$
Step - 5: $x_i = (1 - e^{-y_i})^{\frac{1}{\beta}}; i = 1, 2, \dots, r$ where $\beta = \beta_1$ or β_2.
Step - 6: Calculate ML estimates and Bayes estimates of parameters p, β_1 and β_2 using the respective formulas from section.
Step - 7: Repeat the steps 1 - 5 for S times, thus we have $\widehat{\beta_{1i}}$, $\widehat{\beta_{2i}}$ and $\widehat{p_i}$, $i = 1,2, \dots, S$.
Step - 8: Calculate Bayes estimates of β_1, β_2, and p by taking average of the S values in step 7.
Step - 9: Calculate Root Mean Square Error and PR, using the formula,

$$MSE = \frac{\sum_{i=1}^{S} (\widehat{\theta_i} - \theta)^2}{S}$$

The outputs obtained from the simulations are presented in Table A.1 to Table A.6.

6. Discussion of Numerical Results

From Table A.1 to Table A.6 we observe the following conclusions:

i. The Bayes estimates are better compared to ML estimates based on MSE for both the censoring schemes that are considered.
ii. For the considered values of the parameters β_1, β_2 and p as m increases, the MSE of the estimates decreases for both the censoring schemes for MLEs as well as Bayes estimators.

iii. The Bayes estimator under the K-loss function performs better with respect to MSE compared to squared error loss function for all the values of β_1 and β_2 under both types of censoring schemes adopted.
iv. As the values of the parameters β_1 and β_2 increases, the MSE also increases, for Progressive Type - II censoring scheme.
v. ML estimates and Bayes estimates give almost similar amount of bias.

7. Real Life Example

In this section the analysis of real-life dataset of failure of electrical cables is performed which is presented by Lawless (2003). The test involved 20 cables each with two types of insulations which are considered as Population – I and Population – II respectively. The purpose is to determine whether the estimators have the similar behavior for real life data as it was for simulated data.

The Kolmogorov-Smirnov test is performed to determine whether the data follows log logistic distribution. The calculated value of KS test statistic is 0.1868 for Population – I and 0.0715 for Population – II. The degrees of freedom for Population – I and Population – II are 20 and the test is performed at 5% level of significance. The KS tabulated value for one sample test at 5% level of significance and 20 degrees of freedom is 0.294. The results clearly indicate that the Population – I and Population – II fits well to the log logistic distribution.

From the original data we have prepared progressive type – II censored sample with $n = 40,\ p = 0.5, m_1 = 13, m_2 = 12,\ R = (1,0,0,0,1,1,0,2,0,0,1,0,0,0,0,2,0,0,0,2,0,0,0,0,5)$. The censored mixture data is:

Population – I: 32.0, 35.4, 39.8, 41.2, 45.5, 46.0, 46.2, 46.5, 47.3, 47.3, 49.2, 50.4, 56.3
and
Population – II: 45.3, 49.2, 51.3, 53.2, 53.2, 55.5, 57.1, 57.5, 59.2, 62.4, 63.8, 67.7.

The results are obtained using Bayes and Maximum Likelihood Estimation approaches for the above mentioned real-life dataset and are given in the Table A.7.

The analysis under real-life data supports the findings obtained from the simulation study. The Bayes estimates are better compared to the ML estimates and all the three loss functions SELF, KLF and PLF give similar results. This gives us more confidence to suggest the use of Bayes estimation for Progressive Type - II Censored Data using Mixture of Log Logistic Distributions.

8. Conclusion

In this paper, a two component mixture model based on log logistic distributions has been proposed. Maximum likelihood and Bayesian estimation have been used to estimate the parameters of mixture model under progressive Type – II censoring. Three types of loss functions namely, SELF, KLF and PLF are used. The posterior likelihood based on progressive Type – II censoring has no closed form due to which it is not possible to apply numerical integration. Importance sampling method was used to solve this. Finally, we observe that for precise estimation of the unknown parameters of log logistic distribution, Bayes estimation is preferable over maximum likelihood estimation under all the three types of loss functions and this holds true for real life data as well.

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Appendix A

Table A.1: MLE & Bayes Estimates, PR and MSE for $(\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{\beta_1}, \boldsymbol{\beta_2}, \boldsymbol{p}) = (\mathbf{60}, \mathbf{15}, \mathbf{0.9}, \mathbf{2.0}, \mathbf{0.7})$

MLE / Bayes	Statistic	$\widehat{\beta_1}$	$\widehat{\beta_2}$	$\hat{p}$	Type of Censoring
MLE	Estimate	1.14	2.98	0.60	Progressive Type - II
MLE	MSE	0.11	1.18	0.02	Progressive Type - II
SELF	Estimate	1.11	2.86	0.59	Progressive Type - II
SELF	PR	0.05	0.20	0.03	Progressive Type - II
SELF	MSE	0.07	0.80	0.03	Progressive Type - II
KLF	Estimate	1.08	2.82	0.56	Progressive Type - II
KLF	PR	0.08	0.05	0.20	Progressive Type - II
KLF	MSE	0.06	0.76	0.03	Progressive Type - II
PLF	Estimate	1.13	2.90	0.61	Progressive Type - II
PLF	PR	0.04	0.07	0.04	Progressive Type - II
PLF	MSE	0.07	0.84	0.02	Progressive Type - II
MLE	Estimate	1.03	3.69	0.49	Type - II
MLE	MSE	0.05	3.11	0.06	Type - II
SELF	Estimate	0.97	3.48	0.53	Type - II
SELF	PR	0.04	0.24	0.02	Type - II
SELF	MSE	0.02	2.31	0.04	Type - II
KLF	Estimate	0.95	3.45	0.50	Type - II
KLF	PR	0.08	0.04	0.21	Type - II
KLF	MSE	0.02	2.25	0.05	Type - II
PLF	Estimate	0.99	3.52	0.55	Type - II
PLF	PR	0.04	0.07	0.04	Type - II
PLF	MSE	0.03	2.36	0.04	Type - II

Table A.2: MLE & Bayes Estimates, PR and MSE for $(n, m, \beta_1, \beta_2, p) = (60,\ 15,\ 1.9,\ 3.5,\ 0.35)$

MLE / Bayes	Statistic	$\widehat{\beta_1}$	$\widehat{\beta_2}$	$\hat{p}$	Type of Censoring
MLE	Estimate	2.63	4.66	0.28	Progressive Type - II
MLE	MSE	0.68	1.94	0.01	Progressive Type - II
SELF	Estimate	2.55	4.52	0.31	Progressive Type - II
SELF	PR	0.15	0.60	0.02	Progressive Type - II
SELF	MSE	0.46	1.32	0.01	Progressive Type - II
KLF	Estimate	2.52	4.45	0.26	Progressive Type - II
KLF	PR	0.05	0.06	0.84	Progressive Type - II
KLF	MSE	0.43	1.24	0.01	Progressive Type - II
PLF	Estimate	2.58	4.59	0.35	Progressive Type - II
PLF	PR	0.06	0.13	0.06	Progressive Type - II
PLF	MSE	0.49	1.39	0.01	Progressive Type - II
MLE	Estimate	2.25	5.22	0.25	Type - II
MLE	MSE	0.27	3.44	0.01	Type - II
SELF	Estimate	2.19	4.96	0.28	Type - II
SELF	PR	0.14	0.48	0.02	Type - II
SELF	MSE	0.12	2.40	0.01	Type - II
KLF	Estimate	2.16	4.91	0.24	Type - II
KLF	PR	0.06	0.04	0.77	Type - II
KLF	MSE	0.11	2.33	0.02	Type - II
PLF	Estimate	2.22	5.01	0.31	Type - II
PLF	PR	0.06	0.10	0.06	Type - II
PLF	MSE	0.14	2.47	0.01	Type - II

Table A.3: MLE & Bayes Estimates, PR and MSE for $(\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{\beta_1}, \boldsymbol{\beta_2}, \boldsymbol{p}) = (\mathbf{60},\ \mathbf{25},\ \mathbf{0.9},\ \mathbf{2.0},\ \mathbf{0.7})$

MLE / Bayes	Statistic	$\widehat{\boldsymbol{\beta_1}}$	$\widehat{\boldsymbol{\beta_2}}$	$\widehat{\boldsymbol{p}}$	Type of Censoring
MLE	Estimate	1.02	2.40	0.68	Progressive Type - II
MLE	MSE	0.04	0.45	0.01	Progressive Type - II
SELF	Estimate	1.01	2.34	0.67	Progressive Type - II
SELF	PR	0.02	0.25	0.02	Progressive Type - II
SELF	MSE	0.02	0.20	0.01	Progressive Type - II
KLF	Estimate	1.00	2.29	0.66	Progressive Type - II
KLF	PR	0.04	0.09	0.08	Progressive Type - II
KLF	MSE	0.02	0.18	0.01	Progressive Type - II
PLF	Estimate	1.02	2.40	0.68	Progressive Type - II
PLF	PR	0.02	0.11	0.02	Progressive Type - II
PLF	MSE	0.02	0.23	0.01	Progressive Type - II
MLE	Estimate	1.01	2.62	0.66	Type - II
MLE	MSE	0.03	0.81	0.01	Type - II
SELF	Estimate	0.98	2.50	0.66	Type - II
SELF	PR	0.02	0.33	0.02	Type - II
SELF	MSE	0.02	0.38	0.01	Type - II
KLF	Estimate	0.97	2.43	0.65	Type - II
KLF	PR	0.04	0.11	0.08	Type - II
KLF	MSE	0.01	0.33	0.01	Type - II
PLF	Estimate	0.99	2.56	0.67	Type - II
PLF	PR	0.02	0.13	0.02	Type - II
PLF	MSE	0.02	0.42	0.01	Type - II

Table A.4: MLE & Bayes Estimates, PR and MSE for $(n, m, \beta_1, \beta_2, p) = (60,\ 25,\ 1.9,\ 3.5,\ 0.35)$

MLE / Bayes	Statistic	$\widehat{\beta_1}$	$\widehat{\beta_2}$	$\hat{p}$	Type of Censoring
MLE	Estimate	2.41	3.97	0.34	Progressive Type - II
MLE	MSE	0.44	0.69	0.01	Progressive Type - II
SELF	Estimate	2.32	3.92	0.35	Progressive Type - II
SELF	PR	0.16	0.40	0.02	Progressive Type - II
SELF	MSE	0.22	0.34	0.01	Progressive Type - II
KLF	Estimate	2.29	3.87	0.32	Progressive Type - II
KLF	PR	0.06	0.05	0.36	Progressive Type - II
KLF	MSE	0.20	0.31	0.01	Progressive Type - II
PLF	Estimate	2.35	3.97	0.37	Progressive Type - II
PLF	PR	0.07	0.10	0.04	Progressive Type - II
PLF	MSE	0.25	0.37	0.01	Progressive Type - II
MLE	Estimate	2.25	4.16	0.33	Type - II
MLE	MSE	0.29	0.98	0.01	Type - II
SELF	Estimate	2.18	4.03	0.35	Type - II
SELF	PR	0.15	0.44	0.02	Type - II
SELF	MSE	0.13	0.50	0.01	Type - II
KLF	Estimate	2.14	3.98	0.32	Type - II
KLF	PR	0.06	0.06	0.35	Type - II
KLF	MSE	0.11	0.46	0.01	Type - II
PLF	Estimate	2.21	4.08	0.37	Type - II
PLF	PR	0.07	0.11	0.04	Type - II
PLF	MSE	0.14	0.53	0.01	Type - II

Table A.5: MLE & Bayes Estimates, PR and MSE for $(n, m, \beta_1, \beta_2, p) = (60,\ 35,\ 0.9,\ 2.0,\ 0.70)$

MLE / Bayes	Statistic	$\widehat{\beta_1}$	$\widehat{\beta_2}$	$\widehat{p}$	Type of Censoring
MLE	Estimate	0.96	2.31	0.70	Progressive Type - II
MLE	MSE	0.02	0.40	0.00	Progressive Type - II
SELF	Estimate	0.96	2.32	0.69	Progressive Type - II
SELF	PR	0.01	0.27	0.01	Progressive Type - II
SELF	MSE	0.01	0.24	0.01	Progressive Type - II
KLF	Estimate	0.96	2.27	0.68	Progressive Type - II
KLF	PR	0.03	0.10	0.05	Progressive Type - II
KLF	MSE	0.01	0.22	0.01	Progressive Type - II
PLF	Estimate	0.97	2.38	0.70	Progressive Type - II
PLF	PR	0.01	0.12	0.02	Progressive Type - II
PLF	MSE	0.01	0.26	0.01	Progressive Type - II
MLE	Estimate	0.97	2.24	0.73	Type - II
MLE	MSE	0.02	0.58	0.00	Type - II
SELF	Estimate	0.96	2.14	0.72	Type - II
SELF	PR	0.01	0.35	0.01	Type - II
SELF	MSE	0.01	0.16	0.01	Type - II
KLF	Estimate	0.96	2.06	0.71	Type - II
KLF	PR	0.03	0.16	0.05	Type - II
KLF	MSE	0.01	0.14	0.01	Type - II
PLF	Estimate	0.97	2.22	0.72	Type - II
PLF	PR	0.01	0.16	0.01	Type - II
PLF	MSE	0.01	0.18	0.01	Type - II

Table A.6: MLE & Bayes Estimates, PR and MSE for $(\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{\beta_1}, \boldsymbol{\beta_2}, \boldsymbol{p}) = (\mathbf{60},\ \mathbf{35},\ \mathbf{1.9},\ \mathbf{3.5},\ \mathbf{0.35})$

MLE / Bayes	Statistic	$\widehat{\boldsymbol{\beta_1}}$	$\widehat{\boldsymbol{\beta_2}}$	$\widehat{\boldsymbol{p}}$	Type of Censoring
MLE	Estimate	2.07	3.83	0.35	Progressive Type - II
MLE	MSE	0.14	0.43	0.01	Progressive Type - II
SELF	Estimate	2.07	3.84	0.36	Progressive Type - II
SELF	PR	0.10	0.29	0.01	Progressive Type - II
SELF	MSE	0.06	0.25	0.01	Progressive Type - II
KLF	Estimate	2.04	3.81	0.34	Progressive Type - II
KLF	PR	0.05	0.04	0.23	Progressive Type - II
KLF	MSE	0.06	0.24	0.01	Progressive Type - II
PLF	Estimate	2.09	3.88	0.38	Progressive Type - II
PLF	PR	0.05	0.07	0.03	Progressive Type - II
PLF	MSE	0.07	0.27	0.01	Progressive Type - II
MLE	Estimate	2.14	3.69	0.37	Type - II
MLE	MSE	0.16	0.39	0.01	Type - II
SELF	Estimate	2.11	3.64	0.38	Type - II
SELF	PR	0.09	0.30	0.01	Type - II
SELF	MSE	0.07	0.13	0.01	Type - II
KLF	Estimate	2.09	3.60	0.36	Type - II
KLF	PR	0.04	0.05	0.23	Type - II
KLF	MSE	0.06	0.12	0.01	Type - II
PLF	Estimate	2.13	3.68	0.40	Type - II
PLF	PR	0.04	0.08	0.03	Type - II
PLF	MSE	0.07	0.14	0.01	Type - II

Table A.7: MLE & Bayes Estimates, PR and MSE for $(\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{\beta_1}, \boldsymbol{\beta_2}, \boldsymbol{p}) = (\mathbf{40},\ \mathbf{25},\ \mathbf{0.4054896},\ \mathbf{0.3840652},\ \mathbf{0.5})$

MLE / Bayes	Statistic	$\widehat{\boldsymbol{\beta_1}}$	$\widehat{\boldsymbol{\beta_2}}$	$\widehat{\boldsymbol{p}}$
MLE	Estimate	0.27960	0.26342	0.51278
SELF	Estimate	0.41468	0.39288	0.52065
SELF	PR	0.00005	0.00007	0.00228
KLF	Estimate	0.41462	0.39280	0.51853
KLF	PR	0.00058	0.00090	0.01640
PLF	Estimate	0.41474	0.39298	0.52284
PLF	PR	0.00013	0.00018	0.00438