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Inference on Stress-Strength Reliability for Lomax Exponential Distribution

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Abstract

In this paper, the reliability estimation of single component stress-strength model is studied with strength(X) and stress(Y) of the component follow Lomax exponential distribution. The maximum likelihood and Bayesian estimation methods are applied to derive estimators of reliability. The Bayesian estimators for reliability are constructed under different loss functions such as squared error and linex loss functions with non-informative and gamma priors using Lindley's approximation technique. The simulation experiment is conducted to estimate the mean squared error of the estimators which enable the comparison of different estimators. The construction of asymptotic confidence interval of reliability is also constructed. The real data analysis is done to illustrate the developed procedures.

Key words: Lomax exponential distribution (LED); Stress-strength reliability; maximum likelihood estimation; Bayesian inference; Lindley's approximation technique.

1. Introduction

In the recent years there has been growing interest in defining new generators for univariate continuous distributions by introducing one or more additional shape parameters to the baseline distribution. Some well-known generators are beta-G and gamma-G due to Eugene and Famoye (2002) and Zografos and Balakrishnan (2009), respectively. Torabi and Montazeri (2014) introduced the logistic-G family. Recently, Cordeiro and Pescim (2014) studied a new family of distributions based on the Lomax distribution. The probability density function (pdf) and cumulative distribution function (cdf) of Lomax-G family with two additional parameters α and β are given by

$$f(x) = \alpha\beta^{\alpha} g(x) \left([1 - G(x)] \{\beta - \log[1 - G(x)]\}^{\alpha+1} \right)^{-1}, \ x > 0, \ \alpha, \ \beta > 0$$

and

$$F(x) = 1 - \beta^{\alpha} (\beta - \log[1 - G(x)])^{-\alpha}, \ x > 0, \ \alpha, \ \beta > 0$$

where g(x) and G(x) are the pdf and cdf of parent distribution. The parameters α and β are the shape and scale parameters of the distribution, respectively.

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In this paper, the estimation of stress-strength reliability is considered when X and Y are independently distributed Lomax exponential distribution (LED) which was introduced by Ieren and Kuhe (2018) which is a new generalization of exponential distribution. The LED is constructed by Cordeiro and Pescim (2014) using Lomax-G family. The general form of cumulative distribution function (cdf) and probability density function (pdf) of Lomax G-family with baseline distribution G is given below:
The cumulative distribution function (cdf) and probability density function (pdf) are given by

$$f(x) = \alpha\beta^{\alpha} g(x) \left([1 - G(x)] \{\beta - \log[1 - G(x)]\}^{\alpha+1} \right)^{-1}, \, x > 0, \, \alpha, \, \beta > 0$$

and

$$F(x) = 1 - \beta^{\alpha} (\beta - \log[1 - G(x)])^{-\alpha}, \, x > 0, \, \alpha, \, \beta > 0$$

where g(x) and G(x) are the pdf and cdf of baseline distribution. The parameters α and β are the shape and scale parameters of the distribution, respectively.
In this paper, the problem of estimation of stress-strength reliability is considered when X and Y are independently distributed Lomax exponential distribution (LED) due to Ieren and Kuhe (2018) with exponential distribution as baseline distribution. Then, the cdf and pdf of LED are given by

$$F(x) = 1 - \beta^{\alpha} (\beta + \lambda x)^{-\alpha}, \, \alpha > 0, \, x > 0, \, \beta > 0, \, \lambda > 0$$

and

$$f(x) = \alpha\lambda\beta^{\alpha} (\beta + \lambda x)^{-(\alpha+1)}, \, \alpha > 0, \, x > 0, \, \beta > 0, \, \lambda > 0$$

It is denoted by LED(α, β, λ).
Some particular cases of Lomax exponential distribution are as given below:

1. If λ=1 and β=1, then LED is Pareto type-II distribution.

2. LED is Lomax standard exponential distribution, when λ=1.

3. When β=1, LED is generalized Pareto distribution.

The main focus of the paper is to study the problem of estimating stress-strength reliability when stress and strength variables follow LED. In the literature several authors have studied the estimation of stress-strength reliability for different life time distributions. Awad and Gharraf (1986) considered the estimation of R for Burr distribution. Mokhlis (2005) and Panahi and Asadi (2011) estimated the stress-strength reliability for Burr type-III and Lomax distributions respectively. Abravesh and Mostafaiy (2019) studied the classical and Bayesian estimation of stress-strength reliability based on type II censored sample from Pareto distribution.
The rest of the paper is organised as below. Section 2 deals with derivation of stress-strength reliability when strength X and stress Y follow LED. Maximum likelihood estimation of R and its asymptotic confidence intervals are given in Section 3. In Section 4, the Bayesian estimator of R is presented. The real data analysis is considered in Section 5. Section 6 contains a simulation study and the conclusions are presented in Section 7.

2. Stress-strength reliability

Let X and Y be two independent random variables having LED(α_1, β, λ) and LED(α_2, β, λ) respectively.
Then, the stress-strength reliability is given by

$$R = P(X > Y)$$
$$= \int_0^\infty F(x) f(x)\,dx$$
$$= \frac{\alpha_2}{\alpha_1 + \alpha_2} \tag{1}$$

3. Maximum likelihood estimation (MLE) of reliability

Let $\underline{X} = (X_1, X_2, ..., X_n)$ and $\underline{Y} = (Y_1, Y_2, ..., Y_m)$ be independent random samples from LED(α_1, β, λ) and LED(α_2, β, λ), respectively. Then the likelihood function of α_1, α_2, β and λ given $\left(\underline{x},\, \underline{y}\right)$ is

$$L\left(\alpha_1,\, \alpha_2,\, \beta,\, \lambda|\underline{x},\, \underline{y}\right) = \prod_{i=1}^{n} \alpha_1 \beta^{\alpha_1} \lambda (\beta + \lambda x_i)^{-(\alpha_1+1)} \prod_{j=1}^{m} \alpha_2 \beta^{\alpha_2} \lambda (\beta + \lambda y_j)^{-(\alpha_2+1)} \tag{2}$$

and the log-likelihood function is

$$\log L = n\log\alpha_1 + m\log\alpha_2 + (n\alpha_1 + m\alpha_2)\log\beta + (n+m)\log\lambda - (\alpha_1+1)\textstyle\sum_{i=1}^{n}\log(\beta+\lambda x_i) - (\alpha_2+1)\sum_{j=1}^{m}\log(\beta+\lambda y_j) \tag{3}$$

The likelihood equations are

$$\frac{n}{\alpha_1} + n\log\beta - \sum_{i=1}^{n}\log(\beta+\lambda x_i) = 0 \tag{4}$$

$$\frac{m}{\alpha_2} + m\log\beta - \sum_{j=1}^{m}\log(\beta+\lambda y_j) = 0 \tag{5}$$

$$\frac{n\alpha_1 + m\alpha_2}{\beta} - (\alpha_1+1)\sum_{i=1}^{n}\left(\frac{1}{(\beta+\lambda x_i)}\right) - (\alpha_2+1)\sum_{j=1}^{m}\left(\frac{1}{(\beta+\lambda y_j)}\right) = 0 \tag{6}$$

and

$$\frac{n+m}{\lambda} - (\alpha_1+1)\sum_{i=1}^{n}\left(\frac{x_i}{(\beta+\lambda x_i)}\right) - (\alpha_2+1)\sum_{j=1}^{m}\left(\frac{y_j}{(\beta+\lambda y_j)}\right) = 0 \tag{7}$$

The above equations do not yield solution in closed form. Hence, a popular iterative technique, namely, Newton Raphson technique is used.
Using the invariance property of MLE, the MLE of R is given by

$$\hat{R} = \frac{\hat{\alpha}_2}{\hat{\alpha}_1 + \hat{\alpha}_2}\,.$$

3.1. Asymptotic distribution of R

Under general regularity conditions, the asymptotic distribution of $\left(\underline{\hat{\theta}}-\underline{\theta}\right)$ is multivariate $N_{p+4}\left(\underline{0},\, I\left(\underline{\theta}\right)^{-1}\right)$ distribution, where $I\left(\underline{\theta}\right)$ is the expected information matrix and $\underline{\theta}=[\alpha_1,\alpha_2,\beta,\lambda]^{\intercal}$. Here, $I\left(\underline{\theta}\right)^{-1}$can be approximated by the inverse of observed information matrix $I\left(\underline{\hat{\theta}}\right)^{-1}$ evaluated at $\hat{\theta}$. This distribution is used to construct the 100(1-α)% confidence interval for each parameters.
Asymptotic confidence intervals of the parameters are given by

$$\alpha_1 \pm Z_{\frac{\alpha}{2}}\sqrt{I_{11}^{-1}},\ \alpha_2 \pm Z_{\frac{\alpha}{2}}\sqrt{I_{22}^{-1}}\,,$$

$$\beta \pm Z_{\frac{\alpha}{2}}\sqrt{I_{33}^{-1}}\ and\ \lambda \pm Z_{\frac{\alpha}{2}}\sqrt{I_{44}^{-1}}$$

The asymptotic confidence interval of R is

$$\hat{R} \pm Z_{\frac{\alpha}{2}}\sqrt{AV\left(\hat{R}\right)}\,,$$

where

$$AV\left(\hat{R}\right)=\left[\frac{\partial R}{\partial \alpha_1}\right]^2 I_{11}^{-1}+\left[\frac{\partial R}{\partial \alpha_2}\right]^2 I_{22}^{-1}\,.$$

4. Bayesian estimation of R

In this section, the Bayesian estimation of R under different loss functions and priors is presented. The non-informative and gamma priors are considered to obtain Bayes estimator of R. The prior distribution of α_1, α_2, β and λ are gamma (c_1, d_1), gamma (c_2, d_2), gamma (c_3, d_3) and gamma (c_4, d_4), respectively.
The joint prior distribution of α_1, α_2, β and λ is given by

$$g_1\left(\alpha_1,\,\alpha_2,\,\beta,\,\lambda\right)=g\left(\alpha_1\right)\ g\left(\alpha_2\right)\ g\left(\beta\right)\ g\left(\lambda\right) \tag{8}$$

where

$$g\left(\alpha_1\right)=\frac{d_1^{c_1}}{\Gamma c_1}\exp\left(-d_1\alpha_1\right)\ \alpha_1^{c_1-1},\ \alpha_1>0,\ c_1,\ d_1>0\,,$$

$$g\left(\alpha_2\right)=\frac{d_2^{c_2}}{\Gamma c_2}\exp\left(-d_2\alpha_2\right)\alpha_2^{c_2-1},\ \alpha_2>0,\ c_2,\ d_2>0\,,$$

$$g\left(\beta\right)=\frac{d_3^{c_3}}{\Gamma c_3}\exp\left(-d_3\beta\right)\beta^{c_3-1},\ \beta>0,\ c_3,\ d_3>0$$

and

$$g\left(\lambda\right)=\frac{d_4^{c_4}}{\Gamma c_4}\exp\left(-d_4\lambda\right)\ \lambda^{c_4-1},\ \lambda>0,\ c_4,\ d_4>0$$

If, $c_1=c_2=c_3=c_4=d_1=d_2=d_3=d_4=0$, then it reduces to non-informative prior.
The posterior distribution is

$$\pi\left(\alpha_1,\,\alpha_2,\,\beta,\,\lambda\right)=\frac{L\left(\alpha_1,\,\alpha_2,\,\beta,\,\lambda\right)\ g\left(\alpha_1,\,\alpha_2,\,\beta,\,\lambda\right)}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L\left(\alpha_1,\,\alpha_2,\,\beta,\,\lambda\right)\ g\left(\alpha_1,\,\alpha_2,\,\beta,\,\lambda\right)\,d\alpha_1\,d\alpha_2\,d\beta\,d\lambda}$$

$$\pi\left(\alpha_1,\,\alpha_2,\,\beta,\,\lambda\right)=\frac{G}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty G\,d\alpha_1\,d\alpha_2\,d\beta\,d\lambda} \tag{9}$$

where

$$\begin{aligned}G=&\exp\left(-d_1\alpha_1\right)\alpha_1^{n+c_1-1}\exp\left(-d_2\alpha_2\right)\alpha_2^{m+c_2-1}\exp\left(-d_3\beta\right)\exp\left(-d_4\lambda\right)\beta^{n\alpha_1+m\alpha_2+c_3-1}\\&\lambda^{n+m+c_4-1}\textstyle\prod_{i=1}^{n}\left(\beta+\lambda x_i\right)^{-(\alpha_1+1)}\prod_{j=1}^{m}\left(\beta+\lambda y_j\right)^{-(\alpha_2+1)}\end{aligned}$$

The posterior distribution of R is non-tractable. Hence, the Lindley's approximation technique is used to derive Bayes estimator of R.
For four parameter case, the Lindley's approximation to Bayes estimator of R under squared error loss function is given by

$$\begin{aligned}\hat{R}_S=&u+\left(u_1a_1+u_2a_2+u_3a_3+u_4a_4+a_5+a_6\right)+\\&\tfrac{1}{2}\left[A\left(u_1\sigma_{11}+u_2\sigma_{12}+u_3\sigma_{13}+u_4\sigma_{14}\right)\right]+\\&\tfrac{1}{2}\left[B\left(u_1\sigma_{21}+u_2\sigma_{22}+u_3\sigma_{23}+u_4\sigma_{24}\right)\right]+\\&\tfrac{1}{2}\left[C\left(u_1\sigma_{31}+u_2\sigma_{32}+u_3\sigma_{33}+u_4\sigma_{34}\right)\right]+\\&\tfrac{1}{2}\left[D\left(u_1\sigma_{41}+u_2\sigma_{42}+u_3\sigma_{43}+u_4\sigma_{44}\right)\right],\end{aligned} \tag{10}$$

where,$u=\hat{R},\ u_i,\ i=1,\,2,\,3,\,4$ and $u_{ij},\ i,\ j=1,\,2,\,3,\,4$ are the first and second order derivatives of R, $\sigma_{ij},\ i,\ j=1,\,2,\,3,\,4$ is the (i, j)th element in the inverse of the matrix [-L_{ij}], L_{ij} and L_{ijk} are the second and third order derivatives of log-likelihood function and $\rho_i,\ i=1,\,2,\,3,\,4$ is the first order differentiation of log of prior with respect to α_1, α_2, β and λ.
Here,

$$a_i=\rho_1\sigma_{i1}+\rho_2\sigma_{i2}+\rho_3\sigma_{i3}+\rho_4\sigma_{i4},\ i=1,\,2,\,3,\,4.\,,$$

$$a_5=u_{12}\sigma_{12}+u_{13}\sigma_{13}+u_{14}\sigma_{14}+u_{23}\sigma_{23}+u_{24}\sigma_{24}\,,$$

$$a_6=\frac{1}{2}\left(u_{11}\sigma_{11}+u_{22}\sigma_{22}+u_{33}\sigma_{33}+u_{44}\sigma_{44}\right)\,,$$

$$\begin{aligned}A=&\sigma_{11}L_{111}+2\sigma_{12}L_{121}+2\sigma_{13}L_{131}+2\sigma_{14}L_{141}+2\sigma_{23}L_{231}+2\sigma_{24}L_{241}+\\&\sigma_{22}L_{221}+\sigma_{33}L_{331}+2\sigma_{34}L_{341}+\sigma_{44}L_{441}\,,\end{aligned}$$

$$\begin{aligned}B=&\sigma_{11}L_{112}+2\sigma_{12}L_{122}+2\sigma_{13}L_{132}+2\sigma_{14}L_{142}+2\sigma_{23}L_{232}+2\sigma_{24}L_{242}+\\&\sigma_{22}L_{222}+\sigma_{33}L_{332}+2\sigma_{34}L_{342}+\sigma_{44}L_{442}\,,\end{aligned}$$

$$\begin{aligned}C=&\sigma_{11}L_{113}+2\sigma_{12}L_{123}+2\sigma_{13}L_{133}+2\sigma_{14}L_{143}+2\sigma_{23}L_{233}+2\sigma_{24}L_{243}+\\&\sigma_{22}L_{223}+\sigma_{33}L_{333}+2\sigma_{34}L_{343}+\sigma_{44}L_{443}\,,\end{aligned}$$

and

$$\begin{aligned}D=&\sigma_{11}L_{114}+2\sigma_{12}L_{124}+2\sigma_{13}L_{134}+2\sigma_{14}L_{144}+2\sigma_{23}L_{234}+2\sigma_{24}L_{244}+\\&\sigma_{22}L_{224}+\sigma_{33}L_{334}+2\sigma_{34}L_{344}+\sigma_{44}L_{444}\,.\end{aligned}$$

According to our case,

$$\begin{aligned}\hat{R}_S=&u+\left(u_1a_1+u_2a_2+a_6\right)+\tfrac{1}{2}\left[A\left(u_1\sigma_{11}\right)+B\left(u_2\sigma_{22}\right)\right]+\\&\tfrac{1}{2}\left[C\left(u_1\sigma_{31}+u_2\sigma_{32}\right)+D\left(u_1\sigma_{41}+u_2\sigma_{42}\right)\right],\end{aligned} \tag{11}$$

where

$$a_1=\rho_1\sigma_{11}+\rho_3\sigma_{13}+\rho_4\sigma_{14},\ a_2=\rho_2\sigma_{22}+\rho_3\sigma_{23}+\rho_4\sigma_{24}$$

$$a_5=+u_{14}\sigma_{14}+u_{23}\sigma_{23}+u_{24}\sigma_{24}\,,$$

$$a_6 = \frac{1}{2}\left(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33} + u_{44}\sigma_{44}\right)\,,$$

$$A = \sigma_{11}L_{111} + \sigma_{22}L_{221} + \sigma_{33}L_{331} + 2\sigma_{34}L_{341} + \sigma_{44}L_{441},$$

$$B = \sigma_{22}L_{222} + \sigma_{33}L_{332} + 2\sigma_{34}L_{342} + \sigma_{44}L_{442}\,,$$

$$C = 2\sigma_{13}L_{133} + 2\sigma_{14}L_{143} + 2\sigma_{23}L_{233} + 2\sigma_{24}L_{243} + \sigma_{33}L_{333} + 2\sigma_{34}L_{343} + \sigma_{44}L_{443}\,,$$

$$D = 2\sigma_{13}L_{134} + 2\sigma_{14}L_{144} + 2\sigma_{23}L_{234} + 2\sigma_{24}L_{244} + \sigma_{33}L_{334} + 2\sigma_{34}L_{344} + \sigma_{44}L_{444}\,,$$

$$u = \frac{\hat{\alpha}_2}{\hat{\alpha}_1 + \hat{\alpha}_2},\ u_1 = \frac{-\hat{\alpha}_2}{\left(\hat{\alpha}_1 + \hat{\alpha}_2\right)^2},\ u_2 = \frac{\hat{\alpha}_1}{\left(\hat{\alpha}_1 + \hat{\alpha}_2\right)^2},\ u_3 = u_4 = 0,$$

$$u_{11} = \frac{2\hat{\alpha}_2}{\left(\hat{\alpha}_1 + \hat{\alpha}_2\right)^3},\ u_{22} = \frac{-2\hat{\alpha}_1}{\left(\hat{\alpha}_1 + \hat{\alpha}_2\right)^3},\ u_{12} = u_{21} = \frac{\hat{\alpha}_2 - \hat{\alpha}_1}{\left(\hat{\alpha}_1 + \hat{\alpha}_2\right)^3},$$

$$\rho_1 = \frac{c_1 - 1}{\hat{\alpha}_1} - d_1,\ \rho_2 = \frac{c_2 - 1}{\hat{\alpha}_2} - d_2,\ \rho_3 = \frac{c_3 - 1}{\hat{\beta}} - d_3,\ \rho_4 = \frac{c_4 - 1}{\hat{\lambda}} - d_4\,,$$

$$L_{11} = -\frac{n}{\hat{\alpha}_1^2},\ L_{22} = -\frac{m}{\hat{\alpha}_2^2},$$

$$L_{13} = L_{31} = -\sum_{i=1}^{n}\left(\frac{1}{\hat{\beta} + \hat{\lambda}x_i}\right) + \frac{n}{\hat{\beta}}\,,$$

$$L_{23} = L_{32} = -\sum_{j=1}^{m}\left(\frac{1}{\hat{\beta} + \hat{\lambda}y_j}\right) + \frac{m}{\hat{\beta}}\,,$$

$$L_{14} = L_{41} = -\sum_{i=1}^{n}\left(\frac{x_i}{\hat{\beta} + \hat{\lambda}x_i}\right)\,,$$

$$L_{24} = L_{42} = -\sum_{j=1}^{m}\left(\frac{y_j}{\hat{\beta} + \hat{\lambda}y_j}\right),$$

$$L_{33} = -\frac{(n\hat{\alpha}_1 + m\hat{\alpha}_2)}{\hat{\beta}^2} + (\hat{\alpha}_1 + 1)\sum_{i=1}^{n}\left(\frac{1}{\left(\hat{\beta} + \hat{\lambda}x_i\right)^2}\right) + (\hat{\alpha}_2 + 1)\sum_{j=1}^{m}\left(\frac{1}{\left(\hat{\beta} + \hat{\lambda}y_j\right)^2}\right),$$

$$L_{44} = -\frac{(n + m)}{\hat{\lambda}^2} + (\hat{\alpha}_1 + 1)\sum_{i=1}^{n}\left(\frac{x_i^2}{\left(\hat{\beta} + \hat{\lambda}x_i\right)^2}\right) + (\hat{\alpha}_2 + 1)\sum_{j=1}^{m}\left(\frac{y_j^2}{\left(\hat{\beta} + \hat{\lambda}y_j\right)^2}\right),$$

$$L_{34} = L_{43} = (\hat{\alpha}_1 + 1)\sum_{i=1}^{n}\left(\frac{x_i}{\left(\hat{\beta} + \hat{\lambda}x_i\right)^2}\right) + (\hat{\alpha}_2 + 1)\sum_{j=1}^{m}\left(\frac{y_j}{\left(\hat{\beta} + \hat{\lambda}y_j\right)^2}\right),$$

$$L_{111} = \frac{2n}{\hat{\alpha}_1^2},\ L_{222} = \frac{2m}{\alpha_2^2}\,,$$

$$L_{134} = L_{413} = L_{314} = L_{341} = L_{431} = L_{143} = \sum_{i=1}^{n}\left(\frac{x_i}{\left(\hat{\beta} + \hat{\lambda}x_i\right)^2}\right),$$

$$L_{244} = L_{424} = L_{442} = \sum_{j=1}^{m} \left(\frac{y_j^2}{\left(\hat{\beta} + \hat{\lambda} y_j\right)^2} \right) ,$$

$$L_{234} = L_{423} = L_{324} = L_{342} = L_{432} = L_{243} = \sum_{j=1}^{m} \left(\frac{y_j}{\left(\hat{\beta} + \hat{\lambda} y_j\right)^2} \right),$$

$$L_{144} = L_{414} = L_{441} = \sum_{i=1}^{n} \left(\frac{x_i^2}{\left(\hat{\beta} + \hat{\lambda} x_i\right)^2} \right) ,$$

$$L_{334} = L_{343} = L_{433} = -\left(\hat{\alpha}_1 + 1\right) \sum_{i=1}^{n} \left(\frac{x_i}{\left(\hat{\beta} + \hat{\lambda} x_i\right)^3} \right) - \left(\hat{\alpha}_2 + 1\right) \sum_{j=1}^{m} \left(\frac{y_j}{\left(\hat{\beta} + \hat{\lambda} y_j\right)^3} \right),$$

$$L_{344} = L_{434} = L_{443} = -\left(\hat{\alpha}_1 + 1\right) \sum_{i=1}^{n} \left(\frac{x_i^2}{\left(\hat{\beta} + \hat{\lambda} x_i\right)^3} \right) - \left(\hat{\alpha}_2 + 1\right) \sum_{j=1}^{m} \left(\frac{y_j^2}{\left(\hat{\beta} + \hat{\lambda} y_j\right)^3} \right),$$

$$L_{111} = \frac{2n}{\hat{\alpha}_1^2}, \;\; L_{222} = \frac{2m}{\hat{\alpha}_2^2},$$

$$L_{134} = L_{413} = L_{314} = L_{341} = L_{431} = L_{143} = \sum_{i=1}^{n} \left(\frac{x_i}{\left(\hat{\beta} + \hat{\lambda} x_i\right)^2} \right),$$

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$$L_{234} = L_{423} = L_{324} = L_{342} = L_{432} = L_{243} = \sum_{j=1}^{m} \left(\frac{y_j}{\left(\hat{\beta} + \hat{\lambda} y_j\right)^2} \right),$$

$$L_{144} = L_{414} = L_{441} = \sum_{i=1}^{n} \left(\frac{x_i^2}{\left(\hat{\beta} + \hat{\lambda} x_i\right)^2} \right) ,$$

$$L_{334} = L_{343} = L_{433} = -\left(\hat{\alpha}_1 + 1\right) \sum_{i=1}^{n} \left(\frac{x_i}{\left(\hat{\beta} + \hat{\lambda} x_i\right)^3} \right) - \left(\hat{\alpha}_2 + 1\right) \sum_{j=1}^{m} \left(\frac{y_j}{\left(\hat{\beta} + \hat{\lambda} y_j\right)^3} \right),$$

$$L_{344} = L_{434} = L_{443} = -\left(\hat{\alpha}_1 + 1\right) \sum_{i=1}^{n} \left(\frac{x_i^2}{\left(\hat{\beta} + \hat{\lambda} x_i\right)^3} \right) - \left(\hat{\alpha}_2 + 1\right) \sum_{j=1}^{m} \left(\frac{y_j^2}{\left(\hat{\beta} + \hat{\lambda} y_j\right)^3} \right),$$

$$L_{133} = L_{313} = L_{331} = \sum_{i=1}^{n} \left(\frac{1}{\left(\hat{\beta} + \hat{\lambda} x_i\right)^2} \right) - \frac{n}{\hat{\beta}^2} ,$$

$$L_{233} = L_{323} = L_{332} = \sum_{j=1}^{m} \left(\frac{1}{\left(\hat{\beta} + \hat{\lambda} y_j\right)^2} \right) - \frac{m}{\hat{\beta}^2},$$

$$L_{333} = \frac{2\left(n\hat{\alpha}_1 + m\hat{\alpha}_2\right)}{\hat{\beta}^3} - 2\left(\hat{\alpha}_1 + 1\right) \sum_{i=1}^{n} \left(\frac{1}{\left(\hat{\beta} + \hat{\lambda} x_i\right)^3} \right) - 2\left(\hat{\alpha}_2 + 1\right) \sum_{j=1}^{m} \left(\frac{1}{\left(\hat{\beta} + \hat{\lambda} y_j\right)^3} \right),$$

and

$$L_{444} = \frac{2\left(n + m\right)}{\hat{\lambda}^3} - \left(\hat{\alpha}_1 + 1\right) \sum_{i=1}^{n} \left(\frac{2x_i^3}{\left(\hat{\beta} + \hat{\lambda} x_i\right)^3} \right) - \left(\hat{\alpha}_2 + 1\right) \sum_{j=1}^{m} \left(\frac{2y_j^3}{\left(\hat{\beta} + \hat{\lambda} y_j\right)^3} \right),$$

Under linex loss function,

$$\hat{R}_L = -\frac{1}{\delta} \log \left(\begin{array}{l} v + \left(v_1 a_1 + v_2 a_2 + a_6\right) + \frac{1}{2}\left[A\left(v_1 \sigma_{11}\right) + B\left(v_2 \sigma_{22}\right)\right] + \\ \frac{1}{2}\left[C\left(v_1 \sigma_{31} + v_2 \sigma_{32}\right) + D\left(v_1 \sigma_{41} + v_2 \sigma_{42}\right)\right] \end{array} \right), \tag{12}$$

where

$$v = \exp\left(-\delta \frac{\hat{\alpha}_2}{\hat{\alpha}_1 + \hat{\alpha}_2}\right),\ v_1 = \delta \exp\left(-\delta \frac{\hat{\alpha}_2}{\hat{\alpha}_1 + \hat{\alpha}_2}\right) \frac{\hat{\alpha}_2}{\left(\hat{\alpha}_1 + \hat{\alpha}_2\right)^2},$$

$$v_2 = -\delta \exp\left(-\delta \frac{\hat{\alpha}_2}{\hat{\alpha}_1 + \hat{\alpha}_2}\right) \frac{\hat{\alpha}_1}{\left(\hat{\alpha}_1 + \hat{\alpha}_2\right)^2},$$

$$v_{11} = -\delta \hat{\alpha}_2 \exp\left(-\delta \frac{\hat{\alpha}_2}{\hat{\alpha}_1 + \hat{\alpha}_2}\right) \left[\frac{\hat{\alpha}_2\left(\delta + 2\right) + 2\hat{\alpha}_1}{\left(\hat{\alpha}_1 + \hat{\alpha}_2\right)^4}\right],$$

$$v_{12} = v_{21} = \delta \exp\left(-\delta \frac{\hat{\alpha}_2}{\hat{\alpha}_1 + \hat{\alpha}_2}\right) \left[\frac{\hat{\alpha}_1^2 - \hat{\alpha}_2^2 - \delta \hat{\alpha}_1 \hat{\alpha}_2}{\left(\hat{\alpha}_1 + \hat{\alpha}_2\right)^4}\right]$$

and

$$v_{22} = \delta \hat{\alpha}_1 \exp\left(-\delta \frac{\hat{\alpha}_2}{\hat{\alpha}_1 + \hat{\alpha}_2}\right) \left[\frac{\hat{\alpha}_1\left(\delta + 2\right) + 2\hat{\alpha}_2}{\left(\hat{\alpha}_1 + \hat{\alpha}_2\right)^4}\right].$$

5. Real data analysis

In this section, two real data sets are analysed to illustrate the proposed estimation methods. These data sets are initially used by Nelson (1982). The data sets represent times to breakdown of an insulating fluid between electrodes at different voltage. The failure times (in minutes) for an insulating fluid between two electrodes subject to a voltage of 34kV and 36kV are presented as data set 1 and data set 2, respectively.
Data set 1 (X): 0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, 72.89.
Data set 2 (Y): 0.35, 0.59, 0.96, 0.99, 1.69, 1.97, 2.07, 2.58, 2.71, 2.90, 3.67, 3.99, 5.35, 13.77, 25.50.

To check the fitness for the two data sets, -logL, Akaike information criteria (AIC), Bayesian information criteria (BIC), Akaike information criteria corrected (AICc), Kolmogrov-smirnov (K-S) and Anderson-Darling (A-D) statistics with corresponding p-values are computed and the results for both data sets are given in the Tables below.

Table 1: Estimates of the parameters with corresponding standard error and the values of -logL, AIC, AICc, BIC, K-S and A-D statistics for different distributions for data set 1

Name of the distribution	Estimates of parameters	-logL	AIC	AICc	BIC	$K-S$ $(p-value)$	$A-D$ $(p-value)$
Lomax Exponential	$\alpha = 2.0302(1.6632)$ $\beta = 5.1863(11.6778)$ $\lambda = 0.3101(0.8155)$	68.4234	140.8468	142.4468	143.683	0.12654 (0.5029)	0.32203 (0.9198)
Weibull	$\alpha = 0.7956(0.1561)$ $\beta = 0.1752(0.0380)$	69.1296	142.2592	143.0092	144.1481	0.1613 (0.336)	0.3918 (0.8552)
Exponentiated exponential	$\alpha = 0.6825(0.1941)$ $\beta = 0.0535(0.0180)$	69.3980	142.796	143.546	144.684	0.1886 (0.2292)	0.5057 (0.7388)

Table 2: Estimates of the parameters with corresponding standard error and the values of -logL, AIC, AICc, BIC, K-S and A-D statistics for different distributions for data set 2

Name of the distribution	Estimates of parameters	-logL	AIC	AICc	BIC	$K-S$ $(p-value)$	$A-D$ $(p-value)$
Lomax Exponential	$\alpha = 3.0369(2.889)$ $\beta = 8.5039(3.124)$ $\lambda = 0.8991(1.182)$	36.9792	79.9583	81.5584	77.794	0.14339 (0.4938)	0.45395 (0.7915)
Weibull	$\alpha = 0.8891(0.1635)$ $\beta = 0.2738(0.1151)$	38.0125	81.3828	83.5646	80.7989	0.1917 (0.2941)	0.6271 (0.6199)
Exponentiated exponential	$\alpha = 1.9271(0.5985)$ $\beta = 0.3875(0.0954)$	41.4606	86.9212	87.6712	88.810	0.1979 (0.2724)	1.3637 (0.2125)

The estimate of reliability using MLE is 0.7042. Bayes estimates under different priors and loss functions are presented in Table 3.

Table 3: MLE and the Bayes estimates under different loss functions with different priors.

		Bayes estimaes								
		Non informative prior			Gamma prior					
	MLE	$\hat{R}_S$	$\hat{R}_L$	$\hat{R}_{L1}$	Prior 1			Prior 2		
					$\hat{R}_S$	$\hat{R}_L$	$\hat{R}_{L1}$	$\hat{R}_S$	$\hat{R}_L$	$\hat{R}_{L1}$
λ unknown	0.7042	0.7248	0.7247	0.7248	0.7312	0.7312	0.7312	0.7489	0.7591	0.7467
λ known	0.7046	0.7253	0.7252	0.7253	0.7343	0.7344	0.7343	0.8528	0.8583	0.8477

$\hat{R}_{L1}$ refers to linex loss function with loss parameter $\delta = -0.5$.

6. Simulation study

A simulation study of 10000 observations is conducted by generating samples of different sizes such as (n, m) = (5, 5), (5, 10), (10, 10), (10, 15), (20, 20), (20, 25), (30, 30), (30, 35) and (40, 40). The true values of R which are considered under simulation study are 0.57142 and 0.47058. The parameter values of the prior distribution for squared error and linex loss functions are $c_1 = 1$, $d_1 = 0.8$, $c_2 = 2$, $d_2 = 0.4$, $c_3 = 5$, $d_3 = 0.2$, $c_4 = 1$, $d_4 = 2$ (prior1) and $c_1 = 4$, $d_1 = 3$, $c_2 = 3$, $d_2 = 0.9$, $c_3 = 5$, $d_3 = 5$, $c_4 = 1$ and $d_4 = 2$ (prior 2). The values of loss parameters under linex loss function are 0.5 and -0.5. The proposed estimators are compared using mean squared error (MSE) criteria. The MLEs and Bayes estimates with corresponding MSEs are given in the Tables given in annexure.

7. Conclusions

The estimation of stress-strength reliability (R) is considered, when stress and strength variables follow LED. The maximum likelihood and Bayesian estimation methods are used to estimate stress-strength reliability. MLEs are derived. Bayes estimators under different loss functions such as squared error and linex loss functions with gamma and non-informative priors are obtained. The Lindley's approximation technique is used to approximate the Bayes estimator of R. The real data analysis is conducted to illustrate the developed estimation procedures. A simulation experiment is conducted to study the performance of estimators which are derived in the paper and it reveals that Bayes estimator with non-informative prior is better when compared to MLEs. However, the Bayes estimator with gamma prior is better than that the non-informative prior. The gamma prior under linex loss function is better than the squared error loss function. Specially, the linex with loss parameter -0.5 is better than all.

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ANNEXURE

Table 4: The maximum likelihood and Bayes estimates of R with corresponding MSEs, when R $= 0.47058$, $\alpha_1 = 4.5$, $\alpha_2 = 4$, $\beta = 0.25$, $\lambda = 1$, $c_1 = 1$, $d_1 = 0.8, c_2 = 2$, $d_2 = 0.4$, $c_3 = 5$, $d_3 = 0.2$, $c_4 = 1$ *and* $d_4 = 2$

Sample Size n, m	$\hat{R}$	Bayes estimates					
		Non-informative prior			Gamma prior		
		$\hat{R}_S$	$\hat{R}_L$	$\hat{R}_{L1}$	$\hat{R}_S$	$\hat{R}_L$	$\hat{R}_{L1}$
	0.475778 (0.00935)	0.476434 (0.00921)	0.476434 (0.00921)	0.476432 (0.00921)	0.469151 (0.01326)	0.469537 (0.01319)	0.468764 (0.01333)
	0.484687 (0.00761)	0.484751 (0.00759)	0.484751 (0.00758)	0.484752 (0.00756)	0.498398 (0.00568)	0.498469 (0.00567)	0.498328 (0.00568)
	0.509822 (0.00386)	0.518294 (0.00289)	0.518835 (0.00283)	0.517758 (0.00294)	0.627643 (0.00342)	0.631826 (0.00392)	0.623797 (0.00298)
	0.491192 (0.00646)	0.49118 (0.00645)	0.49119 (0.00646)	0.49128 (0.00648)	0.502418 (0.00482)	0.502452 (0.004812)	0.502386 (0.00482)
	0.499716 (0.00516)	0.502127 (0.004826)	0.501969 (0.004848)	0.501969 (0.00484)	0.5360198 (0.001302)	0.536513 (0.00127)	0.535539 (0.001336)
	0.493795 (0.00604)	0.493784 (0.006042)	0.493785 (0.006043)	0.493784 (0.006042)	0.502259 (0.00481)	0.502278 (0.00485)	0.502241 (0.004810)
	0.498063 (0.00539)	0.499178 (0.005233)	0.499253 (0.005222)	0.499104 (0.005243)	0.517388 (0.002941)	0.513875 (0.002923)	0.517221 (0.002959)
	0.495227 (0.005814)	0.495219 (0.005815)	0.495217 (0.005816)	0.494567 (0.005834)	0.501950 (0.002838)	0.501961 (0.002837)	501939 (0.002840)
	0.497783 (0.005431)	0.498424 (0.005337)	0.498467 (0.005331)	0.49838 (0.005343)	0.510512 (0.002721)	0.510596 (0.002711)	0.510428 (0.002731)

Table 5: The maximum likelihood and Bayes estimates of R with corresponding MSEs, when R $= 0.47058$, $\alpha_1 = 4.5$, $\alpha_2 = 4$, $\beta = 0.25$, $\lambda = 1$, $c_1 = 4$, $d_1 = 3$,$c_2 = 3$, $d_2 = 0.9$, $c_3 = 5$, $d_3 = 5$, $c_4 = 1$ *and* $d_4 = 2$

Bayes estimator						
Gamma prior						
Samplesize n, m	$\hat{R}_S$	$MSE\left(\hat{R}_S\right)$	$\hat{R}_L$	$MSE\left(\hat{R}_L\right)$	$\hat{R}_{L1}$	$MSE\left(\hat{R}_{L1}\right)$
5, 5	0.497594	(0.001201)	0.497486	(0.001202)	0.497486	(0.001199)
10,10	0.496694	(0.0006817)	0.496698	(0.0006819)	0.4966903	(0.0006815)
20, 20	0.495962	(0.0006442)	0.495967	(0.0006445)	0.4959559	(0.0006439)
30, 30	0.493617	(0.000531)	0.493629	(0.000532)	0.493605	(0.0005305)
40, 40	0.491194	(0.000427)	0.491219	(0.000429)	0.49117	(0.000426)
50, 50	0.490183	(0.000419)	0.491835	(0.000422)	0.490092	(0.000418)

Table 6: The maximum likelihood and Bayes estimates of R with corresponding MSEs, when R $= 0.57142$, $\alpha_1 = 1.5$, $\alpha_2 = 2$, $\beta = 0.05$, $\lambda = 1$, $c_1 = 1.5$, $d_1 = 0.8$, $c_2 = 2$, $d_2 = 0.4$, $c_3 = 5$, $d_3 = 0.2$, $c_4 = 1$ *and* $d_4 = 2$

Sample Size n, m	$\hat{R}$	Bayes estimates					
		Non-informative prior			Gamma prior		
		$\hat{R}_S$	$\hat{R}_L$	$\hat{R}_{L1}$	$\hat{R}_S$	$\hat{R}_L$	$\hat{R}_{L1}$
	0.512716 (0.008113)	0.508545 (0.001532)	0.608551 (0.001531)	0.608536 (0.001532)	0.608231 (0.001415)	0.608375 (0.001414)	0.608084 (0.001416)
	0.598296 (0.001438)	0.594557 (0.001177)	0.594561 (0.001178)	0.594554 (0.001176)	0.592429 (0.000554)	0.592497 (0.000556)	0.592359 (0.000552)
	0.584903 (0.001183)	0.582614 (0.001032)	0.582616 (0.001032)	0.582613 (0.001031)	0.584997 (0.000508)	0.585012 (0.000509)	0.584972 (0.000507)
	0.585307 (0.000615)	0.584972 (0.000230)	0.581633 (0.000226)	0.581278 (0.000234)	0.580119 (0.000139)	0.580272 (0.000136)	0.581965 (0.000123)
	0.579466 (0.001084)	0.57843 (0.00098)	0.571844 (0.000973)	0.571843 (0.000981)	0.566447 (0.000118)	0.56646 (0.000125)	0.466435 (0.000111)
	0.578644 (0.000789)	0.573740 (0.000538)	0.573672 (0.000536)	0.573808 (0.000542)	0.572867 (0.000110)	0.573873 (0.000111)	0.573863 (0.000109)
	0.576447 (0.000671)	0.571814 (0.000411)	0.572679 (0.000407)	0.565506 (0.000415)	0.569978 (0.000040)	0.569784 (0.000039)	0.569989 (0.000040)

Table 7: The maximum likelihood and Bayes estimates of R with corresponding MSEs, when R $= 0.57142$, $\alpha_1 = 1.5$, $\alpha_2 = 2$, $\beta = 0.05$, $\lambda = 1$, $c_1 = 4$, $d_1 = 3$,$c_2 = 3$, $d_2 = 0.9$, $c_3 = 5$, $d_3 = 5$, $c_4 = 1$ *and* $d_4 = 2$

Bayes estimator						
Gamma prior						
$Samplesize$ n, m	$\hat{R}_S$	$MSE\left(\hat{R}_S\right)$	$\hat{R}_L$	$MSE\left(\hat{R}_L\right)$	$\hat{R}_{L1}$	$MSE\left(\hat{R}_{L1}\right)$
5, 5	0.614573	(0.001506)	0.614682	(0.001492)	0.614325	(0.001485)
10,10	0.591334	(0.0008835)	0.596698	(0.0008829)	0.5966903	(0.0008821)
20, 20	0.583426	(0.0007566)	0.582466	(0.0007545)	0.581345	(0.0007536)
30, 30	0.581215	(0.0005104)	0.581103	(0.0005102)	0.579996	(0.0005009)
40, 40	0.579855	(0.0004551)	0.578847	(0.0004545)	0.577634	(0.0004495)
50, 50	0.574673	(0.0002486)	0.573321	(0.0002465)	0.572589	(0.0002355)