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Survey of C.R. Rao's Orthogonal Arrays, Balanced Arrays, and Their Applications

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Abstract

This comprehensive review article on orthogonal arrays (OAs), balanced arrays (BAs) and their practical applications serves as a tribute to the life and ground breaking contributions of the legendary statistician, C.R. Rao (1920-2023). It highlights his profound influence on the field of statistical sciences and explores the significant contributions he made to the realms of OAs and BAs. His work in these areas has left an indelible impact on the domains of experimental design, combinatorial mathematics, and statistical analysis. In this article, we delve into some noteworthy applications of OAs and BAs.

Key words: Orthogonal array; Balanced array; Mixed orthogonal array; Balanced incomplete block design; Partially balanced incomplete block design; Association scheme.

AMS Subject Classifications: 62K05, 05B05

1. Introduction

The foundation for the concepts of Latin squares and mutually orthogonal Latin squares was laid in the early 20th century. Later, these foundational ideas were expanded and generalized to include Latin cubes and hypercubes, as well as orthogonal Latin cubes and hypercubes (*cf.* Kishen (1942, 1949)). These developments marked significant progress in the field of experimental design, as they allowed for the exploration of more complex experimental scenarios with multiple factors and levels.

Rao (1946) further extended these concepts by introducing the notion of arrays with a specific strength. These arrays became a versatile tool for designing experiments with various factors, enabling researchers to investigate complex relationships and interactions efficiently. The pivotal moment in the evolution of these ideas came when Rao (1947) introduced the concept of OAs as a unifying framework that generalized and brought together the previously mentioned structures. This marked a significant leap in the field of experimental design and made it more accessible to practitioners in diverse fields.

Furthermore, Rao (1973) continued to expand his contributions by generalizing OAs to mixed orthogonal arrays (MOAs) of strength d. This development allowed researchers

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to work with experiments that involved factors at different levels, thus accommodating a broader range of real world scenarios.

The evolution of experimental design, from the early concepts of Latin squares to the sophisticated OAs, represents a remarkable journey of continuous innovation and generalization. C. R. Rao's pioneering work (1946, 1947, 1949, 1961, 1973) has been instrumental in this evolution, establishing these arrays as fundamental tools in experimental design. Rao's contributions have proven invaluable in both industrial and scientific research by enabling highly efficient experiments that require fewer runs. His brilliance is further exemplified by expanding OAs into higher dimensions, thus broadening their applicability across diverse experimental settings. This expansion has notably enhanced the efficiency of experimentation and optimization in various industries, including manufacturing and quality control. Taguchi's work in the 1980s popularized the use of OAs in industry, known as Taguchi methods, which determine optimum combinations of factors to achieve high output and robustness to environmental changes. An article in Forbes Magazine (March 11, 1996, pp. 114-118) highlighted the significance of OAs, dubbing them a "New Mantra" in various U.S. industrial establishments. This recognition underscores the practical utility of OAs in enhancing efficiency and reducing the number of experimental runs required in industrial research. Beyond industry, OAs have also made profound impacts in agricultural and medical sciences, as discussed by Parsad, Gupta, and Gopinath (2020). Additionally, OAs find applications in coding theory, cryptography, and computer experiments. Comprehensive textbooks on this subject, authored by Dey and Mukerjee (1999), Hedayat, Sloane, and Stufken (1999), and Rosa (2017), provide an extensive exploration of these powerful tools, cementing their place as indispensable resources in the realm of experimental design.

BAs, stemming from the foundational work on partially balanced arrays by Chakravarti (1956, 1961) and further advanced by Srivastava and Chopra (1973), epitomize a sophisticated concept within experimental design. Initially termed as partially balanced arrays, the pioneering research by Chakravarti (1956) laid the groundwork for their exploration. Building upon this foundation, Srivastava and Chopra (1973) made significant strides, advocating for the simplification of the term to "balanced arrays", a change we have embraced. This evolution represents a pivotal moment in the realm of experimental design and statistical methodologies. BAs provide a structured and efficient means to investigate the intricate relationships among multiple factors and their respective levels. By systematically varying factors while minimizing confounding effects, these arrays offer a robust framework for achieving statistical efficiency. In essence, they stand as a testament to the ongoing advancement of experimental design, empowering researchers to uncover insights with clarity and precision.

Within the broader framework of OAs, BAs emerge as a noteworthy and important subset. OAs are a specialized type of BAs. They hold a unique and powerful position in the field of experimental design, as they are specifically designed to ensure that the effects of different factors do not interfere with each other. In other words, OAs allow researchers to explore and quantify the impact of various factors on the outcome of interest without undue influence from unrelated factors. In essence, OAs and BAs are intertwined components of experimental design, with BAs serving as a foundational concept and OAs as a refined and focused tool within this framework. Together, they provide researchers with a comprehensive toolkit to design and execute experiments effectively, ensuring that the results obtained are both reliable and interpretable.

In this review article, we provide definitions of OAs, MOAs, and BAs in Section 2. We focus on the construction of incomplete block designs from BAs and OAs in Section 3. Section 4 discusses the construction of optimum chemical balance weighing designs from BAs. Section 5 covers the construction of second order rotatable designs (SORDs) using BAs. In Section 6, we discuss some methods for constructing BAs. Section 7 considers the construction of orthogonal resolution plans and fractional factorial plans using OAs. Section 8 addresses the application of OAs in Taguchi methods. Section 9 explores other diverse applications of OAs and MOAs in modern research and experimentation. Finally, Section 10 presents the conclusion.

2. Overview of OAs and BAs

In this section, we provide a comprehensive overview of OAs and BAs. While these concepts might be familiar to the audience of this special issue, we briefly revisit them for the sake of completeness and to ensure smooth reading.

2.1. OA

OAs are mathematical structures extensively used in experimental design, coding theory, and quality engineering. They facilitate the systematic testing of different combinations of variables while minimizing the number of experimental runs required.

Definition 1: Consider an array $\mathbf{A}$ of size $k \times N$, where its elements are drawn from a set S comprising s symbols or levels, denoted by $0, 1, \ldots, s-1$. This array $\mathbf{A}$ is termed an OA possessing s levels, with a strength of t, and an index denoted by λ, under the condition that each $t \times N$ subarray within $\mathbf{A}$ contains every t-tuple derived from S precisely λ times as a column.

We denote such an array by OA(N, s^k, t). Clearly, $N = \lambda s^t$.

Definition 2: A MOA OA$(N, s_1^{k_1} s_2^{k_2} \ldots s_v^{k_v}, t)$ is an array of size $k \times N$, where $k = k_1 + k_2 + ... + k_v$ is the total number of factors, in which the first k_1 rows have symbols from $\{0, 1, \ldots, s_1 - 1\}$, the next k_2 rows have symbols from $\{0, 1, \ldots, s_2 - 1\}$, and so on. The array has the property that in any $t \times N$ subarray, every possible t-tuple occurs an equal number of times as a column. Of course, if all s_i's are equal, we get the usual OA(N, s^k, t) as of Definition 1.

For further understanding, readers may refer to authoritative textbooks by Raghavarao (1971), Dey and Mukerjee (1999), Hedayat *et al.* (1999) and Rosa (2017). Additionally, valuable insights can be gained from online resources such as TS-DOC: TS-723 - OAs by WF Kuhfeld, OA testing on Wikipedia, and the design resources server of the Indian Agricultural Statistics Research Institute (IASRI). N. J. A. Sloane's "A Library of OAs" also provides comprehensive information. Furthermore, important references, including works by Bose (1950), Bose and Bush (1952), Bush (1952), Cheng (1980), and Mukhopadhyay (1981), offer deeper insights into the topic.

2.2. BA

BAs are similar to OAs but are specifically designed for applications that require balanced representation across various combinations. This characteristic makes them particularly valuable in the field of combinatorial design.

Definition 3: Let $\mathbf{A}$ be an $k \times N$ array with elements 0,1,2,..., $s-1$. Consider the s^t possible vectors $\mathbf{X}' = (x_1, x_2, \ldots, x_t)$, where each x_i can take any value from $\{0, 1, \ldots, s-1\}$ for $i = 1, 2, \ldots, t$. Associate with each $t \times 1$ vector $\mathbf{X}$ a positive integer $\lambda(x_1, x_2, \ldots, x_t)$, which remains unchanged under permutations of $(x_1, x_2, \ldots, x_t)$. If for every t rowed subarray of $\mathbf{A}$, the s^t distinct $t \times 1$ vectors $\mathbf{X}$ appear as columns exactly $\lambda(x_1, x_2, \ldots, x_t)$ times, then the array $\mathbf{A}$ is called a BA of strength t in N assemblies, with m constraints, s symbols, and the specified $\lambda(x_1, x_2, \ldots, x_t)$ parameters.

It is to be noted that if $\lambda(x_1, x_2, \ldots, x_t) = \lambda$ for all $(x_1, x_2, \ldots, x_t)$, then $\mathbf{A}$ is called an OA of index λ.

The literature on this topic is extensive, making it challenging to cite every relevant work. Therefore, we reference a selection of seminal articles from the early stages, including those by Chakravarti (1956, 1961), Srivastava and Chopra (1973), Rafter and Seiden (1974), and Saha (1981).

3. Constructing incomplete block designs with BAs and OAs

A methodology emerges for constructing BAs, employing the Kronecker product applied to two BAs. This approach leads to the derivation of six distinct balanced incomplete block designs (BIBDs) from a given symmetric balanced incomplete block design (SBIBD). Notably, the method involves operations such as unions, intersections, and difference sets on pairs of blocks of an SBIBD and their complementary designs. Significantly, certain newly generated BIBDs fulfill the minimum replication requirements for specified parameters like v (number of varieties or treatments) and k (block size), showcasing the method's efficiency and efficacy. Expanding beyond its original scope, the study suggests broader applications for this method. It proposes leveraging the new series of SBIBDs to derive additional series of BIBDs, hinting at the potential for an iterative process where new designs build upon established ones, thus enriching the repertoire of available BIBDs. Independent confirmations by Vanstone (1975) and Majindar (1978) regarding the existence of the six BIBDs corresponding to an SBIBD reinforce the method's validity and reliability.

In Saha (1975), the tactical configurations (or t designs) are generalized to G systems of order β, and their equivalence to 2 symbol BAs of strength β is established. This extension confirms their equivalence to 2 symbol BAs of strength β. These findings are then applied to demonstrate that when β is even, $\mathbf{A} \cup \mathbf{A}^c$ yields another 2 symbol BA of strength $\beta + 1$, where $\mathbf{A}$ is a 2 symbol BA of strength β, and $\mathbf{A}^c$ is the complementary array (obtained from $\mathbf{A}$ by interchanging 0s and 1s). This holds true for 2 symbol OAs of strength β when β is even as well. Furthermore, the research identified specific series of 2 symbol OAs with a strength of three, which were obtained from carefully selected 2 symbol BAs with a strength of two. Saha (1975) demonstrated how tactical configurations (t designs) generalize to G systems of order β and established their equivalence to 2 symbol BAs of strength β. It also sheds light on the behavior of combining such arrays and provides insights into obtaining series of 2 symbol OAs with a higher strength from BAs.

Building upon these foundational works, Saha *et al.* (1985) extend the method to generate s symbol BAs with a strength of t. This advancement is then applied to derive diverse partially balanced incomplete block designs (PBIBDs) characterized by m associate classes. Additionally, their research reveals the coexistence of six distinct series of PBIBDs alongside a linked block PBIBD, showcasing the versatility of the method in addressing various experimental design needs and its significant contributions to advancing the field.

For detailed definitions and further reading on BIBD, SBIBD, PBIBD, and association schemes, several excellent textbooks are available, with Raghavarao (1971) being particularly recommended.

4. Optimal chemical balance weighing designs from BAs and BIBDs

Optimal chemical balance weighing designs are experimental frameworks used in chemical experiments to measure the weights of multiple substances simultaneously with high accuracy. These designs aim to minimize the variance of the estimated weights, ensuring precise and unbiased measurements. The key characteristics of optimal chemical balance weighing designs include efficiency, as they maximize the information obtained from a limited number of weighings; balance, by distributing errors evenly across all measurements to reduce systematic biases; and replication, through repeated measurements to enhance reliability. Additionally, these designs often employ combinatorial structures like BAs and BIBDs to systematically arrange substances on the balance, optimizing the weighing process. In essence, these designs provide a structured approach to achieving high precision and minimal error in the measurement of multiple substances. For further reading, refer to Raghavarao (1971), Silvey (1980), Shah and Sinha (1989) and Pukelsheim (1993).

Dey (1971), Saha (1975), Kageyama and Saha (1983), along with other researchers, initially showcased the derivation of optimal chemical balance weighing designs from the incidence matrices of BIBDs.

Dey (1971) utilized the incidence matrices of BIBDs and balanced ternary designs for constructing optimal chemical balance weighing designs.

Regarding the relationship between BIBDs and optimum chemical balance weighing designs, Saha (1975) proved two significant theorems:

Theorem 1: The existence of a BIBD with parameters v, b, r, k, λ satisfying $b \leq 4(r-\lambda)$ implies the existence of an optimum chemical balance weighing design for v objects in $4(r-\lambda)$ weighings.

Theorem 2: The existence of an affine resolvable BIBD with parameters $v, b = 2r, r, k, \lambda$ implies the existence of an optimum chemical balance weighing design for r objects in v weighings.

Kageyama and Saha (1983) investigated a BIBD with parameters v, b, r, k, λ satisfying $b \leq 4(r-\lambda)$ and tabulated the parameters (in the practical range) of BIBDs which validated the above theorems of Saha (1983).

Expanding upon this foundation, Saha and Kageyama (1984) further developed the methodology by illustrating that optimum weighing designs could also be derived from carefully selected two symbol BAs of strength two. Importantly, these arrays were not limited to being incidence matrices of BIBDs, thus widening the potential design scope. To implement this approach, the first step involves identifying two symbol BAs of strength two with desired properties for optimum chemical balance weighing design construction. These arrays must meet specific criteria to ensure suitability. By leveraging the identified arrays, the optimum chemical balance weighing designs can be generated, utilizing the array's structure and properties. This process involves transforming the array into a design that meets the requirements for the optimum chemical balance weighing. Thus, the findings lead us to construct new optimum chemical balance weighing designs other than the above mentioned methods. This research has far reaching implications for the optimum design of chemical balance experiments and provides a more flexible and versatile framework for developing such designs beyond the limitations of traditional BIBDs.

5. Constructing SORDs using BAs for response surface studies

SORDs are a type of experimental design used primarily in response surface methodology (RSM) to model and optimize processes. These designs are particularly effective when the relationship between the factors and the response variable is quadratic. They accommodate a second order (quadratic) polynomial model, encompassing linear, interaction, and squared terms of the input variables. A design is considered rotatable if the variance of the predicted response at any point depends solely on the distance from the design center, rather than the direction, thereby ensuring uniform precision of prediction at all equidistant points from the center. The most common type of SORD is the central composite design (CCD), which combines a factorial or fractional factorial design with center points and axial (or star) points to estimate curvature. These designs efficiently estimate the coefficients of a second order polynomial, enabling the detection of curvature in the response surface and the identification of optimal conditions. Their flexibility and efficiency in handling multiple factors make SORDs indispensable tools in industrial and scientific research for process optimization. For further details, we refer to Khuri and Cornell (1996).

The integration of BAs has significantly expanded the toolkit available to researchers involved in designing SORD for analyzing response surfaces. A pivotal contribution to this field was made by Das and Saha (1973), who demonstrated the successful construction of SORDs under specific conditions. They outlined requirements for 2 symbol BAs of strength two, which enabled the creation of 4 level SORDs. Leveraging these principles, they introduced several novel series of 4 level SORDs. Notably, they uncovered an intriguing finding: a 4 level SORD can be derived for (i) $v - x$ factors from b magnitude sets, and (ii) v factors from $b + c$ magnitude sets, from a BIBD meeting certain criteria, such as $r > 3\lambda$, or $5r - 2b - 3\lambda > 0$ $(x > 0, c > 0)$. Furthermore, these designs can be augmented by selecting appropriate magnitude sets in addition to those derived from the incidence matrices of BIBDs.

Such designs present researchers with a flexible and adaptable framework, facilitating the conduct of response surface experiments and providing a nuanced exploration into the behavior of intricate systems.

6. Construction of BAs

Association schemes with a large number of associate classes have historically been investigated primarily for their combinatorial significance, without a focus on their application in the development of practical experimental designs. However, Saha (1981) introduced a novel approach by utilizing a new class of cyclic association scheme with m associate classes, referred to as $N\mathrm{C}_m$ association scheme. This approach was employed to construct $(m+1)$ symbol BAs of strength two. The resulting BIBDs derived from these arrays were also explored in the same paper.

In a more recent study by Yonglin (2004), association schemes have been employed to investigate their relationship with OAs and frequency squares, which represent a generalization of Latin squares. This research demonstrates the evolving and diverse applications of association schemes in combinatorial design theory, shedding light on their connection to other fundamental structures and concepts.

Researchers made notable contributions for constructing BAs with a strength of two from block designs. For instance, Sinha *et al.* (2002) achieved this by using various types of block designs, including (i) rectangular designs; (ii) group divisible designs; (iii) nested balanced incomplete block designs. These constructions result in BAs, which are useful in experimental design and combinatorial applications.

Balanced nested designs share intricate connections with other combinatorial structures like BAs and balanced n-ary designs. Specifically, the presence of symmetric balanced nested designs mirrors the existence of certain BAs. Delving into this relationship, Fuji-Hara *et al.* (2002) conducted a comprehensive exploration of balanced nested designs. They focused on elucidating the interplay between balanced nested designs and BAs with a strength of two, offering diverse constructions for symmetric balanced nested designs. These constructions proved instrumental in delineating the spectrum of symmetric balanced nested incomplete block designs with block sizes of 3 and 4. Notably, their research unveiled the equivalence between symmetric balanced nested designs and specific categories of BAs. Beyond enriching our understanding of BAs, their work provided invaluable insights into constructing symmetric balanced nested designs, thereby advancing the broader field of combinatorial design theory.

7. Orthogonal resolution and fractional factorial plans with OAs

Orthogonal resolution plans and fractional factorial plans are two types of experimental designs commonly employed in industrial and scientific research to efficiently explore the effects of multiple factors on a response variable while minimizing the number of experimental runs needed. Orthogonal resolution plans are characterized by their ability to provide unbiased estimates of main effects and interactions between factors, even in the presence of confounding. These plans achieve orthogonality by ensuring that each factor is varied independently of the others at different levels, thereby allowing for the unambiguous identification of the effects of individual factors. Additionally, orthogonal resolution plans are designed to have certain desirable properties such as clear aliasing patterns, which aid in the interpretation of results. On the other hand, fractional factorial plans are a subset of orthogonal resolution plans that further reduce the number of experimental runs by systematically selecting a fraction of the total number of possible treatment combinations. Despite this reduction in the number of experimental runs, fractional factorial plans retain the ability to estimate main effects and selected interactions with minimal loss of information. These plans are particularly useful when the number of factors under investigation is large and conducting a full factorial experiment would be impractical or prohibitively expensive. Overall, orthogonal resolution plans and fractional factorial plans are valuable tools in experimental design, offering efficient and cost effective approaches to exploring complex systems and optimizing processes in various fields.

OAs play a crucial role in the construction of orthogonal resolution plans and subclasses of fractional factorial plans, which are essential for optimizing experimental efficiency and reliability. In orthogonal resolution plans, OAs help organize experiments to ensure clarity and precision in identifying the effects of different factors by minimizing confounding and enhancing the interpretability of results. These plans are categorized by their resolution, with higher resolutions indicating clearer distinctions between main effects and interactions. OAs also aid in constructing fractional factorial plans, which allow researchers to study the most significant factors and interactions using a fraction of the total runs required in a full factorial design. This systematic approach significantly reduces the number of experimental runs needed, saving time and resources while maintaining experimental integrity. By ensuring balanced representation and systematic variation of factor levels, OAs enhance the efficiency and robustness of experimental designs, making them indispensable tools across various scientific and industrial fields. An excellent textbook in this area is authored by Dey and Mukerjee (1999), offering comprehensive insights into the construction and application of OAs and fractional factorial designs in experimental design.

8. Application of OAs in Taguchi methods

Taguchi methods, pioneered by Japanese engineer and statistician Genichi Taguchi, have profoundly influenced quality engineering and process optimization. These methods prioritize robust design, focusing on making products and processes resistant to variations, thus enhancing quality and performance without significant cost increases. Taguchi methods are extensively applied to improve the quality of manufactured goods and refine product and process design. Central to this approach is the optimization of designs to make them robust against various sources of variation, such as manufacturing inconsistencies or environmental changes. This robustness ensures that products and processes perform consistently under diverse conditions. Robust design in Taguchi methods emphasizes reducing the sensitivity of products to variations by identifying and optimizing controllable factors, thus minimizing the impact of uncontrollable noise factors.

A fundamental aspect of Taguchi methods is the design of experiments, which utilizes OAs, a type of fractional factorial design. These arrays enable the efficient study of multiple factors simultaneously, allowing for the identification of main effects and interactions with a minimal number of experimental runs. This efficiency makes Taguchi methods particularly valuable in industries where improving quality and reducing costs are critical, such as automotive, electronics, telecommunications, and manufacturing. Applications of these methods range from enhancing product design robustness to optimizing process parameters for high quality outputs with minimal variability. In quality improvement, Taguchi methods systematically identify and address sources of defects and inconsistencies in production processes.

Balanced repeated replications (BRRs) are crucial for obtaining reliable and generalizable results in experimental design. Taguchi methods inherently support this through the use of OAs, ensuring balanced and systematic experimentation. These arrays are designed to test each factor level an equal number of times across the experiment, thus preventing data skew from imbalance. Incorporating replication and randomization into the experimental design controls for random variations and ensures that observed effects are due to the studied factors rather than external influences. Analysis of variance is often employed to analyze experimental results, identifying significant factors and interactions, thereby ensuring conclusions are based on balanced and reliable data.

In conclusion, Taguchi methods offer a powerful approach to quality engineering and process optimization by emphasizing robust design and systematic experimental designs. The application of BRRs within these methods ensures that experimental results are reliable and generalizable, making them highly valuable across various industries. The use of OAs allows for the efficient examination of multiple parameters in a condensed set of experiments. Determining optimal parameter levels requires an in depth understanding of the process and the consideration of the cost of conducting experiments. By selecting the appropriate OA, based on the number of parameters and levels, researchers can ensure that each variable and setting is tested equally, thereby achieving reliable and comprehensive experimental results. Key references in this field include works by Gupta *et al.* (1982), Taguchi (1987), Taguchi and Konishi (1987), Kacker *et al.* (1991), Sitter (1993) and Rosa (2017).

9. Other diverse applications of OAs and MOAs in modern research

Venturing beyond the discussed domains, let us delve into the diverse realms where OAs and MOAs leave their lasting impression. From coding theory and cryptography to computer experiments and beyond, OAs and MOAs emerge as indispensable tools, enriching modern research and experimentation with precision and efficiency. Join us in this section as we unravel the intricate tapestry of applications where these mathematical constructs play pivotal roles, shaping the landscape of information science, technology, and the design of experiments.

9.1. Coding theory, cryptography and computer experiments

Coding theory, cryptography and computer experiments are three distinct yet interconnected domains at the intersection of mathematics, computer science, and engineering. Coding theory, a fundamental component of information theory, focuses on the design and analysis of error detecting and error correcting codes essential for reliable data transmission and storage in the presence of noise or errors. By systematically encoding data into a form that can withstand errors, coding theory enhances the robustness and integrity of digital communication systems like telecommunications networks and data storage devices. Cryptography, on the other hand, stands as the guardian of communication and information security, employing sophisticated mathematical techniques and algorithms to develop cryptographic protocols and algorithms, orchestrating secure communication and data storage by encoding sensitive information. Cryptography's paramount mission lies in upholding the pillars of confidentiality, integrity, and authenticity within digital communications, serving as a formidable barrier against unauthorized access and malicious intrusions. For deeper insights, readers can explore the works of Kahn (1996) and Stinson and Paterson (2018), along with the references cited herein. Computer experiments represent a complementary domain leveraging computational methods, simulations, and modeling techniques to study complex systems and phenomena. By designing and conducting simulations or computational models, researchers can explore the behavior, performance, and characteristics of systems under various conditions, providing a cost effective and efficient means of investigating complex systems, enabling validation of theoretical models, optimization of system designs, and exploration of theoretical concepts across diverse fields from engineering and physics to biology and economics. In summary, coding theory, cryptography, and computer experiments each contribute unique insights and methodologies to the broader landscape of information science and technology, forming essential pillars supporting the development of robust and secure communication systems and the exploration and optimization of complex systems across various domains.

OAs serve as invaluable assets in diverse domains, including coding theory, cryptography, and computer experiments. In coding theory, OAs play a pivotal role in the design and analysis of error correcting codes, crucial for reliable data transmission in communication systems. By systematically varying parameters and configurations, OAs aid in constructing codes that can detect and correct errors efficiently, enhancing the robustness and reliability of communication channels. In cryptography, OAs contribute to the development of secure encryption methods by ensuring the resilience of cryptographic algorithms against various attack vectors. Their systematic approach facilitates the design and testing of cryptographic protocols, strengthening the confidentiality and integrity of sensitive information in digital communications. Furthermore, in computer experiments, OAs provide a structured framework for algorithm testing and simulation studies. By enabling systematic exploration of different algorithmic configurations and scenarios, OAs facilitate comprehensive evaluation and optimization of algorithm performance across diverse computational environments. Through their versatility and systematic variation of factors, OAs play a pivotal role in advancing coding theory, cryptography, and computational research, ensuring the development of robust and efficient solutions in today's digital landscape. For further exploration of these topics, notable references include works by Bose and Shrikhande (1959), Niederreiter (1992), Kahn (1996), Hedayat *et al.* (1999), Massey (2002), Adhikari and Bose (2004), Adhikari *et al.* (2007), Bose and Mukerjee (2006, 2010, 2013), Bose *et al.* (2013) and Stinson and Paterson (2018).

9.2. OA based Latin hypercube designs (OALHDs)

OA based Latin hypercube designs (OALHDs) are advanced statistical tools used in computer experiments to ensure space filling properties, which are crucial for comprehensive exploration of the experimental space. OALHDs combine the strengths of OAs and Latin hypercube sampling, facilitating the creation of experimental designs that uniformly cover the entire parameter space. This uniformity ensures that the design points are spread out evenly, preventing clustering and enhancing the reliability of simulation outcomes. The space filling properties of OALHDs are particularly valuable in computer experiments, where they allow researchers to efficiently sample a wide range of input configurations and explore the performance of complex systems under various conditions. By ensuring a thorough and balanced exploration of the input space, OALHDs help in constructing accurate surrogate models, optimizing system performance, and validating theoretical models. Their application spans numerous fields, including engineering, physics, and environmental science, where robust and efficient design of experiments is critical for gaining insights into complex phenomena and making informed decisions. For detail, readers may consider Sacks *et al.* (1989), Koehler and Owen (1996) and Lin and Tang (2022).

9.3. OAs as BRR structures for variance estimation

A BRR structure is a sampling design commonly used in survey sampling for variance estimation. In BRR, the sample is divided into several balanced replicates, ensuring that each replicate represents the population equally well. Within each replicate, the same survey weights and adjustments are applied as in the original sample. Variance estimation is then performed by computing the variance across these replicates, taking into account both within replicate and between replicate variations. BRR helps to improve the efficiency and accuracy of variance estimation, especially in complex survey designs where traditional methods may be inadequate.

OAs serve as invaluable tools for variance estimation, particularly in the context of large scale complex survey designs where non linear statistics are involved. Acting as BRR structures, OAs provide a systematic and efficient approach to estimating the variance of non linear statistics derived from survey data. By systematically varying factors and configurations within the survey design, OAs ensure balanced representation and systematic variation, thereby capturing the complexities inherent in the survey data. This balanced approach is crucial for accurately estimating the variance of non linear statistics, which may exhibit complex relationships and interactions among survey variables. Additionally, OAs offer the advantage of reducing the computational burden associated with variance estimation in large scale surveys, allowing for efficient and reliable estimation of variance even in complex survey designs. Overall, the utilization of OAs as BRR structures enhances the precision and robustness of variance estimation methods, thereby improving the reliability of survey data analysis in diverse fields. Notable references in this area include works by Gupta *et al.* (1982), Gupta and Nigam (1987), Wu (1991), Sitter (1993), and Parsad and Gupta (2007).

9.4. Optimum covariate designs

In recent years, the quest for experimental units with precisely defined covariate values to achieve optimal precision in regression parameter estimation has garnered significant interest among researchers. The pioneering work by Troya (1982a, 1982b) introduced the concept of optimal covariates designs (OCDs), laying the groundwork for exploring optimal designs to estimate regression parameters associated with controllable covariates. OCDs, renowned for their capacity to offer the most efficient estimation of covariate effects within a presumed linear model, have emerged as indispensable tools in experimental design. Building upon Troya's ground breaking contributions, Das *et al.* (2003) delved into combinatorial solutions, particularly focusing on the estimability of regression coefficients in randomized block designs and certain series of BIBDs.

Rao *et al.* (2003) further elucidated the construction of OCDs derived from MOAs, unraveling the intrinsic relationship between OCDs and experimental designs like completely randomized designs and randomized block designs, both grounded in MOAs. This revelation not only underscores the versatility of MOAs but also expands their application horizons into experimental design realms. For an in depth exploration of this captivating subject, Das, Dutta, Mandal, and Sinha (2015) offer a comprehensive textbook, serving as an invaluable reference for enthusiasts and practitioners alike in the domain of experimental design.

9.5. Optimizing super absorbent composites: leveraging OAs

At the Indian Agricultural Research Institute (IARI) in New Delhi, a ground breaking experiment was devised to engineer super absorbent composites with optimized water absorption characteristics and improved stability in plant growth media. The objective is to maximize absorbency while minimizing the concentrations of monomer, cross linker, and alkali. This intricate experiment encompassed a multitude of factors, including the nature and concentration of alkali, duration and temperature of exposure, backbone clay ratio, monomer concentration, cross linker concentration, initiator concentration, volume of water, and more. With 3 factors at 3 levels and 6 factors at 5 levels, the experiment constituted a daunting $3^3 \times 5^6$ factorial design, necessitating a staggering 421,875 runs for a single replication $-$ an impractical endeavor given limited resources.

In light of the experimenter's interest in orthogonal estimation of main effects and constrained resources, a MOA of strength two emerged as a pragmatic solution, slashing the number of runs to a manageable 225. Although sacrificing intra effect orthogonality, the MOA ensured sufficient resolution and interaction detection. Furthermore, modifications to the experimental objectives led to the creation of a $3^5 \times 6^8$ factorial design, accommodating additional factors and selected interactions, all within the confines of 72 runs. The strategic utilization of MOAs empowered the experimenter to efficiently explore a diverse array of factors and interactions while upholding the integrity of the experiment.

Additionally, IASRI has harnessed OAs for orthogonal main effect plans in asymmetrical factorials and for variance estimations in large scale complex survey data. These endeavors underscore the versatility and utility of OAs across diverse experimental settings. For further insights, we encourage readers to explore the institute's websites.

10. Conclusion

This review article pays homage to the enduring legacy and profound contributions of the legendary statistician, C. R. Rao (1920-2023), across the realms of experimental design, information science, technology, and industry. Delving into the intricate interplay of OAs and BAs, this article offers readers profound insights into the transformative influence of these arrays, as conceptualized by Professor Rao. Referencing seminal works by Parsad, Gupta, Gopinath (2020), Rao (2020), Kannan and Kundu (2021), and Peddada and Khattree (2023), it invites deeper exploration into Prof. Rao's extraordinary contributions and his profound impact on statistical sciences.

Professor Rao's visionary insights have indelibly shaped experimental design, combinatorial mathematics, and statistical analysis, profoundly influencing these disciplines. This article meticulously navigates through the multifaceted applications of OAs and BAs, eloquently showcasing their versatility and paramount importance across various domains. From the intricate construction of BIBDs to the precision of optimum chemical balance weighing designs, and from SORDs to Taguchi methods, orthogonal resolution plans, fractional factorial plans, coding theory, cryptography, computer experiments, OALHDs, and OCDs, this article unveils the methodological advancements fostered by BAs, OAs, and MOAs.

Through meticulous examination, it elucidates the nuanced relationships between association schemes, OAs, and BAs, revealing their immense potential in both experimental design and combinatorial theory. While acknowledging the remarkable strides made thus far, the article passionately underscores the imperative for ongoing research endeavors to fully unlock the latent capabilities of these abstract mathematical structures and their practical applications in experimental design. Indeed, further exploration and analysis in this domain hold the promise of ushering in more advanced and potent experimental design techniques and strategies, thereby enriching the fabric of scientific inquiry and discovery.

In this article, we choose not to delve into mathematical intricacies, recognizing the extensive literature available on the subject. Condensing such a vast topic into a few pages presents a daunting task, and we are mindful of the challenges it entails. Nevertheless, our objective remains clear to offer a lucid exposition that captivates readers beyond this specialized field, sparking their curiosity and nurturing a deeper interest in the subject matter.

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References

Adhikari, A. and Bose, M. (2004). Construction of new visual threshold schemes using combinatorial designs. *IEICE Transactions*, **E87-A**, 1198-1202.

Adhikari, A., Bose, M., Kumar, D., and Roy, B. (2007). Applications of partially balanced incomplete block designs in developing (2,n) Visual cryptographic Schemes. *IEICE Transactions*, **E90-A**, 949-951.

Bose, M., Dey, A., and Mukerjee, R. (2013). Key predistribution schemes distributed sensor networks via block designs. *Design, Codes and Cryptography*, **467**, 111–136.

Bose, M. and Mukerjee, R. (2006). Optimal (2, n) visual cryptographic schemes. *Design, Codes and Cryptography*, **40**, 255-267.

Bose, M. and Mukerjee, R. (2010). Optimal (k, n) visual cryptographic schemes for general k. *Designs, Codes and Cryptography*, **55**, 19–35.

Bose, M. and Mukerjee, R. (2013). Union distinct families of sets, with an application to cryptography. *Ars Combinatoria*, **110**, 179–192.

Bose, R. C. (1950). A note on orthogonal arrays. *The Annals of Mathematical Statistics*, **21**, 304–305.

Bose, R. C. and Bush, K.A. (1952). Orthogonal arrays of strength two and three. *The Annals of Mathematical Statistics*, **23**, 508–524.

Bose, R. C. and Shrikhande, S.S. (1959). A note on result in the theory of code construction. *Information and Control*, **2**, 183–194.

Bush, K. A. (1952). Orthogonal arrays of index unity. *The Annals of Mathematical Statistics*, **23**, 426–434.

Chakravarti, I. M. (1956). Fractional replication in asymmetrical factorial designs and partially balanced arrays. *Sankhya*, **17**, 143–164.

Chakravarti, I. M. (1961). On some methods of construction of partially balanced arrays. *The Annals of Mathematical Statistics*, **32**, 1181–1185.

Cheng, C. S. (1980). Orthogonal arrays with variable numbers of symbols. *Annals of Statistics*, **8**, 447–453.

Das, K., Mandal, N. K., and Sinha, B. K. (2003). Optimal experimental designs for models with covariates. *Journal of Statistical Planning and Inference*, **115**, 273-285.

Das, P., Dutta, G., Mandal, N. K., and Sinha, B. K. (2015). *Optimal Covariate Designs: Theory and Applications*. Springer, New York.

Design Resources Server, *Indian Agricultural Statistics Research Institute (ICAR), New Delhi 110 012, India*. Available at: `https://drs.icar.gov.in/Oarray/oa/default.htm`.

Dey, A. (1971). On some chemical balance weighing designs. *Austrian Journal of Statistics*, **13**, 137–141.

Dey, A. and Mukerjee, R. (1999). *Fractional Factorial Plans*. John Wiley and Sons, New York.

Fuji-Hara, R., Kageyama, S., Kuriki, S., Miao, Y., and Shinohara, S. (2002). Balanced nested designs and balanced arrays. *Discrete Mathematics*, **259**, 91–119.

Gupta, V. K., Nigam, A. K., and Dey, A. (1982). Orthogonal main effect plans for asymmetrical factorials. *Technometrics*, **24**, 135–137.

Gupta, V. K. and Nigam, A. K. (1985). A class of asymmetrical orthogonal resolution IV designs. *Journal of Statistical Planning and Inference*, **12**, 381–383.

Gupta, V. K. and Nigam, A. K. (1987). Mixed orthogonal arrays for variance estimation with unequal numbers of primary selections per stratum. *Biometrika*, **74**, 735–742.

Hedayat, A. S., Sloane, N. J. A., and Stufken, J. (1999). *Orthogonal Arrays: Theory and Applications*. Springer, New York.

Kacker, R. N., Lagergren, E. S., and Filliben, J. J. (1991). Taguchi's Orthogonal Arrays are Classical Designs of Experiments. *Journal of Research of the National Institute of Standards and Technology*, **96**, 577–591.

Kageyama, S. and Saha, G. M. (1983). Note on the construction of optimum chemical balance weighing designs. *Annals of the Institute of Statistical Mathematics*, **35**, Part A, 447–452.

Kahn, D. (1996). *The Codebreakers: The Comprehensive History of Secret Communication from Ancient Times to the Internet*. Scribner, New York.

Kannan, N. and Kundu, D. (2021). C. Radhakrishna Rao: A Century in Statistical Science. *International Statistical Review*, **89**, DOI:10.1111/insr.12467.

Khuri, A. I. and Cornell, J. A. (1996). *Response Surface, Design and Analysis* (2nd ed.). Marcel Dekker Inc., New York.

Kishen, K. (1942). On Latin and hyper-graeco-Latin cubes and hyper-cubes. *Current Science*, **11**, 98–99.

Kishen, K. (1949). On the construction of latin and hyper-graeco-latin cubes and hypercubes. *Journal of the Indian Society of Agricultural Statistics*, **2**,20–48.

Koehler, J. R. and Owen, A. B. (1996). 9 Computer experiments, *Handbook of Statistics*, edited by S. Ghosh and C. R. Rao, Elsevier, Amsterdam.

Kuhfeld, W. F. *TS-DOC: TS-723 - Orthogonal Arrays*, Lists of orthogonal arrays, difference schemes, and experimental design tools maintained by Warren F. Kuhfeld. Available at: `https://support.sas.com/techsup/technote/ts723.html`.

Lin, D. and Tang, B. (2022). Latin Hypercubes and Space-filling Designs, arXiv, eprint: 2203.06334 [Online]. Available: `https://arxiv.org/abs/2203.06334`.

Majindar, K. N. (1978). Coexistence of some B.I.B. Designs. *Canadian Mathematical Bulletin*, **21**, 73–77.

Massey, J. L. (2002). Randomness, Arrays, Differences and Duality. *IEEE Transactions on Information Theory*, **48**, 1698–1703.

Mukhopadhyay, A. C. (1981). Construction of some series of orthogonal arrays. *Sankhya B*, **43**, 81–92.

Niederreiter, H. (1992). Orthogonal arrays and other combinatorial aspects in the theory of uniform point distributions in unit cubes. *Discrete Mathematics*, **106-107**, 361–367.

Parsad, R. and Gupta, V. K. (2007). Variance estimation from complex surveys using balanced repeated replication. `https://shorturl.at/aCOid`.

Parsad, R., Gupta, V. K., and Gopinath, P. P. (2020). *Calyampudi Radhakrishna Rao's Life Sketch and its Influence on Designing of Experiments with a Special Reference to Agricultural Sciences*. ICAR-IASRI, New Delhi, 1–18. `http://krishi.icar.gov.in/jspui/handle/123456789/41295`.

Peddada, S. D. and Khattree, R. (2023). C. R. Rao, statistician who transformed data analytics (1920–2023). *Nature*, Oct; 622(7984):691, doi: 10.1038/d41586-023-03200-5.

Pukelsheim, F. (1993). *Optimal Design of Experiments*. John Wiley, New York.

Rafter, J. A. and Seiden, E. (1974). Contributions to the Theory and Construction of Balanced Arrays. *Annals of Statistics*, **2**, 1256– 1273.

Raghavarao, D. (1971). *Constructions and Combinatorial Problems in Design of Experiments*. Wiley, New York.

Rao, B. L. S. P. (2020). C. R. Rao: A Life in Statistics. *Bhāvanā*, 4, `https://bhavana.org.in/c-r-rao-a-life-in-statistics/`.

Rao, C. R. (1946). Hypercubes of strength "d" leading to confounded designs in factorial experiments. *Bulletin of the Calcutta Mathematical Society*, **38**, 67–78.

Rao, C. R. (1947). Factorial experiments derivable from combinatorial arrangements of arrays. *Journal of the Royal Statistical Society* (Supplement), **9**, 128–139.

Rao, C. R. (1949). On a class of arrangements. *Proceedings of the Edinburgh Mathematical Society*, **8**, 119–125.

Rao, C. R. (1961). Combinatorial arrangements analogous to orthogonal arrays. *Sankhya A*, **23**, 283–286.

Rao, C. R. (1973). Some combinatorial problems of arrays and applications to design of experiments. 349-359 of: Srivastava, J. N. (ed.), *A Survey of Combinatorial Theory*, North Holland.

Rao, P. S. S. N. V. P., Rao, S. B., Saha, G. M., and Sinha, B. K. (2003). Optimal designs for covariates' models and mixed orthogonal arrays. *Electronic Notes in Discrete Mathematics*, **15**, 157–160.

Rosa, P. J. (2017). *Taguchi Techniques for Quality Engineering* (2nd ed.). McGraw Hill.

Sacks, J., Welch, W. J., Mitchell, T. J., and Wynn, H. P. (1989). Design and Analysis of Computer Experiments, *Statistical Science*, **4**, 409–423.

Saha, G. M. and Das, A. R. (1973). Four level second order rotatable designs from partially balanced arrays. *Journal of the Indian Society of Agricultural Statistics*, **25**, 97–102.

Saha, G. M. (1975). Some results in tactical configurations and related topics. *Utilitas Mathematica*, **7**, 223–240.

Saha, G. M. (1975). A note on relations between incomplete block and weighing designs. *Annals of the Institute of Statistical Mathematics*, **27**, 387–390.

Saha, G. M. (1981). Balanced arrays from association schemes and some related results. In: Rao, S.B. (eds) Combinatorics and Graph Theory. *Lecture Notes in Mathematics*, vol 885. Springer, Berlin, Heidelberg.

Saha, G. M. and Kageyama, S. (1984). Balanced arrays and weighing designs. *Austrian Journal of Statistics*, **26**, 119–124.

Saha, G. M. and Samanta, B. K. (1985). A construction of balanced arrays of strength t and some related incomplete block designs. *Annals of the Institute of Statistical Mathematics*, **37**, 337–345.

Shah, K. R. and Sinha, B. K. (1989). *Theory of Optimal Designs*. Lecture Notes in Statistics, vol 54. Springer, New York.

Silvey, S. D. (1980). *Optimal Design*. Chapman and Hall, London.

Sinha, K., Dhar, V., Saha, G. M., and Kageyama, S. (2002). Balanced arrays of strength two from block designs. *Journal of Combinatorial Designs*, **10**, 303–312.

Sitter, R. R. (1993). Balanced repeated replications based on orthogonal multi-arrays. *Biometrika*, **80**, 211–221.

Sloane, N. J. A. (2007). *A Library of Orthogonal Arrays*. Available at: `http://neilsloane.com/oadir/`.

Srivastava, J. N. and Chopra, D. V. (1973). Balanced arrays and orthogonal arrays. *A Survey of Combinatorial Theory*, 411–428, North Holland, Amsterdam.

Stinson, D. R. and Paterson, M. B. (2018). *Cryptography: Theory and Practice*. 4th ed. CRC Press.

Taguchi, G. (1986). *Introduction to Quality Engineering: Design Quality into Products and Processes*. Tokyo: Asian Productivity Organization.

Taguchi, G. (1987). *System of Experimental Design: Engineering Methods to Optimize Quality and Minimize Costs*. MI, American Supplier Institute Inc.

Taguchi, G. and Konishi, S. (1987). *Taguchi Methods Orthogonal Arrays and Linear Graphs: Tools for Quality Engineering*. American Supplier Institute, Dearborn, Michigan.

Troya, L. J. (1982a). Optimal designs for covariates models. *Journal of Statistical Planning and Inference*, **6**, 373-419.

Troya, L. J (1982b). Cyclic designs for covariates models. *Journal of Statistical Planning and Inference*, **7**, 49-75.

Vanstone, S. A. (1975). A note on a construction of b.i.b.d.'s. *Utilitas Mathematica*, **7**, 321–322.

Wikipedia, *Orthogonal array testing - Wikipedia, the free encyclopedia*, Orthogonal array testing is a systematic, statistical way of testing. Orthogonal arrays can be applied in user interface testing, system testing. Available at: `https://en.wikipedia.org/wiki/Orthogonal_array_testing`.

Wu, C. F. J. (1991). Balanced repeated replications based on mixed orthogonal arrays. *Biometrika*, **78**, 181–188.

Yonglin, Z. (2004). *Combinatorial Designs via Association Scheme*. Ph.D. Thesis, Hong Kong Baptist University, Hong Kong.