Statistics and Applications {ISSN 2454-7395 (online)}
Volume 21, No. 1, 2023 (New Series), pp 179–185

Properties of Partial Product Processes

Babu D., Sutha M., Neelakandan R. and Elumalai P.
Department of Mathematics
Thanthai Periyar Government Institute of Technology, Vellore, India

Received: 14 January 2022; Revised: 07 May 2022; Accepted: 08 July 2022

Abstract

The partial product process is a sequence of non-negative random variables $X_1, X_2, X_3, ...$ such that the distribution function of X_1 is $F(x)$ and the distribution function of X_{i+1} is $F(\beta_i x) \quad (i = 1, 2, 3, ...)$ where $\beta_i > 0$ are constants and $\beta_i = \beta_0\beta_1\beta_2...\beta_{i-1}$. It is a monotone process. In this paper, the probabilistic properties of the partial product process are studied and some limit theorems are also established.

Key words: Geometric process; Partial product process; Renewal process.

AMS Subject Classifications: 60K10, 90B25

1. Introduction

The mathematical theory of reliability has put forth a great effort to issues of life-testing, machine support, replacement, order statistics, and so on. The maintenance problems are concerned about the circumstance that emerges about the reduction of the productivity level of items or breakdown. The problem of replacement is to recognize the best policy which enables determination of ideal replacement time that is generally economical. One of the most interesting and critical topics to study in reliability is the study of maintenance problems.

A common assumption in the initial period of studying maintenance issues is that repair is perfect, a repairable framework after the repair is *as good as new.* This assumption clearly has the effect of a natal way. In practice, most repairable systems deteriorate because of the combined wear and tear impact. Barlow (1960) thusly presented a minimal repair model in which a system after the repair has the same failure rate and effective age as it was when it failed. Brown(1983) proposes an imperfect repair model, in which the repair is perfect with likelihood p, and the repair is minimal with a probability of $1 - p$.

Deteriorating systems have a different problem as the one portrayed above. For instance, in machine maintenance problems, after every repair, the working time of a machine will end up shorter and shorter, so the absolute working time or the existence of the machine must be limited. However, in perspective on the aging and aggregate wear, the repair time will turn out to be longer and tend to increase so that at the end the machine is non-repairable. Therefore, there is need to consider a repair replacement model for deteriorating systems, the progressive survival times are diminishing, while the consecutive repair times are expanding.

Lam (1988) first presented a Geometric Process Repair model to model a deterio-

Corresponding Author: Babu D.
Email: babudurai15@gmail.com

rating system with the above characteristics.

Definition 1: The sequence $\{X_n,\ n = 1, 2, 3, \ldots\}$ of non negative independent random variables is called a geometric process, if the distribution function of X_n is given by $F\left(a^{n-1}x\right)$ for $n = 1, 2, 3, \ldots$, where $a(> 0)$ is a constant.

In Geometric process, the operating times and repair times of a system are uniformly decreasing. But practically, it is not uniform. To overcome this, the partial product process was introduced by Babu et. al (2018).

Definition 2: Let $\{X_n,\ n = 1, 2, ...\}$ be a sequence of non-negative independent random variables and let $F(x)$ be the distribution function of X_1. Then $\{X_n,\ n = 1, 2, ...\}$ is called a partial product process, if the distribution function of X_{i+1} is $F(\beta_i x)\quad (i = 1, 2, ...)$ where $\beta_i > 0$ are constants and $\beta_i = \beta_0\beta_1\beta_2...\beta_{i-1}$.

By induction, it is clear that for real β_i $(i = 1, 2, ...)$, $\beta_i = {\beta_0}^{2^{i-1}}$. Thus, the distribution function of X_{i+1} is $F\left({\beta_0}^{2^{i-1}}x\right)$ for $i = 1, 2, \ldots$.

The remainder of this paper are structured as follows. In section 2, some probabilistic properties of the partial product process are studied and in section 3, some limit theorems are established.

2. Probabilistic properties of partial product process

Let F and f be the distribution function and density function of X_1 respectively, and denote $E(X_1) = \lambda$ and $Var(X_1) = \sigma^2$.

Then for $i = 1, 2, ...$, we have

$$E(X_{i+1}) = \frac{\lambda}{\beta_0^{2^{i-1}}}$$

and

$$Var(X_{i+1}) = \frac{\sigma^2}{\beta_0^{2^i}}.$$

Thus, $\beta_0,\ \lambda$ and σ^2 are three important parameters of partial product process.

Note that when $F(0) < 1$, then $\lambda > 0$.

Define $S_0 = 0$ and

$$S_n = \sum_{i=1}^{n} X_i$$

Let $\mathbb{F}_n = \sigma(X_1, X_2, ..., X_n)$ be the σ- algebra generated by $\{X_i,\ i = 1, 2, ..., n\}$.

Theorem 1: If $\beta_0 > 1$, then $\{S_n,\ n = 1, 2, ...\}$ is a nonnegative submartingale with respect to $\mathbb{F}_n = \sigma(X_1, X_2, ..., X_n)$.

Proof: Obviously, $\{S_n,\ n = 1, 2, ...\}$ is a sequence of increasing non-negative random variables with

$$E[S_{n+1}|\mathbb{F}_n] = S_n + E[X_{n+1}] \geqslant S_n \tag{1}$$

Also,

$$\begin{aligned}\underset{n\geqslant 0}{Sup}\, E\left[|S_n|\right] &= \lim_{n\to\infty} E\left[S_n\right] \\ &= \lim_{n\to\infty} E\left[\sum_{i=1}^{n} X_i\right] \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} E\left(X_i\right) \\ &= \lim_{n\to\infty} \left[\lambda + \sum_{i=2}^{n} \frac{\lambda}{{\beta_0}^{2^{i-2}}}\right] \\ &= \lambda\left[1 + \sum_{i=2}^{\infty}\left(\frac{1}{\beta_0}\right)^{2^{i-2}}\right] < \infty, \end{aligned} \tag{2}$$

where equation (2) is due to the fact that the series $\sum_{i=2}^{\infty}\left(\frac{1}{\beta_0}\right)^{2^{i-2}}$ is convergent by comparing with geometric series if $\frac{1}{\beta_0} < 1$.

Theorem 2: If $\beta_0 > 1$, there exists a random variable S such that

$$S_n \xrightarrow{a.s.} S \; as \; n \to \infty.$$

Proof: By Theorem 1 and *Martingale Convergence Theorem*(Ross, 1996), we have $S_n \xrightarrow{a.s.} S \; as \; n \to \infty$.

Theorem 3: If $\beta_0 > 1$, $\{S_n,\ n = 1, 2, ...\}$ has a unique decomposition such that

$$S_n = L_n - A_n \tag{3}$$

where $\{L_n,\ n = 1, 2, ...\}$ is a martingle, $\{A_n,\ n = 1, 2, ...\}$ is decreasing with $A_1 = 0$ and $A_n \in \mathbb{F}_{n-1}$.

Proof: Let $L_1 = S_1$ and $A_1 = 0$. For $n \geqslant 2$, define

$$L_n = L_{n-1} + \left(S_n - E\left[S_n|\mathbb{F}_{n-1}\right]\right), \tag{4}$$

$$A_n = A_{n-1} + \left(S_{n-1} - E\left[S_n|\mathbb{F}_{n-1}\right]\right). \tag{5}$$

From equations (4) and (5), we have

$$L_n - A_n = \sum_{i=2}^{n}\left(S_i - S_{i-1}\right) + S_1 - A_1 = S_n$$

and (3) follows. It is easy to check $\{L_n,\ n = 1, 2, ...\}$ and $\{A_n,\ n = 1, 2, ...\}$ satisfy the requirements. Next, we need to prove such a decomposition is unique.

Suppose $S_n = {L_n}' - {A_n}'$ is another decomposition. Then,

$$L_n - {L_n}' = A_n - {A_n}'$$

Since $A_1 = {A_1}' = 0$, it is clear that $L_1 = {L_1}'$. Also as $L_2 - {L_2}' = A_2 - {A_2}' \in \mathbb{F}_1$,, we have

$$L_2 - {L_2}' = E\left[L_2 - {L_2}'|\mathbb{F}_1\right] = L_1 - {L_1}' = 0.$$

This implies that $L_2 = {L_2}'$. Then, by induction, we can prove that $L_n = {L_n}'$ and hence $A_n = {A_n}'$. Thus the uniqueness is proved.

Theorem 4: If $\beta_0 > 1$, then $\{S_n,\ n = 1, 2, ...\}$ has a unique Riesz decomposition such that

$$S_n = Y_n + Z_n \tag{6}$$

where $\{Y_n,\ n = 1, 2, ...\}$ is a non-negative martingale and $\{Z_n,\ n = 1, 2, ...\}$ is a non-positive submartingale with $\lim_{n\to\infty} E[Z_n] = 0$.

Proof: From equation (2),

$$\lim_{n\to\infty} E[S_n] = \lambda \left[1 + \sum_{i=2}^{\infty} \left(\frac{1}{\beta_0}\right)^{2^{i-2}}\right] < \infty.$$

Thus the proof is complete by Riesz decomposition theorem(Ross, 1996).

Theorem 5: If $\beta_0 > 1$, then $\{S_n,\ n = 1, 2, ...\}$ has a Krickeberg decomposition such that

$$S_n = L_n - M_n$$

where $\{L_n,\ n = 1, 2, ...\}$ is a non-positive submartingale and $\{M_n,\ n = 1, 2, ...\}$ is a non-positive martingale. Moreover, such a decomposition has the maximality such that for any other decomposition $S_n = L_n{}' - M_n{}'$ where $L_n{}'$, $M_n{}'$ are nonpositive submartingale and nonpositive martingale respectively, then

$$L_n \geq L_n{}' \quad and \quad M_n \geq M_n{}'$$

Proof: Note that,

$$\underset{n}{Sup}\, E\left[S_n{}^+\right] = \underset{n}{Sup}\, E[S_n] = \lambda \left[1 + \sum_{i=2}^{\infty} \left(\frac{1}{\beta_0}\right)^{2^{i-2}}\right] < \infty.$$

Thus the proof is complete by Krickeberg decomposition(Ross, 1996).

3. Limit theorems for partial product process

In renewal process, Wald's equation plays a key role. The following theorem is a generalization of the Wald's equation to a *partial product process*, it is called as Wald's equation for *partial product process.*

Theorem 6 (Wald's equation for *partial product process*)**:** Suppose that $\{X_n,\ n = 1, 2, ...\}$ forms a *partial product process* with $E[X_1] = \lambda < \infty$, then for $t > 0$, we have

$$E\left[S_{N(t)+1}\right] = \lambda E\left[1 + \sum_{n=2}^{N(t)+1} \frac{1}{\beta_0{}^{2^{n-2}}}\right], \tag{7}$$

where $N(t)$ is the counting process which represents the number of occurrences of an event up to time t.

Proof: Let I_A be the indicator function of event A. Then $I_{\{S_{n-1}\le t\}} = I_{\{N(t)+1\ge n\}}$ and X_n are independent. Accordingly, for $t > 0$, we have

$$\begin{aligned} E\left[S_{N(t)+1}\right] &= E\left[\sum_{n=1}^{N(t)+1} X_n\right] \\ &= \sum_{n=1}^{\infty} E\left[X_n I_{\{N(t)+1\ge n\}}\right] \\ &= \sum_{n=1}^{\infty} E\left[X_n\right] P\left(N\left(t\right)+1\ge n\right) \qquad (8) \\ &= \lambda E\left[1+\sum_{n=2}^{N(t)+1}\frac{1}{{\beta_0}^{2^{n-2}}}\right]. \end{aligned}$$

Corollary 1:

$$E\left[S_{N(t)+1}\right]\begin{cases} > \lambda\left[E\left(N\left(t\right)\right)+1\right], & 0<\beta_0<1 \\ = \lambda\left[E\left(N\left(t\right)\right)+1\right], & \beta_0=1 \\ < \lambda\left[E\left(N\left(t\right)\right)+1\right], & \beta_0>1. \end{cases}$$

Proof: Let $\beta_0 > 1$. Then,

$$\begin{aligned} \frac{1}{\beta_0}<1 \;&\Rightarrow\; E\left[1+\sum_{n=2}^{N(t)+1}\frac{1}{{\beta_0}^{2^{n-2}}}\right] < E\left[1+\sum_{n=2}^{N(t)+1}(1)\right] \\ &\Rightarrow\; \lambda E\left[1+\sum_{n=2}^{N(t)+1}\frac{1}{{\beta_0}^{2^{n-2}}}\right] < \lambda E\left[N\left(t\right)+1\right] \\ &\Rightarrow\; E\left[S_{N(t)+1}\right] < \lambda\left[E\left(N\left(t\right)\right)+1\right] \quad (\textit{by theorem } 6). \end{aligned}$$

Similarly, we can prove that $E\left[S_{N(t)+1}\right] > \lambda\left[E\left(N\left(t\right)\right)+1\right]$ if $\beta_0 < 1$. For $\beta_0 = 1$, the result is trivial.

Note that if $\beta_0 = 1$, the *partial product process* reduces to a renewal process, while corollary 1 gives $E\left[S_{N(t)+1}\right] = \lambda\left[E\left(N\left(t\right)\right)+1\right]$. This is Wald's equation for the renewal process.

Theorem 7: If a stochastic process $\{X_n,\ n = 1, 2, 3, ...\}$ is a *partial product process*, then

$$(1)\quad \lim_{t\to\infty}\frac{1}{t}E\left[S_{N(t)+1}\right] = 0 \quad \text{if } \beta_0 > 1.$$

$$(2)\quad \lim_{t\to\infty}\frac{1}{t}E\left[S_{N(t)+1}\right] = 1 \quad \text{if } \beta_0 = 1.$$

Proof: Let $\beta_0 > 1$. Then from equation (8), we have

$$\begin{aligned}\lim_{t\to\infty} E\left[S_{N(t)+1}\right] &= \lim_{t\to\infty}\sum_{n=1}^{\infty} E\left[X_n\right] P\left(N\left(t\right)+1 \geq n\right)\\ &= \sum_{n=1}^{\infty} E\left[X_n\right] P\left(S_n < \infty\right)\\ &= \sum_{n=1}^{\infty} E\left[X_n\right](1)\\ &= \lambda\left(1+\sum_{n=2}^{\infty}\frac{1}{{\beta_0}^{2^{n-2}}}\right) < \infty.\end{aligned}$$

Thus,

$$\lim_{t\to\infty}\frac{1}{t}E\left[S_{N(t)+1}\right] = 0.$$

This completes the proof of part (1).

Now, assume that $\beta_0 = 1$. Then,

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}E\left[S_{N(t)+1}\right] &= \lim_{t\to\infty}\frac{\lambda}{t}\left[E\left(N\left(t\right)\right)+1\right]\\ &= \lambda\lim_{t\to\infty}\left[\frac{E\left(N\left(t\right)\right)}{t}\right]\\ &= \lambda\times\frac{1}{\lambda} = 1, \qquad (9)\end{aligned}$$

where (9) due to *Elementary Renewal Theorem.* This completes the proof of part (2) and theorem.

4. Conclusion

In this paper, some limit theorems and probability properties of the partial product process are established. Since it is monotone process, it can be applied to the maintenance model for a deteriorating system.

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