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Some Existence on Ordered Multi-designs

Kazuki Matsubara[1] and Sanpei Kageyama[2]
[1]*Faculty of Commerce, Chuo Gakuin University, Abiko, Japan*
[2]*Hiroshima University, Higashi-Hiroshima, Japan*

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Abstract

Two variants of an orthogonal array, orthogonal arrays of type I and of type II, were introduced by Rao in 1961. Furthermore, as generalizations of an orthogonal array and an orthogonal array of type II, an orthogonal multi-array and a perpendicular multi-array have been introduced by Brickell in 1984 and by Li *et al.* in 2018, respectively. In this paper, as a generalization of the orthogonal array of type I, an ordered multi-design is newly introduced from a combinatorial viewpoint. Necessary conditions for the existence of an ordered multi-design are discussed and several constructions of the ordered multi-design are provided by use of group divisible designs and self-orthogonal latin squares, through a difference technique. As main results, the existence of a family of ordered multi-designs is provided and also the sufficiency of necessary conditions for existence is shown for a class of ordered multi-designs with one possible exception.

Key words: Ordered multi-design; Perpendicular multi-array; Self-orthogonal Latin square; Group divisible design.

AMS Subject Classifications: 05B15, 05B05

1. Introduction

An *ordered multi-design* of size $N \times k$, denoted by $\mathrm{OMD}_\lambda(k \times c, v)$, is an $N \times k$ multi-array, $\mathcal{A} = (A_{ij})$, on a set V of v points, which satisfies the following conditions:

(C1) each entry A_{ij} ($|A_{ij}| = c$) is a c-subset of V and kc distinct points occur in k entries of each row of $\mathcal{A}$, and

(C2) for any ordered pair (j_1, j_2) of integers with $1 \le j_1 < j_2 \le k$ and for any ordered pair (x_1, x_2) of distinct points in V, there are exactly λ rows of $\mathcal{A}$ such that the points x_1 and x_2 appear in the j_1th and the j_2th entries, *i.e.*, in the j_1th and the j_2th columns, of each of the λ rows, respectively.

Note that the conditions (C1) and (C2) lead to $N = \lambda v(v-1)/(c^2)$. Moreover, $k \ge 2$ is assumed at least to validate the condition (C2).

Let us illustrate the definition of the $\mathrm{OMD}_\lambda(k \times c, v)$ by the following example.

Corresponding Author: Kazuki Matsubara
Email: additivestructure@gmail.com

Example 1: An $\mathrm{OMD}_2(3 \times 2, 6)$ on $V = \mathbb{Z}_5 \cup \{\infty\}$ is given by

$$\begin{pmatrix} \infty,0 & 1,4 & 2,3 \\ \infty,1 & 2,0 & 3,4 \\ \infty,2 & 3,1 & 4,0 \\ \infty,3 & 4,2 & 0,1 \\ \infty,4 & 0,3 & 1,2 \\ 2,3 & \infty,0 & 1,4 \\ 3,4 & \infty,1 & 2,0 \\ 4,0 & \infty,2 & 3,1 \\ 0,1 & \infty,3 & 4,2 \\ 1,2 & \infty,4 & 0,3 \\ 1,4 & 2,3 & \infty,0 \\ 2,0 & 3,4 & \infty,1 \\ 3,1 & 4,0 & \infty,2 \\ 4,2 & 0,1 & \infty,3 \\ 0,3 & 1,2 & \infty,4 \end{pmatrix}$$

with $k = 3$ (three columns), $c = 2$, $N = 15$ (fifteen rows), entries of the first row $A_{11} = \{\infty, 0\}$, $A_{12} = \{1, 4\}$, $A_{13} = \{2, 3\}$, entries of the second row $A_{21} = \{\infty, 1\}$, $A_{22} = \{2, 0\}$, $A_{23} = \{3, 4\}, \ldots$, entries of the sixth row $A_{61} = \{2, 3\}$, $A_{62} = \{\infty, 0\}$, $A_{63} = \{1, 4\}$, etc. The condition (C2) with $\lambda = 2$ can be checked, *e.g.*, 0 and 1 occur in the first and the second columns, respectively, of the first and the last rows.

From now on, each row of an $\mathrm{OMD}_\lambda(k \times c, v)$ is separately displayed in the form of

$$(a_{11}, a_{12}, \ldots, a_{1c} \mid a_{21}, a_{22}, \ldots, a_{2c} \mid \ldots \mid a_{k1}, a_{k2}, \ldots, a_{kc})$$

by use of kc points on V or $(A_{i1} \mid A_{i2} \mid \ldots \mid A_{ik})$ by use of k entries A_{ij} $(1 \leq i \leq N)$.

It is clear that the $\mathrm{OMD}_\lambda(k \times 1, v)$ coincides with the ordered design, denoted by $\mathrm{OD}_\lambda(k, v)$, defined in Rao (1961), who call the ordered design by the other name "an orthogonal array of Type I". An orthogonal array and a perpendicular array (called by the other name "an orthogonal array of Type II" in Rao, 1961) have been generalized to an orthogonal multi-array (OMA) in Brickell (1984) and a perpendicular multi-array (PMA) in Li *et al.* (2018), respectively. Furthermore, applications of the OMA and the PMA to design of experiments and coding theory are discussed in Brickell (1984), Li *et al.* (2015), Li *et al.* (2018), Mukerjee (1998) and Sitter (1993). On the other hand, as far as the authors know, the ordered multi-design has never been discussed in literature.

In this paper, the existence on an $\mathrm{OMD}_\lambda(k \times 2, v)$, *i.e.*, $c = 2$, is mainly discussed from a viewpoint of combinatorics. In Section 2, a construction and a fundamental property of the OMD, and combinatorial structures used in later sections are presented. In Section 3, necessary conditions for the existence of an $\mathrm{OMD}_\lambda(k \times c, v)$ are discussed. In Section 4, constructions of a cyclic $\mathrm{OMD}_\lambda(3 \times 2, v)$ are provided by use of difference techniques. In Sections 5 and 6, methods of constructing an OMD are presented by use of a group divisible design (GDD) and self-orthogonal latin squares (SOLS), respectively. In Section 7, the existence of an $\mathrm{OMD}_\lambda(k \times 2, q)$ for any prime power q is provided. Furthermore, it is shown that the necessary conditions discussed in Section 3 are also sufficient for the existence of an $\mathrm{OMD}_\lambda(3 \times 2, v)$ with one possible exception, as in the following main results.

Theorem 1: There exists an $\mathrm{OMD}_\lambda(k \times 2, q)$ for any prime power q, any $\lambda \equiv 0 \pmod 2$ and any k with $2 \le k \le \lceil (q-1)/2 \rceil$.

Theorem 2: Let v be a positive integer with $v \ge 6$. Then there exists an $\mathrm{OMD}_\lambda(3 \times 2, v)$ if and only if $v \equiv 1 \pmod 4$ or $\lambda \equiv 0 \pmod 2$ with a possible exception of $(v, \lambda) = (9, 1)$.

As the appendix, some individual examples, which cannot be obtained by methods in this paper, will be presented to be utilized in the proof of Theorem 2 in Section 7.

2. Preliminaries

At first, the perpendicular multi-array discussed in Li *et al.* (2018) and Matsubara and Kageyama (2021) is reviewed. The perpendicular multi-array $\mathcal{A} = (A_{ij})$, denoted by $\mathrm{PMA}_\lambda(k \times c, v)$, is defined by the condition (C1) and the following condition (C3):

(C3) for any two columns of $\mathcal{A}$ and for any unordered pair $\{x_1, x_2\}$ of distinct points in V, there are exactly λ rows of $\mathcal{A}$ such that the points x_1 and x_2 separately appears in the two entries of each of the λ rows.

Since the condition (C2) involves the condition (C3), it follows that any $\mathrm{OMD}_\lambda(k \times c, v)$ can be regarded as a $\mathrm{PMA}_{2\lambda}(k \times c, v)$.

On the other hand, it is known (see Bierbrauer, 2007) that the existence of an $\mathrm{OD}_1(k, v)$, *i.e.*, $\mathrm{OMD}_1(k \times 1, v)$, is equivalent to the existence of $k-2$ idempotent mutually orthogonal latin squares. The review of results on the existence and applications of the $\mathrm{OD}_\lambda(k, v)$ can be found in Bierbrauer (2007), Bierbrauer and Edel (1994), Kunert and Martin (2000) and Majumdar and Martin (2004). Especially, the following result will be useful for the construction of an $\mathrm{OMD}_\lambda(k \times c, v)$ described in Section 5.

Lemma 1 (Bierbrauer, 2007)**:** There exists an $\mathrm{OD}_1(k, k)$ for any prime power k.

A direct construction of an $\mathrm{OMD}_2(k \times 2, v)$ can be obtained as follows.

Lemma 2: Let q be an odd prime power. Then there exists an $\mathrm{OMD}_2(k \times 2, q)$ with $k = (q-1)/2$.

Proof: Let $V = GF(q)$. Then a direct sum decomposition of $GF(q)$ can be given by

$$GF(q) = \{0\} \cup B_1 \cup B_2 \cup \ldots \cup B_{\frac{q-1}{2}},$$

where $B_j = \{a_j, -a_j\}$ with $a_j \in GF(q)$ $(1 \le j \le (q-1)/2)$. Now consider $(q-1)/2$ rows:

$$(\alpha^\ell B_1 \mid \alpha^\ell B_2 \mid \ldots \mid \alpha^\ell B_{\frac{q-1}{2}}),\ 1 \le \ell \le \frac{q-1}{2},$$

where $\alpha^\ell B_j = \{\alpha^\ell a_j, -\alpha^\ell a_j\}$ and α is a primitive element of $GF(q)$. Hence any two entries $\{\alpha^\ell a_{j_1}, -\alpha^\ell a_{j_1}\}, \{\alpha^\ell a_{j_2}, -\alpha^\ell a_{j_2}\}$ in the same row yield four pairs as

$$(\alpha^\ell a_{j_1}, \alpha^\ell a_{j_2}), (-\alpha^\ell a_{j_1}, -\alpha^\ell a_{j_2}), (\alpha^\ell a_{j_1}, -\alpha^\ell a_{j_2}), (-\alpha^\ell a_{j_1}, \alpha^\ell a_{j_2})$$

for the condition (C2). Furthermore, for any pair (x,y) it holds that

$$\{(x+t, y+t) \mid t \in GF(q)\} = \{(x', y') \mid x', y' \in GF(q), x'-y' = x-y\}.$$

Since $\alpha^{(q-1)/2} = -1$ and $\{\alpha^\ell, -\alpha^\ell \mid 1 \le \ell \le (q-1)/2\} = GF(q)\backslash\{0\}$, both of

$$\{(\alpha^\ell a_{j_1}+t, \alpha^\ell a_{j_2}+t), (-\alpha^\ell a_{j_1}+t, -\alpha^\ell a_{j_2}+t) \mid 1 \le \ell \le \frac{q-1}{2}, t \in GF(q)\}$$

and

$$\{(\alpha^\ell a_{j_1}+t, -\alpha^\ell a_{j_2}+t), (-\alpha^\ell a_{j_1}+t, \alpha^\ell a_{j_2}+t) \mid 1 \le \ell \le \frac{q-1}{2}, t \in GF(q)\}$$

are equal to $\{(x,y) \mid x, y \in GF(q), x \neq y\}$. Therefore the required $\mathrm{OMD}_2(k \times 2, q)$ with $k = (q-1)/2$ can be obtained from the following $(q-1)q/2$ rows:

$$(\alpha^\ell B_1 + t \mid \alpha^\ell B_2 + t \mid \ldots \mid \alpha^\ell B_{\frac{q-1}{2}} + t),\ 1 \le \ell \le \frac{q-1}{2},\ t \in GF(q),$$

where $\alpha^\ell B_j + t = \{\alpha^\ell a_j + t, -\alpha^\ell a_j + t\}$. □

Next a fundamental property of the OMD, which is useful to construct OMDs for various values of k, is provided as follows.

Lemma 3: Any subarray obtained by deleting any k' $(k' < k)$ columns of an $\mathrm{OMD}_\lambda(k \times c, v)$ is an $\mathrm{OMD}_\lambda((k-k') \times c, v)$.

Proof: Since an $\mathrm{OMD}_\lambda(k \times c, v)$ satisfies the conditions (C1) and (C2), it is clear that any two columns of the $\mathrm{OMD}_\lambda((k-k') \times c, v)$ also satisfy (C1) and (C2). □

Now, a combinatorial design used in later sections is introduced. Let v, k, λ be positive integers. A *group divisible design*, denoted by (k,λ)-GDD, is a triplet $(V, \mathcal{G}, \mathcal{B})$, where V is a set of v points, $\mathcal{G}$ is a partition of V into subsets (called groups) and $\mathcal{B}$ $(|\mathcal{B}| = b)$ is a family of subsets (called blocks) of size k each of V such that

(G1) every pair of distinct points $x, y \in V$ in different groups occurs in exactly λ blocks, and
(G2) every pair of distinct points $x, y \in V$ in the same group does not occur in any block.

The *group type* of a (k,λ)-GDD is a multi-set $\{|G| \mid G \in \mathcal{G}\}$. The usual exponential notation is used to describe group types. Thus the notation $h_1^{t_1} h_2^{t_2} \cdots h_n^{t_n}$ means that there are t_i groups of size h_i for $1 \le i \le n$ (cf. Ge, 2007).

The following proposition on GDDs is known.

Lemma 4 (Ge, 2007)**:** Let g, u and m be non-negative integers. Then there exists a $(3,1)$-GDD of type $g^u m^1$ if and only if the following conditions are all satisfied:

(a) if $g > 0$, then $u \ge 3$, or $u = 2$ and $m = g$, or $u = 1$ and $m = 0$, or $u = 0$;
(b) $m \le g(u-1)$ or $gu = 0$;
(c) $g(u-1) + m \equiv 0 \pmod 2$ or $gu = 0$;
(d) $gu \equiv 0 \pmod 2$ or $m = 0$; and
(e) $\frac{1}{2}g^2u(u-1) + gum \equiv 0 \pmod 3$.

The GDD will be utilized for a method of constructing OMDs discussed in Section 7.

3. Necessary Conditions

Necessary conditions for the existence of an $\mathrm{OMD}_\lambda(k\times c, v)$ are considered. It is obvious by the conditions (C1) and (C2) that for any $\mathrm{OMD}_\lambda(k\times c, v)$ of size $N\times k$

$$v \geq kc \tag{1}$$

holds. Since N is a positive integer,

$$c^2 \mid \lambda v(v-1) \tag{2}$$

holds. Furthermore, every point must occur equally r $(= cN/v)$ times in each column. Hence it is seen that

$$c \mid \lambda(v-1) \tag{3}$$

holds.

The sufficiency of these necessary conditions (1), (2), (3) for the existence when $(c, v) = (2, q)$ with any prime power q and $(k, c) = (3, 2)$, will be proved with some exceptions as in Theorems 1 and 2, respectively, in Section 7.

Furthermore, another necessary condition for the existence of an $\mathrm{OMD}_\lambda(k\times c, v)$ of size $N\times k$ can be presented by use of the following result.

Lemma 5 (Matsubara and Kageyama, 2021)**:** In a $\mathrm{PMA}_\lambda(k\times c, v)$ of size $N\times k$, it holds that

$$N \geq v-1.$$

In particular, $N = v-1$ implies $v = 2c$.

Theorem 3: In an $\mathrm{OMD}_\lambda(k\times c, v)$ of size $N\times k$, it holds that

$$N \geq v. \tag{4}$$

Proof: Since any $\mathrm{OMD}_\lambda(k\times c, v)$ is a $\mathrm{PMA}_{2\lambda}(k\times c, v)$, $N \geq v-1$ holds. For the proof, it is sufficient to show the non-existence of an $\mathrm{OMD}_\lambda(2\times c, v)$ with $N = v-1$. When $N = v-1$, Lemma 5 implies $v = 2c$, that is, v is even and N is odd. On the other hand, $v = 2c$ and (1) imply that $k = 2$ holds and each point appears in all of N rows of the $\mathrm{OMD}_\lambda(2\times c, v)$. Hence, each point cannot occur equally in each of the two columns. □

The existence of an $\mathrm{OMD}_1(2\times c, c^2+1)$, which satisfies $N = v = c^2+1$ and $k = 2$, for any $c \geq 2$ is known in Matsubara and Kageyama (2021) as a $\mathrm{PMA}_2(2\times c, c^2+1)$. Hence the inequality (4) is best possible when $k = 2$. However, any existence result on an $\mathrm{OMD}_\lambda(k\times c, v)$ with $N = v, k \geq 3$ and $c \geq 2$ is not known in literature as far as the authors know.

The minimality of λ is also discussed here. An $\mathrm{OMD}_\lambda(k\times c, v)$ is said to be *minimal* if there exists no $\mathrm{OMD}_{\lambda'}(k\times c, v)$ for any $\lambda' < \lambda$. Especially, it is clear that any OMD with $N = v$ and any OMD with $\lambda = 1$ are minimal. On the other hand, by taking u copies of each row of $\mathcal{A}$, it is clear that the existence of an $\mathrm{OMD}_\lambda(k\times c, v)$ implies the existence of an $\mathrm{OMD}_{\lambda u}(k\times c, v)$. In fact, the existence of a minimal $\mathrm{OMD}_\lambda(3\times 2, v)$ plays an important role in Section 7. Some minimal $\mathrm{OMD}_\lambda(k\times 2, v)$ are exhaustively listed within the scope of $4 \leq v \leq 20$ in Table 1 of Appendix.

4. OMD with a Cyclic Automorphism

Combinatorial multi-arrays (OMA, PMA, OMD) are regarded as a pair $(V, \mathcal{R})$ of a point set V and a set $\mathcal{R}$ of rows. When $V = \mathbb{Z}_v$ (or $V = \mathbb{Z}_{v-1} \cup \{\infty\}$) and $\mathcal{R} = \{\boldsymbol{R} + t \mid \boldsymbol{R} \in \mathcal{R}\}$ with $\boldsymbol{R} + t = (a_{11} + t, \ldots, a_{1c} + t \mid \ldots \mid a_{k1} + t, \ldots, a_{kc} + t)$ for any $t \in \mathbb{Z}_v$ (or any $t \in \mathbb{Z}_{v-1}$), the array is said to be *cyclic* (or *1-rotational*, where ∞ is a fixed point with $\infty + t = \infty$ for any $t \in \mathbb{Z}_{v-1}$). Then a *row orbit* of $\boldsymbol{R} \in \mathcal{R}$ is defined by $\{\boldsymbol{R} + t \mid t \in \mathbb{Z}_v\}$ (or $\{\boldsymbol{R} + t \mid t \in \mathbb{Z}_{v-1}\}$). Note that the length of any row orbit on $\mathbb{Z}_v$ is assumed to be v in this paper. Choose an arbitrary row from each row orbit and call it a *base row*. Hence, for a cyclic multi-array, the array can be represented simply by displaying base rows. For example, the $\mathrm{OMD}_2(3 \times 2, 6)$ given in Example 1 is presented by

$$(\infty, 0 \mid 1, 4 \mid 2, 3), (2, 3 \mid \infty, 0 \mid 1, 4), (1, 4 \mid 2, 3 \mid \infty, 0) \mod 5.$$

For two points x and y in the j_1th and the j_2th $(1 \le j_1 < j_2 \le k)$ entries, respectively, of each base row, $x - y \equiv d \pmod v$ implies that in the orbit of the base row there exists a row containing x' and y' in the j_1th and the j_2th entries, respectively, for any distinct points x', y' in $\mathbb{Z}_v$ with $x' - y' \equiv d \pmod v$. Hence, it is seen that the multi-array obtained from orbits on $\mathbb{Z}_v$ of m base rows $(A^*_{i1} \mid \ldots \mid A^*_{ik})$, $1 \le i \le m$, satisfies the condition (C2) of an $\mathrm{OMD}_\lambda(k \times c, v)$ and the condition (C3) of a $\mathrm{PMA}_\lambda(k \times c, v)$ if

$$\bigcup_{1 \le i \le m} \{d - d' \mid d \in A^*_{ij_1}, d' \in A^*_{ij_2}\} = \lambda\,(\mathbb{Z}_v \backslash \{0\}) \tag{5}$$

and

$$\bigcup_{1 \le i \le m} \{\pm(d - d') \mid d \in A^*_{ij_1}, d' \in A^*_{ij_2}\} = \lambda\,(\mathbb{Z}_v \backslash \{0\}) \tag{6}$$

holds, respectively, for any j_1, j_2 with $1 \le j_1 < j_2 \le k$, where λS means a multi-set containing each element of the set S exactly λ times. Furthermore, m base rows with a 1-rotational automorphism on $\mathbb{Z}_{v-1} \cup \{\infty\}$ yield a multi-array satisfying the condition (C2) if

$$\bigcup_{1 \le i \le m} \{d - d' \mid d \in A^*_{ij_1}, d' \in A^*_{ij_2}\} = \lambda\,((\mathbb{Z}_{v-1} \cup \{\infty\}) \backslash \{0\}), \tag{7}$$

where $\infty - t = t - \infty = \infty$ for any $t \in \mathbb{Z}_{v-1}$.

In fact, it can be checked that the base rows given in Examples 7, 8 (for cyclic OMDs), Examples 3 to 6 (for cyclic PMAs) and Examples 1 and 9 to 12 (for 1-rotational OMDs) satisfy the conditions (5), (6) and (7), respectively.

At first, a direct construction of an $\mathrm{OMD}_2(k \times 2, v)$ is provided as follows.

Lemma 6: Let v be odd and p be the smallest prime factor of v. Then there exists a cyclic $\mathrm{OMD}_2(k \times 2, v)$ with $k = (p-1)/2$.

Proof: Let $\mathcal{R}^*$ be a set of the following $(v-1)/2$ rows:

$$\boldsymbol{R}^*_t = (t, -t \mid 2t, -2t \mid \ldots \mid \frac{p-1}{2}t, -\frac{p-1}{2}t),\ 1 \le t \le \frac{v-1}{2}.$$

Since p is the smallest prime factor of the odd v, $\boldsymbol{R}_t^*$ contains $p-1$ different elements in $\mathbb{Z}_v$ for each t. Moreovre, both $\gcd(j_2-j_1,v)=1$ and $\gcd(j_1+j_2,v)=1$ hold for each j_1,j_2 with $1\le j_1<j_2\le (p-1)/2$. Hence it also holds that

$$\{\pm(j_1t-j_2t)\mid 1\le t\le \frac{v-1}{2}\}=\{\pm(j_1t+j_2t)\mid 1\le t\le \frac{v-1}{2}\}=\mathbb{Z}_v\backslash\{0\}.$$

Since two entries $\{j_1t,-j_1t\}$ and $\{j_2t,-j_2t\}$ yield four differences $\pm(j_1t-j_2t)$ and $\pm(j_1t+j_2t)$, it is shown that $\mathcal{R}^*$ yields the required cyclic $\mathrm{OMD}_2(k\times 2,v)$ with $k=(p-1)/2$. □

Next another method of constructing a cyclic $\mathrm{OMD}_\lambda(k\times c,v)$ from a cyclic $\mathrm{PMD}_\lambda(k\times c,v)$ is presented as follows.

Lemma 7: The existence of a cyclic $\mathrm{PMA}_\lambda(k\times c,v)$ implies the existence of a cyclic $\mathrm{OMD}_\lambda(k\times c,v)$.

Proof: Let a set of m base rows of the cyclic $\mathrm{PMA}_\lambda(k\times c,v)$ be

$$\mathcal{R}^*=\{(A_{i1}^*\mid\dots\mid A_{ik}^*)\mid 1\le i\le m\}.$$

Then take the set $\mathcal{R}^*\cup\mathcal{R}^{**}$ of rows with

$$\mathcal{R}^{**}=\{(-A_{i1}^*\mid\dots\mid -A_{ik}^*)\mid 1\le i\le m\}.$$

Since $\mathcal{R}^*$ satisfies (6), $\mathcal{R}^*\cup\mathcal{R}^{**}$ satisfies (5). Hence $\mathcal{R}^*\cup\mathcal{R}^{**}$ yields the required cyclic $\mathrm{OMD}_\lambda(k\times c,v)$. □

For an odd prime p, a cyclic $\mathrm{OMD}_1(k\times 2,p)$ can be constructed when there exists a point set satisfying the following condition on $\mathbb{Z}_p$:

(L) for any distinct points x,y in the set,

$$\binom{x+y}{p}\binom{x-y}{p}=-1,$$

where $\binom{a}{p}$ is the Legendre symbol of a at p.

Lemma 8: Let $p\equiv 1 \pmod 4$ be an odd prime and α be a primitive element of $\mathbb{Z}_p$. If there exists a k-set S on $\mathbb{Z}_p$ satisfying the condition (L), then a cyclic $\mathrm{OMD}_1(k\times 2,p)$ exists.

Proof: Let $S=\{a_1,a_2,\dots,a_k\}$ be a set satisfying (L) on $\mathbb{Z}_p$. Then, for any x,y satisfying (L), it is seen that $\pm(x+y)\alpha^{2t}$ $(1\le t\le (p-1)/4)$ yield a set of quadratic residues or a set of non-quadratic residues, according as $x+y$ is a quadratic residue or not. The same holds for the case of $\pm(x-y)\alpha^{2t}$.

Hence, for any j_1,j_2 with $1\le j_1<j_2\le k$, it holds that

$$\bigcup_{1\le t\le \frac{p-1}{4}}\{\pm(a_{j_1}+a_{j_2})\alpha^{2t},\pm(a_{j_1}-a_{j_2})\alpha^{2t}\}=\mathbb{Z}_p\backslash\{0\}.$$

On the other hand, two entries $\{a_{j_1}\alpha^{2t}, -a_{j_1}\alpha^{2t}\}$ and $\{a_{j_2}\alpha^{2t}, -a_{j_2}\alpha^{2t}\}$ yield four differences $\pm(a_{j_1}+a_{j_2})\alpha^{2t}$ and $\pm(a_{j_1}-a_{j_2})\alpha^{2t}$ for any t with $1 \le t \le (p-1)/4$ and $1 \le j_2 < j_2 \le k$. Therefore the base rows

$$(a_1\alpha^{2t}, -a_1\alpha^{2t} \mid \cdots \mid a_k\alpha^{2t}, -a_k\alpha^{2t})$$

with $1 \le t \le (p-1)/4$ can yield the required $\mathrm{OMD}_1(k \times 2, p)$. □

Examples of such k-set S are presented as follows.

Example 2: The following sets on $\mathbb{Z}_p$ satisfy the condition (L)

$$\{1,3,4\} \text{ on } \mathbb{Z}_{13},\ \{1,2,7\} \text{ on } \mathbb{Z}_{17},\ \{1,2,4\} \text{ on } \mathbb{Z}_{29},\ \{1,4,17\} \text{ on } \mathbb{Z}_{37},\ \{1,7,8\} \text{ on } \mathbb{Z}_{41}.$$

In the case where $\lambda = 2$ and even v, the 1-rotational automorphism is useful to construct an $\mathrm{OMD}_2(k \times c, v)$. Examples 9 to 12 (for 1-rotational OMDs) are used for the proof of Theorem 2.

5. GDD Construction

For combinatorial multi-arrays with fixed k and c, the GDD construction in the literature (*e.g.*, Li *et al.*, 2018; Matsubara and Kageyama, 2021) is useful to show the complete existence of multi-arrays for any v. Now, the GDD construction of an $\mathrm{OMD}_\lambda(k \times c, v)$ is presented.

Lemma 9: The existence of a (k,λ)-GDD of type $h_1^{t_1}h_2^{t_2}\cdots h_n^{t_n}$, an $\mathrm{OD}_1(k,k)$ and an $\mathrm{OMD}_\lambda(k \times c, h_i c + 1)$ for each i $(1 \le i \le n)$ implies the existence of an $\mathrm{OMD}_\lambda(k \times c, v^*)$ with $v^* = c(h_1 t_1 + \cdots + h_n t_n) + 1$.

Proof: Let G_ℓ be a group of a (k,λ)-GDD of type $h_1^{t_1}h_2^{t_2}\cdots h_n^{t_n}$ on $\mathbb{Z}_v$ with $1 \le \ell \le u$, $v = \sum_{i=1}^n h_i t_i$ and $u = \sum_{i=1}^n t_i$. Then, we take the direct product $\mathbb{Z}_v \times \mathbb{Z}_c$, and let $V = (\mathbb{Z}_v \times \mathbb{Z}_c) \cup \{\infty\}$ be a point set of the required $\mathrm{OMD}_\lambda(k \times c, v^*)$.

Further let the ith block of the (k,λ)-GDD of type $h_1^{t_1}h_2^{t_2}\cdots h_n^{t_n}$ be

$$\{v_{i1}, v_{i2}, \ldots, v_{ik}\},\ 1 \le i \le b,$$

where b is the number of blocks of the (k,λ)-GDD. Let the jth row of an $\mathrm{OD}_1(k,k)$ on $\mathbb{Z}_k$ be

$$(a_{j1}, a_{j2}, \ldots, a_{jk}),\ 1 \le j \le k(k-1).$$

Then replace each point $v_{ii'} \in \mathbb{Z}_v$ with a subset $B_{ii'} = \{(v_{ii'}, e) \mid e \in \mathbb{Z}_c\}$ for $1 \le i \le b$ and $1 \le i' \le k$. In this case the following row set:

$$\mathcal{R}_0 = \{\left(B_{ia_{j1}} \mid B_{ia_{j2}} \mid \ldots \mid B_{ia_{jk}}\right) \mid 1 \le i \le b, 1 \le j \le k(k-1)\}$$

is at first considered.

Furthermore, let $\mathcal{R}_\ell$ on $(G_\ell \times \mathbb{Z}_c) \cup \{\infty\}$ with $1 \le \ell \le u$ be the row sets obtained from the $\mathrm{OMD}_\lambda(k \times c, h_i c + 1)$ with $|G_\ell| = h_i$ for some i $(1 \le i \le n)$. Then, any ordered pair of two points (x, c_1) and (y, c_2) with $x, y \in G_\ell$ and $c_1, c_2 \in \mathbb{Z}_c$ for any ℓ appears in λ rows of any ordered two columns of $\mathcal{R}_\ell$, and does not appear in different entries of any row of other row sets. Moreover, any ordered pair of two points (x, c_1) and (y, c_2) with $x \in G_\ell, y \in G_{\ell'}$ and $c_1, c_2 \in \mathbb{Z}_c$ for any ℓ, ℓ' $(\ell \neq \ell')$ appears in λ rows of any ordered two columns of $\mathcal{R}_0$, while it does not appear in different entries of any row of other row sets.

Hence, the union of these row sets $\mathcal{R}_0 \cup \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_u$ can yield the required $\mathrm{OMD}_\lambda(k \times c, v^*)$. □

Moreover, the following result can also be obtained.

Lemma 10: The existence of a (k, λ)-GDD of type $h_1^{t_1} h_2^{t_2} \cdots h_n^{t_n}$, an $\mathrm{OD}_1(k, k)$ and an $\mathrm{OMD}_\lambda(k \times c, h_i c)$ for each i $(1 \leq i \leq n)$ implies the existence of an $\mathrm{OMD}_\lambda(k \times c, v^*)$ with $v^* = c(h_1 t_1 + \cdots + h_n t_n)$.

Proof: Let G_ℓ $(1 \leq \ell \leq u)$ and $\mathcal{R}_0$ be the same as in the proof of Lemma 9. Moreover, let $\mathcal{R}_\ell$ on $G_\ell \times \mathbb{Z}_c$ with $1 \leq \ell \leq u$ be the row sets obtained from the $\mathrm{OMD}_\lambda(k \times c, h_i c)$ with $|G_\ell| = h_i$ for some i $(1 \leq i \leq n)$.

By discussion similar to the proof of Lemma 9, the union of these row sets $\mathcal{R}_0 \cup \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_u$ can yield the required $\mathrm{OMD}_\lambda(k \times c, v^*)$. □

The following existence results on GDDs are obtained by checking that the parameters satisfy the conditions described in Lemma 4.

Lemma 11: There exist a $(3, 1)$-GDD of type $6^u 6^1$, a $(3, 1)$-GDD of type $6^u 8^1$ and a $(3, 1)$-GDD of type $6^u 10^1$ for any $u \geq 3$.

Lemma 12: There exist a $(3, 1)$-GDD of type 3^3, a $(3, 1)$-GDD of type 4^3, a $(3, 1)$-GDD of type 5^3 and a $(3, 1)$-GDD of type $3^4 5^1$.

Note that a (k, λ)-GDD with $\lambda \geq 1$ can be obtained from a $(k, 1)$-GDD by taking λ copies of each block.

6. Construction from k-SOLS(v)

Let $L = (a_{ij})$ and $L' = (a'_{ij})$ are two latin squares of order v. The latin squares L and L' are said to be *orthogonal* if all ordered pairs (a_{ij}, a'_{ij}) are distinct. A set of latin squares $L_1, \ldots, L_s$ is called *mutually orthogonal latin squares* of order v, denoted by s-MOLS(v), if they are orthogonal in each pair. A *self-orthogonal latin square* of order v is a latin square that is orthogonal to its transpose. A set $\{L_1, \ldots, L_s\}$ of self-orthogonal latin squares of order v is denoted by s-SOLS(v), if $\{L_1, L_2^T, \ldots, L_s, L_s^T\}$ is a $2s$-MOLS(v). Without loss of generality, any latin square in an s-SOLS(v) can be replaced by a latin square with $a_{ii} = i$, by renaming the symbols.

Lemma 13 (Abel and Bennet, 2012)**:** There exists a 2-SOLS(v) for any positive integer v, except for $v \in \{2, 3, 4, 5, 6\}$ and possibly for $v \in \{10, 12, 14, 18, 21, 22, 24, 30, 34\}$.

Lemma 14 (Finizio and Zhu, 2007)**:** There exists a $(2^{n-1} - 1)$-SOLS(2^n) for any $n \geq 2$.

It is well known (see Bierbrauer, 2007) that the existence of a k-MOLS(v), all of whose squares satisfy $a_{ii} = i$ with $1 \leq i \leq v$, is equivalent to the existence of an $\mathrm{OD}_1(k+2, v)$. Moreover, in Matsubara and Kageyama (2015) and Sawa *et al.* (2007), some type of combinatorial designs, called pairwise additive BIB designs, are constructed by use of a k-SOLS(v). In a manner similar to Matsubara and Kageyama (2015) and Sawa *et al.* (2007), the following construction is presented.

Lemma 15: The existence of a k-SOLS(v) implies the existence of an $\mathrm{OMD}_2((k+1)\times 2, v)$.

Proof: Let a set of $2k$-MOLS(v) derived from the k-SOLS(v) be $\{L_h, L_h^T \mid 1 \leq h \leq k\}$, where $L_h = (a_{ij}^{(2h-1)})$, $L_h^T = (a_{ij}^{(2h)}) = (a_{ji}^{(2h-1)})$ and $a_{ii}^{(2h-1)} = a_{ii}^{(2h)} = i$ $(1 \leq i \leq v)$. Further let $\mathcal{R}$ be a set of the following $v(v-1)/2$ rows:

$$(i, j \mid a_{ij}^{(1)}, a_{ij}^{(2)} \mid a_{ij}^{(3)}, a_{ij}^{(4)} \mid \cdots \mid a_{ij}^{(2k-1)}, a_{ij}^{(2k)})$$

with $1 \leq i < j \leq v$.

Then $(a_{ij}^{(2h_1-1)}, a_{ij}^{(2h_2-1)})$ and $(a_{ij}^{(2h_1)}, a_{ij}^{(2h_2)})$, for $1 \leq i < j \leq v$ and each h_1, h_2 of $1 \leq h_1 < h_2 \leq k$, yield all of pairs of distinct points in V, since L_{h_1} and L_{h_2} are orthogonal. Moreover, $(a_{ij}^{(2h_1)}, a_{ij}^{(2h_2-1)})$ and $(a_{ij}^{(2h_1-1)}, a_{ij}^{(2h_2)})$ for $1 \leq i < j \leq v$ also yield all of pairs of distinct points in V, since $L_{h_1}^T$ and L_{h_2} are orthogonal. Hence it is seen that the above-mentioned $\mathcal{R}$ yields an $\mathrm{OMD}_2((k+1)\times 2, v)$. □

Now, two families of an $\mathrm{OMD}_2(k\times 2, v)$ can be constructed by taking Lemma 15 with Lemmas 13 and 14 as the following shows.

Lemma 16: There exists an $\mathrm{OMD}_2(3\times 2, v)$ for any $v \geq 7$ except for $v \in \{10, 12, 14, 18, 21, 22, 24, 30, 34\}$.

Lemma 17: There exists an $\mathrm{OMD}_2(2^{n-1}\times 2, 2^n)$ for any $n \geq 2$.

7. Proof of Main Results

We are now in a position to prove Theorems 1 and 2.

Proof of Theorem 1: For an odd prime power q, the existence of the required $\mathrm{OMD}_\lambda(k\times 2, q)$ with $2 \leq k \leq (q-1)/2$ is shown by taking Lemmas 2 and 3 with some copies of rows. On the other hand, the existence of the required $\mathrm{OMD}_\lambda(k\times 2, 2^n)$ with $n \geq 2$ and $2 \leq k \leq 2^{n-1}$ is shown by use of Lemmas 3 and 17 and taking copies of rows. □

Proof of Theorem 2: For the complete proof, it is enough to show the existence of the following cases:

(I) $v \equiv 1 \pmod 4$ and $v \neq 9$ when $\lambda \geq 1$,
(II) $v \equiv 0, 2, 3 \pmod 4$ when $\lambda \equiv 0 \pmod 2$,
(III) $v = 9$ and $\lambda \geq 2$.

In Cases (I) and (II), minimal $\mathrm{OMD}_\lambda(3\times 2, v)$, *i.e.*, $\lambda = 1$ and $\lambda = 2$, respectively, are firstly constructed and then the existence for any λ is shown by taking copies of rows of the OMD. Since the existence of a minimal $\mathrm{OMD}_\lambda(3\times 2,9)$, *i.e.*, $\lambda=1$, is unknown, the existence of $\mathrm{OMD}_\lambda(3\times 2,9)$ for any $\lambda\geq 2$ is shown in Case (III) by using examples with $\lambda=2,3$.

Case (I): Lemma 7 with Examples 5 and 6 shows the existence of an $\mathrm{OMD}_1(3\times 2,v)$ with $v=25,33$. Lemma 8 with Example 2 shows the existence of an $\mathrm{OMD}_1(3\times 2,v)$ with $v=13,17,29,37,41$. Moreover, Examples 7 and 8 show the existence of an $\mathrm{OMD}_1(3\times 2,v)$ with $v=21,45$.

On the other hand, by Lemma 11, there exist a $(3,1)$-GDD of type 6^u6^1, a $(3,1)$-GDD of type 6^u8^1 and a $(3,1)$-GDD of type 6^u10^1 for any $u\geq 3$. Now consider the $\mathrm{OD}_1(3,3)$ given in Lemma 1 and the $\mathrm{OMD}_1(3\times 2,v)$ with $v=6\cdot 2+1,8\cdot 2+1,10\cdot 2+1=13,17,21$ given above. Then Lemma 9 yields (i) an $\mathrm{OMD}_1(3\times 2,v)$ with $v\geq 49$ and $v\equiv 1 \pmod{12}$ from the $(3,1)$-GDD of type 6^u6^1, (ii) an $\mathrm{OMD}_1(3\times 2,v)$ with $v\geq 53$ and $v\equiv 5 \pmod{12}$ from the $(3,1)$-GDD of type 6^u8^1, and (iii) an $\mathrm{OMD}_1(3\times 2,v)$ with $v\geq 57$ and $v\equiv 9 \pmod{12}$ from the $(3,1)$-GDD of type 6^u10^1.

Hence, for Case (I), the required multi-arrays are constructed by taking copies of rows of the $\mathrm{OMD}_1(3\times 2,v)$.

Case (II): Lemma 16 gives an $\mathrm{OMD}_2(3\times 2,v)$ with $v\equiv 0,2,3 \pmod 4$ except for $v\in\{6,10,12,14,18,22,24,30,34\}$. Examples 1 and 9 to 12 yield an $\mathrm{OMD}_2(3\times 2,v)$ with $v\in\{6,10,12,14,22\}$.

On the other hand, by Lemma 12 with use of two copies of rows, there exist a $(3,2)$-GDD of type 3^3, a $(3,2)$-GDD of type 4^3, a $(3,2)$-GDD of type 5^3 and a $(3,2)$-GDD of type 3^45^1. Now consider the $\mathrm{OD}_1(3,3)$ and the $\mathrm{OMD}_2(3\times 2,v)$ with $v=3\cdot 2,4\cdot 2,5\cdot 2=6,8,10$ given above. Then Lemma 10 yields an $\mathrm{OMD}_2(3\times 2,v)$ with $v\in\{18,24,30,34\}$. Thus, for Case (II), the required multi-arrays are constructed by taking copies of rows of the $\mathrm{OMD}_2(3\times 2,v)$.

Case (III): Lemma 7 with Examples 3 and 4 shows the existence of an $\mathrm{OMD}_\lambda(3\times 2,9)$ with $\lambda=2,3$. Hence, for Case (III), the required multi-arrays are constructed by combining u copies and u' copies of rows of the $\mathrm{OMD}_2(3\times 2,9)$ and the $\mathrm{OMD}_3(3\times 2,9)$, respectively, with $\lambda=2u+3u'$ $(u\geq 0,u'\geq 0)$. □

8. Concluding Remark

Theorem 1 shows the existence of an $\mathrm{OMD}_\lambda(k\times 2,q)$ for any prime power q except possibly for $q\equiv 1 \pmod 4$ and $\lambda\equiv 1 \pmod 2$. Moreover, Theorem 2 shows that the necessary conditions (1) (2) and (3) are also sufficient for the existence of an $\mathrm{OMD}_\lambda(3\times 2,v)$ except possibly for an $\mathrm{OMD}_1(3\times 2,9)$. Unfortunately, the existence of the $\mathrm{OMD}_1(k\times 2,q)$ with $k\geq 4,q\equiv 1 \pmod 4$ and the $\mathrm{OMD}_1(3\times 2,9)$ cannot be proved by any method in this paper.

Lemma 7 together with the asymptotic existence results on a cyclic $\mathrm{PMA}_1(k\times 2,v)$ given in Li *et al.* (2018) and Matsubara and Kageyama (2021) can provide some asymptotic existence of a cyclic $\mathrm{OMD}_1(k\times 2,v)$ which is minimal. However, it seems difficult to show both of the exact and asymptotic existence of an $\mathrm{OMD}_\lambda(k\times c,v)$ with $N=v$, $k\geq 3$ and $c\geq 2$.

Finally, though we can find some applications of combinatorial structures (OMA, PMA, OD) related to the OMD as stated in Sections 1 and 2, any application of the OMD is not presented anywhere, including this paper. It will be discussed in a forthcoming paper.

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Appendix

Some individual examples which can be found by use of a computer are presented. Note that each of such examples cannot be presented by use of the construction methods provided in this paper.

Example 3: A cyclic $\text{PMA}_2(3 \times 2, 9)$ on $\mathbb{Z}_9$ is given by

$$(0,1 \mid 2,4 \mid 3,6), (0,7 \mid 1,2 \mid 5,8) \quad \text{mod } 9.$$

Example 4: A cyclic $\text{PMA}_3(3 \times 2, 9)$ on $\mathbb{Z}_9$ is given by

$$(0,8 \mid 2,3 \mid 1,5), (0,7 \mid 1,2 \mid 3,6), (0,6 \mid 5,7 \mid 2,8) \quad \text{mod } 9.$$

Example 5: A cyclic $\text{PMA}_1(3 \times 2, 25)$ on $\mathbb{Z}_{25}$ is given by

$$(0,12 \mid 3,23 \mid 17,18), (0,22 \mid 12,21 \mid 13,24), (0,24 \mid 5,7 \mid 3,14) \quad \text{mod } 25.$$

Example 6: A cyclic $\text{PMA}_1(3 \times 2, 33)$ on $\mathbb{Z}_{33}$ is given by

$$\begin{aligned}&(0,16 \mid 17,27 \mid 12,13), (0,1 \mid 14,24 \mid 2,16), (0,1 \mid 8,30 \mid 7,24),\\ &(0,3 \mid 15,31 \mid 22,28) \quad \text{mod } 33.\end{aligned}$$

Example 7: A cyclic $\text{OMD}_1(3 \times 2, 21)$ on $\mathbb{Z}_{21}$ is given by

$$\begin{aligned}&(10,11 \mid 5,16 \mid 9,12), (9,12 \mid 8,13 \mid 2,19), (10,11 \mid 8,13 \mid 5,16),\\ &(2,19 \mid 9,12 \mid 10,11), (10,11 \mid 2,19 \mid 7,14) \quad \text{mod } 21.\end{aligned}$$

Example 8: A cyclic $\text{OMD}_1(3 \times 2, 45)$ on $\mathbb{Z}_{45}$ is given by

$$\begin{aligned}&(19,26 \mid 17,28 \mid 3,42), (20,25 \mid 21,24 \mid 6,39), (17,28 \mid 6,39 \mid 11,34),\\ &(11,34 \mid 6,39 \mid 10,35), (9,36 \mid 21,24 \mid 11,34), (21,24 \mid 11,34 \mid 13,32),\\ &(17,28 \mid 9,36 \mid 10,35), (4,41 \mid 10,35 \mid 1,44), (22,23 \mid 2,43 \mid 10,35),\\ &(17,28 \mid 10,35 \mid 13,32), (13,32 \mid 16,29 \mid 22,23) \quad \text{mod } 45.\end{aligned}$$

Example 9: A 1-rotational $\text{OMD}_2(3 \times 2, 10)$ on $\mathbb{Z}_9$ is given by

$$\begin{aligned}&(0,\infty \mid 1,5 \mid 6,8), (2,7 \mid 0,\infty \mid 1,4), (2,4 \mid 5,7 \mid 0,\infty), (0,6 \mid 4,8 \mid 1,7),\\ &(0,7 \mid 4,6 \mid 2,3) \quad \text{mod } 9.\end{aligned}$$

Example 10: A 1-rotational $\text{OMD}_2(3 \times 2, 12)$ on $\mathbb{Z}_{11}$ is given by

$$\begin{aligned}&(0,\infty \mid 4,7 \mid 1,10), (2,9 \mid 0,\infty \mid 5,6), (4,7 \mid 1,10 \mid 0,\infty), (2,9 \mid 3,8 \mid 1,10),\\ &(4,7 \mid 5,6 \mid 2,9), (2,9 \mid 5,6 \mid 4,7) \quad \text{mod } 11.\end{aligned}$$

Example 11: A 1-rotational $\text{OMD}_2(3 \times 2, 14)$ on $\mathbb{Z}_{13}$ is given by

$$\begin{aligned}&(0,\infty \mid 1,12 \mid 6,7), (5,8 \mid 0,\infty \mid 3,10), (4,9 \mid 3,10 \mid 0,\infty), (4,9 \mid 6,7 \mid 5,8),\\ &(6,7 \mid 2,11 \mid 4,9), (2,11 \mid 4,9 \mid 5,8), (3,10 \mid 6,7 \mid 2,11) \quad \text{mod } 13.\end{aligned}$$

Example 12: A 1-rotational $\mathrm{OMD}_2(3 \times 2, 22)$ on $\mathbb{Z}_{21}$ is given by

$$(0,\infty \mid 7,14 \mid 5,16), (7,14 \mid 0,\infty \mid 10,11), (5,16 \mid 8,13 \mid 0,\infty), (9,12 \mid 8,13 \mid 1,20),$$
$$(7,14 \mid 9,12 \mid 5,16), (5,16 \mid 1,20 \mid 7,14), (7,14 \mid 4,17 \mid 1,20), (5,16 \mid 4,17 \mid 2,19),$$
$$(9,12 \mid 3,18 \mid 8,13), (7,14 \mid 9,12 \mid 8,13), (9,12 \mid 1,20 \mid 2,19) \quad \mathrm{mod}\ 21.$$

Finally, a table of the existence of a minimal $\mathrm{OMD}_\lambda(k \times 2, v)$ shown by our methods is presented for $4 \leq v \leq 20$. When $c = 2$ is fixed, N and λ are uniquely determined by v. For v, N and λ, values of k are indicated about known or unknown existence of the OMD. Note that values of bold k represent the upper bound of k obtained from (1) and "–" in the column of unknown implies that the complete existence of an $\mathrm{OMD}_\lambda(k \times c, v)$ is shown. Moreover, for two minimal OMDs of Nos. 2 and 6 which cannot be obtained by Theorems 1 and 2, base rows are newly given.

Table 1: Minimal $\mathrm{OMD}_\lambda(k \times c, v)$ with $4 \leq v \leq 20, c = 2$

No	v	N	λ	known	unknown	Source
1	4	6	2	$k = \mathbf{2}$	–	Theorem 1
2	5	5	1	$k = \mathbf{2}$	–	$(1,4 \mid 2,3)$ mod 5
3	6	15	2	$2 \leq k \leq \mathbf{3}$	–	Theorem 2
4	7	21	2	$2 \leq k \leq \mathbf{3}$	–	Theorem 1
5	8	28	2	$2 \leq k \leq \mathbf{4}$	–	Theorem 1
6	9	18	1	$k = 2$	$3 \leq k \leq \mathbf{4}$	$(0,1 \mid 2,4), (2,4 \mid 0,1)$ mod 9
7	10	45	2	$2 \leq k \leq 3$	$4 \leq k \leq \mathbf{5}$	Theorem 2
8	11	55	2	$2 \leq k \leq \mathbf{5}$	–	Theorem 1
9	12	66	2	$2 \leq k \leq 3$	$4 \leq k \leq \mathbf{6}$	Theorem 2
10	13	39	1	$2 \leq k \leq 3$	$4 \leq k \leq \mathbf{6}$	Theorem 2
11	14	91	2	$2 \leq k \leq 3$	$4 \leq k \leq \mathbf{7}$	Theorem 2
12	15	105	2	$2 \leq k \leq 3$	$4 \leq k \leq \mathbf{7}$	Theorem 2
13	16	120	2	$2 \leq k \leq \mathbf{8}$	–	Theorem 1
14	17	68	1	$2 \leq k \leq 3$	$4 \leq k \leq \mathbf{8}$	Theorem 2
15	18	153	2	$2 \leq k \leq 3$	$4 \leq k \leq \mathbf{9}$	Theorem 2
16	19	171	2	$2 \leq k \leq \mathbf{9}$	–	Theorem 1
17	20	190	2	$2 \leq k \leq 3$	$4 \leq k \leq \mathbf{10}$	Theorem 2