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Change Point Detection using Kumaraswamy Power Lomax Distribution

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Abstract

The Kumaraswamy Power Lomax distribution is an extension of Power Lomax distribution which can be applied many fields engineering, finance and medical research. In this paper, we study a change point problem of this distribution. A procedure based on Modified Information Criterion (MIC) is proposed to detect change point(s) in parameters of this distribution through binary segmentation. The practical applications are provided to illustrate the detection of multiple change points.

Key words: Kumaraswamy power lomax; Change point; Information criterion; Binary segmentation.

AMS Subject Classifications: 92B15, 62P10

1. Introduction

Lomax distribution (Lomax, 1954) is one of the more versatile forms of the Pareto distributional forms. The extension works of Lomax distribution are carried out by many researchers like Kumaraswamy-generalized Lomax distribution (Shams, 2013), type II Topp-Leone Power Lomax (TIITLPL) distribution (Al-Marzouki, Jamal, Chesneau and Elgarhy, 2020), Marshall-Olkin exponential Lomax distribution (Nagarjuna VBV and Vishnu Vardhan, 2020), and Sine Power Lomax (Nagarjuna, Vardhan and Chesneau, 2021b) to utilizing generalized family of distributions by adding additional parameters to the model.

In generalized family of distributions, Kumaraswamy generalized family of distributions is a well know family and has been utilized by several researchers to come out with new functional forms. To mention a few, there are the Kumaraswamy-Weibull distribution (Cordeiro, Ortega and Nadarajah, 2010), Kumaraswamy-Burr XII (KBXII) distribution (Paranaiba, Ortega, Cordeiro and Pascoa, 2013) and Kumaraswamy generalized Power Lomax distribution(KPL)(Nagarjuna, Vardhan and Chesneau, 2021a).

Recently, the attractive properties of Power Lomax distributions and its mathematical tractability was presented by Nagarjuna et al. (2021a). The cumulative distribution

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function (cdf) and probability density function (pdf) of KPL distribution are

$$F_{KPL}(x;\xi) = 1 - \left\{1 - \left[1 - \left(\frac{\lambda}{\lambda + x^{\beta}}\right)^{\alpha}\right]^{a}\right\}^{b}, \quad x > 0, \tag{1}$$

where $\xi = (\alpha, \beta, \lambda, a, b) > 0$, and

$$f_{KPL}(x;\xi) = \frac{ab\alpha\beta}{\lambda} x^{\beta-1} \left(\frac{\lambda}{\lambda + x^{\beta}}\right)^{\alpha+1} \left[1 - \left(\frac{\lambda}{\lambda + x^{\beta}}\right)^{\alpha}\right]^{a-1} \left\{1 - \left[1 - \left(\frac{\lambda}{\lambda + x^{\beta}}\right)^{\alpha}\right]^{a}\right\}^{b-1}. \tag{2}$$

The different shapes of the KPL distribution has been observed at several parameters of the distribution and very nicely depicted in Figure (1). From this Figure (1), we can observe that the density curves of the KPL distributional are decreasing or uni-modal shapes with very flexible to the skewness too.

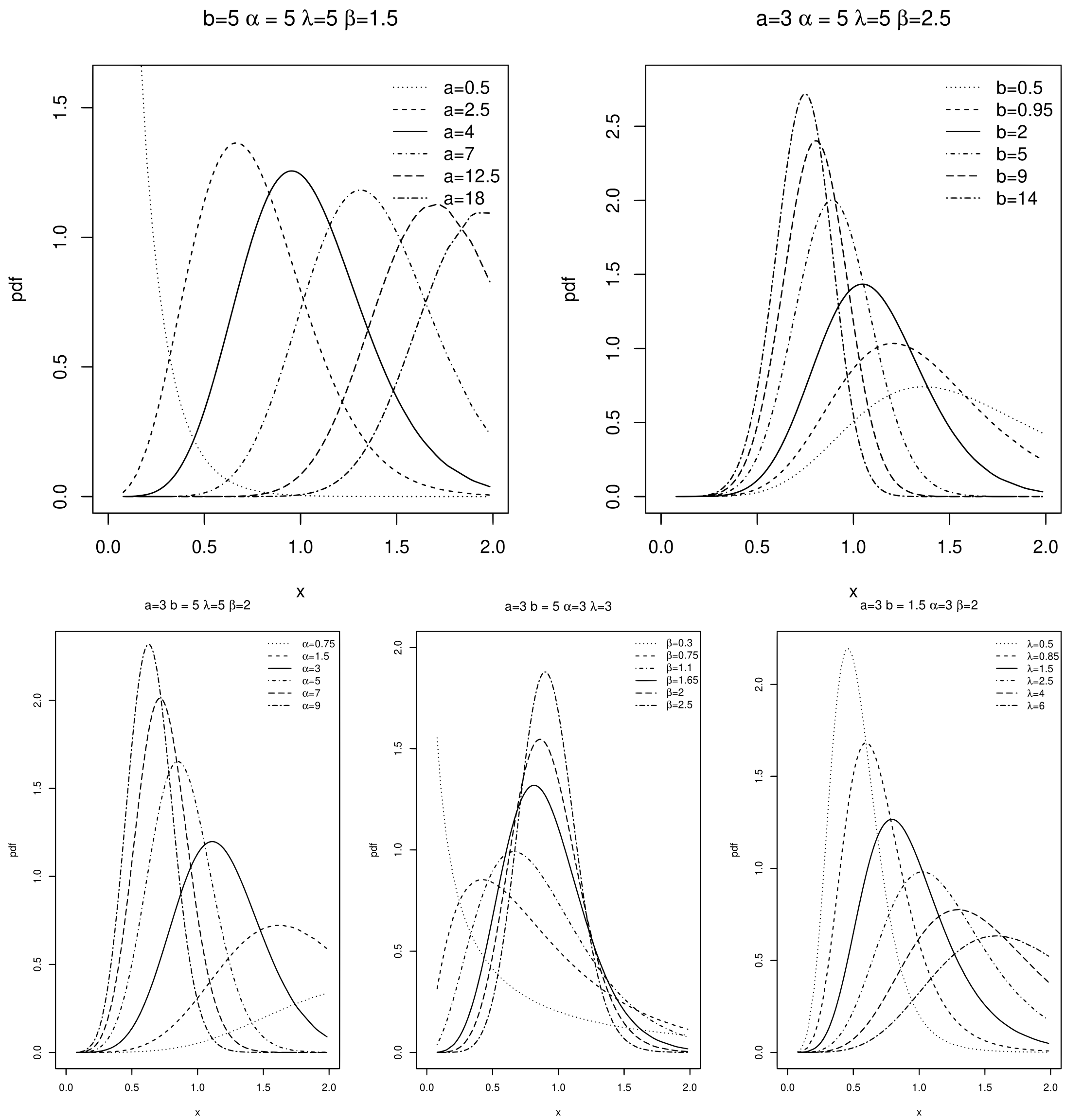

Figure 1: Density curves of KPL distribution at different parameter values

Now-a-days, the study of Change Point (CP) problem is has grabbed the attention of many researchers from industry, weather, quality control etc. Briefly, the CP problem is the problem to study a change or changes in data. A change in data represents the point before which the data follows a distribution and follows a different distribution after that point. Initial works in CP detection was by Page (1954, 1955), where the methods for the detection of single and multiple points were addressed.

In general, the change point problem involves two steps: estimation and hypothesis testing. In hypothesis testing step, we test the null hypothesis of no change versus the alternative of at least one change in data. If the null hypothesis is rejected, we move from the estimation step to estimate the location of change. Otherwise we stop and conclude there is no change in data. To know about the theoritical developments and applications of change point problems, readers can refer to the work of Chen and Gupta (2011). and applications in this field.

2. Change Point Methodology

Let $x_1, x_2, \ldots, x_n$ be a sequence of independent observations that follows a particular distribution. We would like to test the null hypothesis (H_0) versus the alternative hypothesis (H_1), which refers in testing the presence of atmost one change point in the data. For this problem, the binary segmentation procedure and MIC are used to search for all possible change points.
We define the H_0 and H_1 as

$$H_0 : \xi_1 = \xi_2 = \ldots = \xi_n \ vs \ H_1 : \xi_1 = \xi_2 = \ldots = \xi_k \neq \xi_{k+1} = \ldots = \xi_n$$

here $\xi_i; i = 1, 2, \ldots n$ is a parameter set of a particular distribution and 'k' is the position of the change point. If a change point is detected then we choose H_1, otherwise. Let us assume that 'k' is the change point location and at this point the data gets divided into segments, *first segment* will be from 1 to k^{th} point and the *second segment* will start from $(k+1)^{th}$ and n.

Generally, for finding a change point problem there are two popular methods such as, likelihood ratio test (LRT) and Bayesian procedures. Apart from these methods, researchers have also shown interest in the well known for model selection that is Akaike Information Criterion (AIC) and Bayes or Schwarz Information Criterion (BIC).

Chen and Gupta (1997) proposed a test to locate the change in variances of the normal distribution using BIC. Later, Chen et al. (2006) pointed out, that the BIC do not concentrate more on penalty term and it needs some modifications related to the concepts of change point problems. They proposed a new information criterion and named it as the Modified Information Criterion (MIC) which is the modification of the approach based on BIC. This is done for refining the model complexity as a function of the change location in the context of change point problem.

3. Binary Segmentation Procedure

Vostrikova (1981) developed the binary segmentation procedure which was shown to be consistent. Such a procedure transfers the detection of multiple changes to a sequence of consecutive steps of at most one change in each step. Before starting the procedure, at each iteration we need to test for the goodness of fit of data to a particular distribution. The steps involved in binary segmentation procedure are as follows:

1. Under the H_0 and H_1, the general form of log-likelihood functions are given by $\log L(X,\xi)$ and $\log L(X,\xi',\xi'')$. Here $X \sim f_x(.)$; ξ' is the parameter set of first segment and ξ'' is the parameter set of second segment.
2. Compute the MIC values under H_0 and H_1. The general expressions of MIC under H_0 and H_1 are

$$H_0: \quad MIC_n = -2\,\log L_{H_0} + m\,\log n$$

$$H_1: \quad MIC_k = -2\,\log L_{H_1} + \left(2m + \left(\frac{2m}{n} - 1\right)^2\right)\log n$$

here 'm' is the number of parameters; 'n' is the number of observations at each iteration.
3. If $MIC_n < \underset{1\le k<n}{min} MIC_k$, then we accept H_0, i.e., there is no change point in the data, otherwise we accept H_1 meaning to that there exits a change point. At this change point location, the dataset divide into two segments.
4. This will continue until the condition given in step-3 is satisfied, i.e., H_0 is accepted.

As per the steps of the binary segmentation procedure, the log-likelihood functions and MIC are presented below.

Step-1: Let us define the H_0 and H_1 based on the parameter set of KPL distribution.

$$H_0: \xi_1 = \xi_2 = \ldots = \xi_n \;\; vs \;\; H_1: \xi_1 = \xi_2 = \ldots = \xi_k \neq \xi_{k+1} = \ldots = \xi_n$$

here $\xi_i = (a_i,\, b_i,\, \alpha_i, \beta_i, \lambda_i); i = 1, 2, \ldots n$ which are the parameters of KPL distribution.

Under H_0 and H_1, the log-likelihood functions of KPL distributions are

$$\log L_{H_0}(x;\xi) = n\log\left(\frac{ab\alpha\beta}{\lambda}\right) + (\beta-1)\sum_{i=1}^{n}\log x_i + (\alpha+1)\sum_{i=1}^{n}\log\left(\frac{\lambda}{\lambda + x_i^{\beta}}\right) + (a-1)\sum_{i=1}^{n}\log\Upsilon_i + (b-1)\sum_{i=1}^{n}\log(1-\Upsilon_i^{a})$$

$$\begin{aligned}\log L_{H_1}(x;\xi',\xi'') = {} & k\log\left(\frac{a'b'\alpha'\beta'}{\lambda'}\right) + (\beta'-1)\sum_{i=1}^{n}\log x_i + (\alpha'+1)\sum_{i=1}^{k}\log\left(\frac{\lambda'}{\lambda' + x_i^{\beta'}}\right) \\ & + (a'-1)\sum_{i=1}^{k}\log\Upsilon_i + (b'-1)\sum_{i=1}^{k}\log(1-\Upsilon_i^{a'}) \\ & + (n-k)\log\left(\frac{a''b''\alpha''\beta''}{\lambda''}\right) + (\beta''-1)\sum_{i=k+1}^{n}\log x_i + (\alpha''+1)\sum_{i=k+1}^{n}\log\left(\frac{\lambda''}{\lambda'' + x_i^{\beta''}}\right) \\ & + (a''-1)\sum_{i=k+1}^{n}\log\Upsilon_i + (b''-1)\sum_{i=k+1}^{n}\log(1-\Upsilon_i^{a''})\end{aligned}$$

where it is set $\Upsilon_i = 1 - \left[\lambda/(\lambda + x_i^{\beta})\right]^{\alpha}$.

The maximum likelihood estimators (MLEs) of KPL distribution for H_0 and H_1 are

$$\frac{\partial}{\partial a}\log L_{H_0} = \frac{n}{a} + \sum_{i=1}^{n}\log\Upsilon_i + \sum_{i=1}^{n}(1-b)\frac{\log\Upsilon_i}{(\Upsilon_i^{-a}-1)} = 0$$

$$\frac{\partial}{\partial b}\log L_{H_0}=\frac{n}{b}+\sum_{i=1}^{n}\log(1-\Upsilon_i^{a})=0$$

$$\frac{\partial}{\partial \alpha}\log L_{H_0}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left(\frac{\lambda}{\lambda+x_i^{\beta}}\right)+\sum_{i=1}^{n}\left(\frac{\lambda}{\lambda+x_i^{\beta}}\right)^{\alpha}\log\left(\frac{\lambda}{\lambda+x_i^{\beta}}\right)\left[\frac{a(b-1)}{\Upsilon_i(\Upsilon_i^{-a}-1)}-\frac{(a-1)}{\Upsilon_i}\right]=0$$

$$\frac{\partial}{\partial \beta}\log L_{H_0}=\frac{n}{\beta}+\sum_{i=1}^{n}\log x_i-(\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i^{\beta}}{\lambda+x_i^{\beta}}\right)\left[\left(\frac{\lambda}{\lambda+x_i^{\beta}}\right)^{\alpha-1}\frac{\lambda x_i^{\beta}\log x_i}{(\lambda+x_i^{\beta})^2}\left[\frac{\alpha(a-1)}{\Upsilon_i}-\frac{a\alpha(b-1)}{\Upsilon_i(\Upsilon_i^{-a}-1)}\right]\log x_i=0$$

$$\frac{\partial}{\partial \lambda}\log L_{H_0}=-\frac{n}{\lambda}+\left(\frac{\alpha+1}{\lambda}\right)\sum_{i=1}^{n}\left(\frac{x_i^{\beta}}{\lambda+x_i^{\beta}}\right)+\sum_{i=1}^{n}x_i^{\beta}\left(\frac{\lambda}{\lambda+x_i^{\beta}}\right)^{\alpha+1}\left[\frac{a\alpha(b-1)}{\lambda^2\Upsilon_i(\Upsilon_i^{-a}-1)}-\frac{\alpha(a-1)}{\lambda^2\Upsilon_i}\right]=0$$

$$\frac{\partial}{\partial a'}\log L_{H_1}=\frac{k}{a'}+\sum_{i=1}^{k}\log\Upsilon_i'+\sum_{i=1}^{k}(1-b')\frac{\log\Upsilon_i'}{(\Upsilon_i'^{-a'}-1)}=0$$

$$\frac{\partial}{\partial b'}\log L_{H_1}=\frac{k}{b'}+\sum_{i=1}^{k}\log(1-\Upsilon_i'^{a'})=0,$$

$$\frac{\partial}{\partial \alpha'}\log L_{H_1}=\frac{k}{\alpha'}+\sum_{i=1}^{k}\log\left(\frac{\lambda'}{\lambda'+x_i^{\beta'}}\right)+\sum_{i=1}^{k}\left(\frac{\lambda'}{\lambda'+x_i^{\beta'}}\right)^{\alpha'}\log\left(\frac{\lambda'}{\lambda'+x_i^{\beta'}}\right)\left[\frac{a'(b'-1)}{\Upsilon_i'(\Upsilon_i'^{-a'}-1)}-\frac{(a'-1)}{\Upsilon_i'}\right]=0$$

$$\frac{\partial}{\partial \beta'}\log L_{H_1}=\frac{k}{\beta'}+\sum_{i=1}^{k}\log x_i-(\alpha'+1)\sum_{i=1}^{k}\left(\frac{x_i^{\beta'}}{\lambda'+x_i^{\beta'}}\right)\left[\left(\frac{\lambda'}{\lambda'+x_i^{\beta'}}\right)^{\alpha'-1}\frac{\lambda' x_i^{\beta'}\log x_i}{(\lambda'+x_i^{\beta'})^2}\right]\left[\frac{\alpha'(a'-1)}{\Upsilon_i'}-\frac{a'\alpha'(b'-1)}{\Upsilon_i'(\Upsilon_i'^{-a'}-1)}\right]\log(x_i)=0$$

$$\frac{\partial}{\partial \lambda'}\log L_{H_1}=-\frac{k}{\lambda'}+\left(\frac{\alpha'+1}{\lambda'}\right)\sum_{i=1}^{k}\left(\frac{x_i^{\beta'}}{\lambda'+x_i^{\beta'}}\right)+\sum_{i=1}^{k}x_i^{\beta'}\left(\frac{\lambda'}{\lambda'+x_i^{\beta}{}'}\right)^{\alpha'+1}\left[\frac{a'\alpha'(b'-1)}{\lambda'^2\Upsilon_i'(\Upsilon_i'^{-a'}-1)}-\frac{\alpha(a'-1)}{\lambda'^2\Upsilon_i'}\right]=0$$

where $\Upsilon_i' = 1 - \left[\lambda'/(\lambda' + x_i^{\beta'})\right]^{\alpha'}$ and

$$\frac{\partial}{\partial a''}\log L_{H_1} = \frac{(n-k)}{a''} + \sum_{i=k+1}^{n}\log\Upsilon_i'' + \sum_{i=k+1}^{n}(1-b'')\frac{\log\Upsilon_i''}{(\Upsilon_i''^{-a''}-1)} = 0,$$

$$\frac{\partial}{\partial b''}\log L_{H_1} = \frac{(n-k)}{b''} + \sum_{i=k+1}^{n}\log(1-\Upsilon_i''^{a''}) = 0$$

$$\frac{\partial}{\partial \alpha''}\log L_{H_1} = \frac{(n-k)}{\alpha''} + \sum_{i=k+1}^{n}\log\left(\frac{\lambda''}{\lambda''+x_i^{\beta''}}\right) + \sum_{i=1}^{k}\left(\frac{\lambda''}{\lambda''+x_i^{\beta''}}\right)^{\alpha''}\log\left(\frac{\lambda''}{\lambda''+x_i^{\beta''}}\right)\left[\frac{a''(b''-1)}{\Upsilon_i''(\Upsilon_i''^{-a''}-1)} - \frac{(a''-1)}{\Upsilon_i''}\right] = 0,$$

$$\frac{\partial}{\partial \beta''}\log L_{H_1} = \frac{(n-k)}{\beta''} + \sum_{i=k+1}^{n}\log x_i - (\alpha''+1)\sum_{i=k+1}^{n}\left(\frac{x_i^{\beta''}}{\lambda''+x_i^{\beta''}}\right)\left[\left(\frac{\lambda''}{\lambda''+x_i^{\beta''}}\right)^{\alpha''-1}\frac{\lambda'' x_i^{\beta''}\log x_i}{(\lambda''+x_i^{\beta''})^2}\right]\left[\frac{\alpha''(a''-1)}{\Upsilon_i''} - \frac{a''\alpha''(b''-1)}{\Upsilon_i''(\Upsilon_i''^{-a''}-1)}\right]\log x_i = 0$$

$$\frac{\partial}{\partial \lambda''}\log L_{H_1} = -\frac{(n-k)}{\lambda''} + \left(\frac{\alpha''+1}{\lambda''}\right)\sum_{i=k+1}^{n}\left(\frac{x_i^{\beta''}}{\lambda''+x_i^{\beta''}}\right) + \sum_{i=k+1}^{n} x_i^{\beta''}\left(\frac{\lambda''}{\lambda''+x_i^{\beta''}}\right)^{\alpha''+1}\left[\frac{a''\alpha''(b''-1)}{\lambda''^2\Upsilon_i''(\Upsilon_i''^{-a''}-1)} - \frac{\alpha''(a''-1)}{\lambda''^2\Upsilon_i''}\right] = 0$$

Step-2: The MIC under the null hypothesis (H_0) is defined as

$$H_0 : MIC_n = -2\,\log L_{H_0} + 5\log n \tag{3}$$

where '5' is the number of parameters to be estimated in the KPL distribution. The MIC under the alternative hypothesis (H_1) is defined as

$$H_1 : MIC_k = -2\,\log L_{H_1} + \left(10 + \left(\frac{10}{n} - 1\right)^2\right)\log n \tag{4}$$

for a fixed change at location k.

Once, the MIC_n and MIC_k are obtained, we check for the condition given in step-3 of binary segmentation procedure. *In CPA, the change point is the index of each data point from 1 to n samples. Even after partition, the detection of change point will be indicated by its actual index of 'n' samples.*

4. Applications of KPL distribution in Change Point Detection

The practical applications of Change point detection was considered to the following real data sets as Floyd river data (Mudholkar and Hutson, 1996) and Eruption data (da Silva, de Andrade, Maciel, Campos and Cordeiro, 2013).

Floyd River Data: The dataset is about the recordings of annual flood discharge rates (in ft^3/s) from the Floyd River at James, Iowa. There are total of 39 samples measured between the period of 1935-1973 with 10 years of split.

Table 1: The Annual Flood Discharge Rates of Floyd River

Years	Flood Discharge in (ft^3/s)
1935-1944	1460, 4050, 3570, 2060, 1300, 1390, 1720, 6280, 1360, 7440,
1945-1954	**5320**, 1400, 3240, 2710, 4520, 4840, 8320, **13900**, 71500, 6250,
1955-1964	2260,318, 1330, 970, **1920**, 15100, 2870, 20600, 3810, 726,
1965-1973	7500, **7170**, 2000, 829, 17300, 4740, 13400, 2940, 5660.

Now applying the binary segmentation procedure, the following points are observed.

The $MIC_n > \min_{1 \leq k < n} MIC_k$ i.e., 771.167 > 743.3059. So, this indicates that there is a shift/change in the floods of the years.

1. The first change point location is at $k = 18\,(MIC_k = 742.3649)$. So, the total sample ($n = 39$) gets divided into two segments. First segment is from 1 to 18 data points and rest of them fall under second segment.

2. Continuing the procedure with 18 samples and 21 samples, two more change points are observed. One in the first segment ($n = 18$) and other in the second segment ($n = 21$).

3. The second change point is detected at 11^{th} location ($MIC_k = 297.6745$) of the first 18 samples. Similarly, the 3^{rd} change point is observed at 32^{nd} location of the second segment ($n = 21$) with $MIC_k = 421.9489$.

4. One more change point is noticed between the 19^{th} and 32^{rd} location, i.e., at $k = 25$ with $MIC_k = 248.5449$.

5. In the above step, the condition given in step-3 of binary segmentation procedure is satisfied, hence the procedure will get terminated.

6. In total, in this dataset, four change points are detected at $k = 18$ (Flood discharge rate=13900ft^3/s); $k = 11$ (Flood discharge rate=5320ft^3/s); $k = 32$ (Flood discharge rate=7170ft^3/s) and $k = 25$ (Flood discharge rate=248.5449ft^3/s).

The entire description is depicted in Figures (2-4). The distribution fit at every iteration is computed and the same is presented in Figure (4).

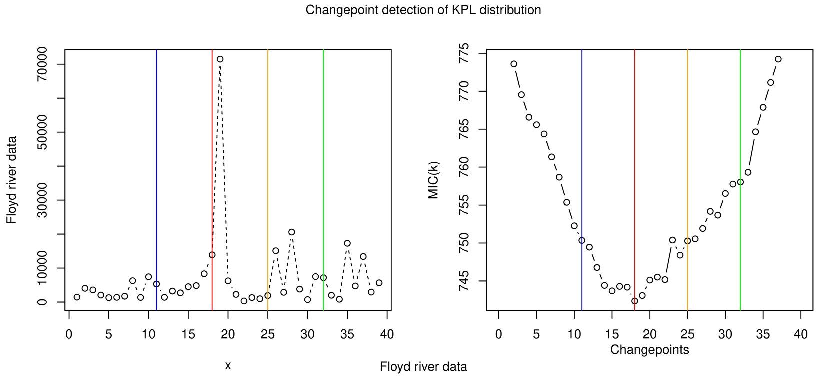

Figure 2: Change point detection using KPL for Floyd river data

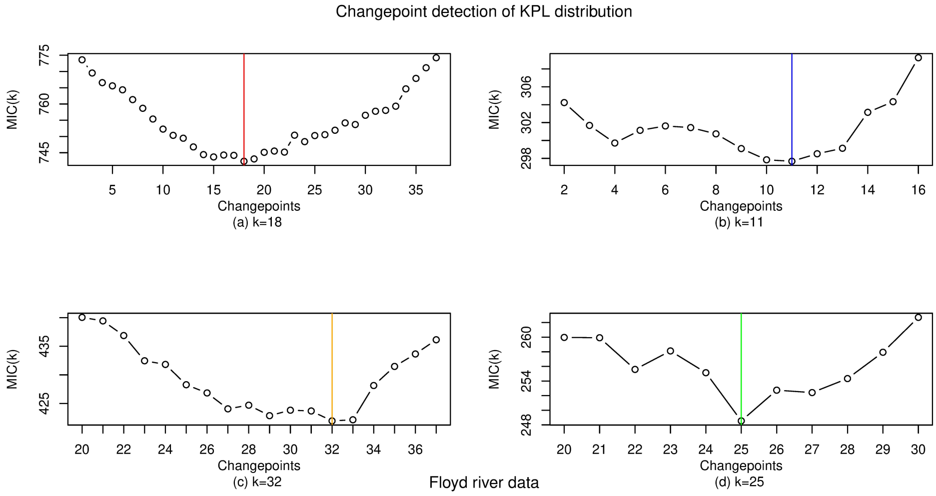

Figure 3: Change point detection values using KPL distribution

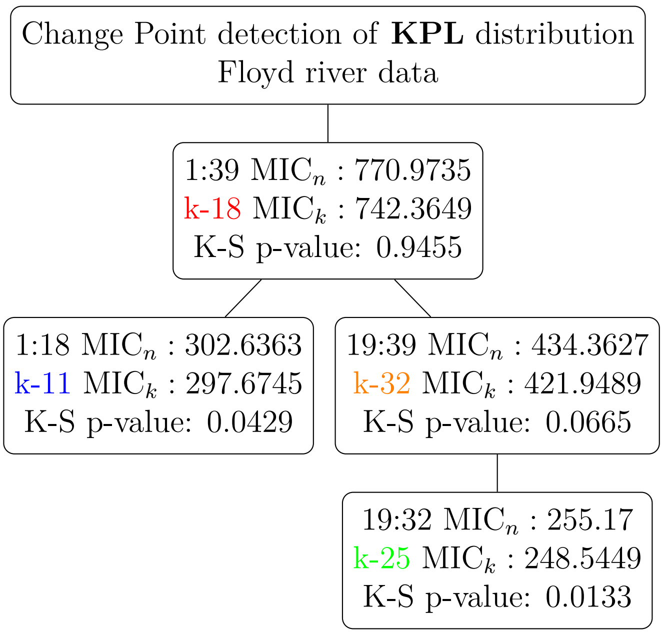

Figure 4: Floyd river data - Decision tree

Eruption Data:

This dataset is about the waiting times (in seconds), between 65 successive eruptions of the Kiama Blowhole. These values were recorded with the aid of digital watch on Jim Irish and the data values are: 83, 51, 87, 60, 28,**95**, 8, 27, 15, 10, 18, 16, 29, 54,**91**, 8, 17, 55, 10, 35, 47, 77, 36, 17, 21,**36**, 18, 40, 10, 7, 34, 27, 28, 56, **8**, 25, 68, 146, 89, 18, 73, **69**, 9,37, 10, 82, 29, 8, 60, 61, **61**, 18, 169, 25, 8, 26, 11, 83, 11, 42, 17, 14, 9, 12.

Now applying the binary segmentation procedure, the following points are observed.

The $MIC_n > \min_{1\leq k<n} MIC_k$ i.e.,609.2285 > 541.1450. So, this indicates that there is a shift/change in the eruption waiting times.

1. The first change point location is at $k = 26\,(MIC_k = 541.1450)$. So, the total sample ($n = 64$) gets divided into two segments. First segment is from 1 to 26 data points and rest of them fall under second segment.

2. Continuing the procedure with 26 samples and 38 samples, two more change points are observed. One in the first segment ($n = 26$) and other in the second segment ($n = 38$).

3. The second change point is detected at 15^{th} location ($MIC_k = 128.2241$) of the first 25 samples. Similarly, the 3^{rd} change point is observed at 42^{nd} location of the second segment ($n = 38$) with $MIC_k = 334.2032$.

4. Here, again the second segment gets divided into two segments. We observed that, two more change points are detected i.e., at $k = 35$ (between 27^{th} and 42^{th} data points) ($MIC_k = 94.8768$) and $k = 60$ (between 43^{rd} and 64^{th} data points) ($MIC_k = 201.8235$).

5. One more change point is noticed between the 1^{st} and 15^{th} location, i.e., at $k = 6\,(MIC_k = 55.5510)$.

6. In the above step, the condition given in step-3 of binary segmentation procedure is satisfied, hence the procedure will get terminated.

7. In total, in this dataset, six change points are detected at $k = 26$ (Eruption waiting time=36s); $k = 15$ (Eruption waiting time=91s); $k = 42$ (Eruption waiting time=9s); $k = 35$ (Eruption waiting time=8s); $k = 51$ (Eruption waiting time=61s); and $k = 6$ (Eruption waiting time=95s).

The entire description is depicted in Figures (5-7). The distribution fit at every iteration is computed and the same is presented in Figure (7).

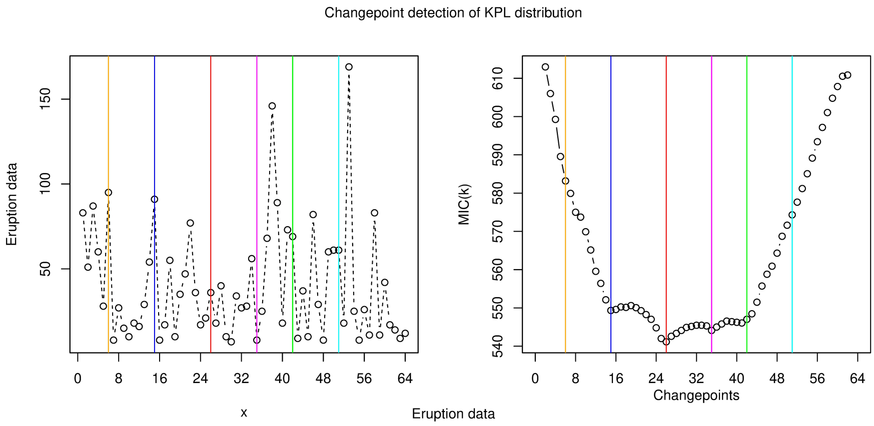

Figure 5: Change point detection using KPL CPD for Eruption data

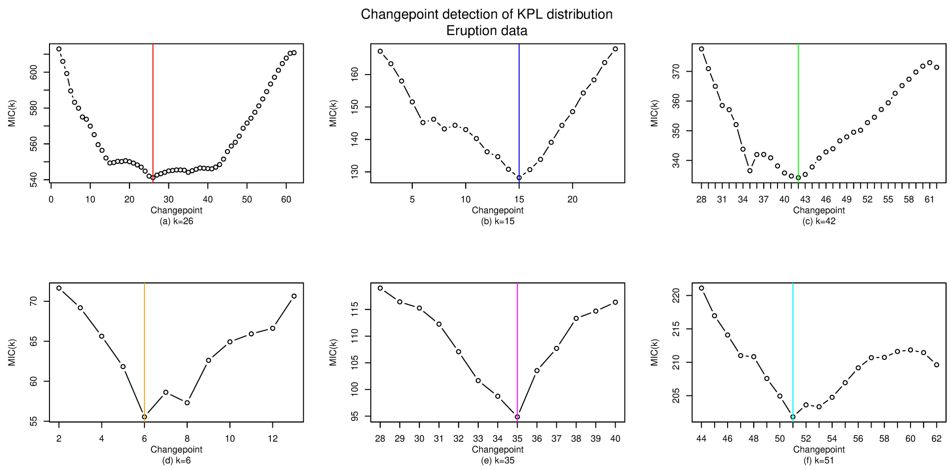

Figure 6: Change point detection values using KPL distribution

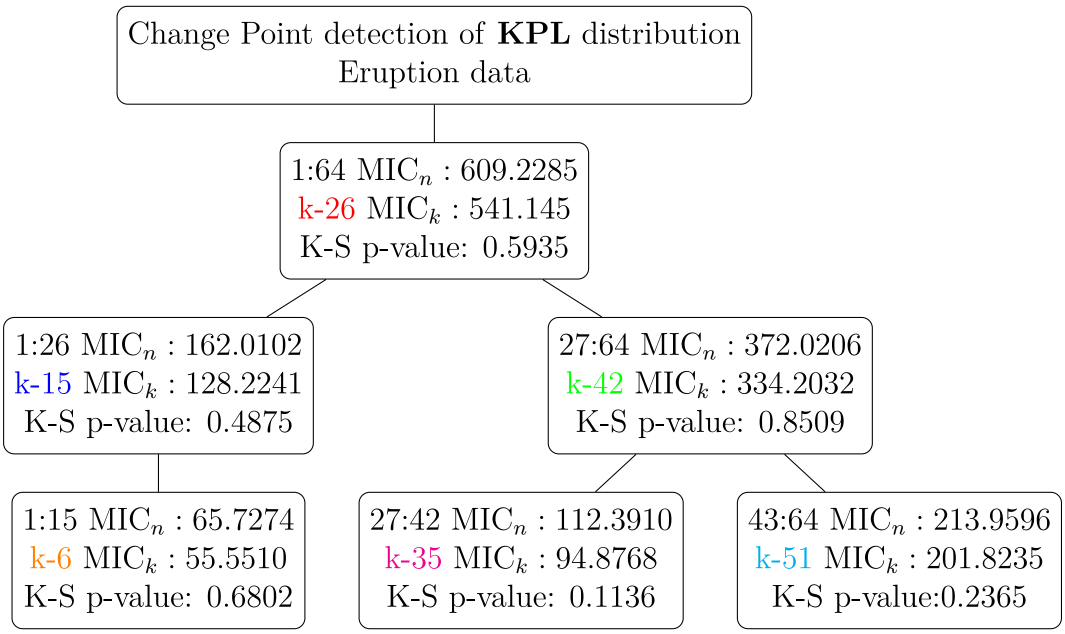

Figure 7: Eruption data - Decision tree

5. Discussions

In this paper, we discussed that, the KPL distribution plays an important role to detect multiple change points. For the datasets considered, at each and every iteration test for goodness of fit and computed MIC values are computed. The tree diagram representation is based on the algorithm of binary segmentation.

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