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On the Bivariate Generalized Chen Distribution

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Abstract

A new model of bivariate distributions is presented in this paper. The model introduced here is of the Marshall–Olkin type. The joint survival function, the joint probability density function and the joint hazard function of the bivariate generalized Chen (BGCh) distribution are obtained. The maximum likelihood and Bayesian methods are used to estimate the unknown parameters of the BGCh distribution. Numerical methods are required to calculate the desired estimates.

Key words: Marshall–Olkin type; Generalized Chen (BGCh) distribution; The maximum likelihood and Bayesian methods; MCMC.

AMS Subject Classifications: 62K05, 05B05

1. Introduction

A suitable parametric model is often of interest in the analysis of survival data, as it provides insight into the characteristics of the failure times and hazard functions that may not be available with non-parametric methods. The Weibull distribution is one of the most commonly used families for modeling such data. However, only monotonically increasing and decreasing hazard functions can be generated from the classic two-parameter Weibull distribution. As such this two-parameter model is inadequate when the true hazard shape is of bathtub nature. Models with bathtub-shaped hazard rate are needed in reliability analysis and decision making when the complete life cycle of the system is to be modeled. Many authors have proposed models with bathtub-shaped failure rates. For example, Smith and Bain (1975) proposed the exponential power distribution. Mudholkar and Srivastava (1993) suggested the exponentiated Weibull distribution. Chen (2000) provided a two-parameter lifetime distribution with bathtub shape or increasing failure function, now known as Chen distribution. Xie *et al.* (2002) modified the Chen distribution to include a scale parameter named modified Weibull extension and also referred to as the generalized Chen distribution. They discussed the parameters' estimation using maximum likelihood method. For more generalizations and modifications of Weibull distribution, see Murthy *et al.* (2004) and Pham and Lai (2007).

Bivariate lifetime data arise frequently in many practical problems and in these situations it is important to consider different bivariate models that could be used to model

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such bivariate lifetime data. There are a number of papers dealing with bivariate models of type of Marshal-Olkin. For example, Sarhan and Balakrishnan (2007) introduced a bivariate distribution using exponential and generalized exponential distributions, now known as Sarhan-Balakrishnan bivariate (SBBV) distribution. Although, they derived several interesting properties of this distribution, the marginal distributions of SBBV distribution are not in known forms. Kundu *et al.* (2012) modified the SBBV distribution to include a scale parameter and discussed the estimation of parameters using maximum likelihood method. Kundu and Gupta followed the idea using the generalized exponential to introduce the bivariate generalized exponential (BVGE) distribution so that the marginal distributions are generalized exponential distributions. They derived several interesting properties of this distribution and discussed the maximum likelihood estimation of the unknown parameters. Also, they re-analyzed a real data set that was analyzed by Meintanis (2007) and concluded that the BVGE distribution provides a better fit than the bivariate Marshall-Olkin distribution. Sarhan (2019) noted that none of the marginal distributions of the SBBV and the BVGE provide a bathtub shape of the hazard function and this lack of the bathtub property limits the application of these distributions. Thus he introduced a new bivariate distribution named the bivariate generalized Rayleigh (BVGR) distribution. The BVGR distribution has generalized Rayleigh marginal distributions. The hazard rate functions of the marginals of the BVGR can be either increasing or decreasing or bathtub shaped, and with this property the BVGR distribution has wider applicability than other distributions. Sarhan (2019) investigated several interesting properties of this distribution and estimated the unknown parameters by using the maximum likelihood and Bayes methods. Many authors discussed the Marshal-Olkin idea for different distributions; see for example; El-Gohary *et al.* (2015), Kundu and Gupta (2017), Azizi *et al.* (2019), Muhammed (2019) and others.

Using the idea of Marshal-Olkin, we propose a new bivariate generalized Chen (BGCh) distribution. The BGCh distribution has generalized Chen marginal distributions. The joint survival function, the joint probability density function and the joint hazard function of the BGCh distribution are obtained. The maximum likelihood and Bayesian methods are used to estimate the unknown parameters of the BGCh distribution. Numerical methods are required to calculate these estimates.

2. The bivariate generalized Chen distribution

In this section, we define a new bivariate distribution, shortly denoted by BGCh. We start with the joint survival function of the distribution and then we derive the corresponding joint probability density function.

2.1. The joint survival function

Chen (2000) introduced a two-parameter lifetime distribution with either bathtub-shaped or increasing failure rate with the survival function

$$S_{Ch}(t) = exp(\lambda(1 - e^{(t)^{\beta}})), \quad t \geq 0, \lambda \text{ and } \beta > 0.$$

and the corresponding probability density function

$$f_{Ch}(t) = \lambda\beta(t)^{\beta-1}exp((t)^{\beta} + \lambda(1 - e^{(t)^{\beta}})), \quad t \geq 0, \lambda \text{ and } \beta > 0.$$

Xie *et al.* (2002) modified the Chen distribution to include a scale parameter named the generalized Chen distribution. The survival function of the univariate generalized Chen (GCh) distribution is

$$S_{GCh}(t) = exp(\lambda\alpha(1 - e^{(t/\alpha)^\beta})), \quad t \geq 0, \lambda, \alpha \text{ and } \beta > 0. \tag{1}$$

with probability density function (pdf)

$$f_{GCh}(t) = \lambda\beta(t/\alpha)^{\beta-1} exp((t/\alpha)^\beta + \lambda\alpha(1 - e^{(t/\alpha)^\beta})), \quad t \geq 0, \lambda, \alpha \text{ and } \beta > 0. \tag{2}$$

Now, suppose that $T_j, j = 1, 2, 3$ are independent random variables with T_i having GCh distributions with scale parameters α, and $\lambda_j, j = 1, 2, 3$ and shape parameter β; i.e. $T_i \sim$ GCh$(\alpha, \beta, \lambda_j), j = 1, 2, 3$. Define $X_i = min(T_i, T_3), i = 1, 2$. Then one can say that the vector (X_1, X_2) follows the bivariate generalized Chen distribution with scale parameters α,and $\lambda_j, j = 1, 2, 3$ and shape parameter β. We will denote it by BGCh$(\alpha, \beta, \lambda_1, \lambda_2, \lambda_3)$ and to simplify we write $\lambda_{123} = \lambda_1 + \lambda_2 + \lambda_3$ and $\lambda_{i3} = \lambda_i + \lambda_3, i = 1, 2$.

Theorem 1: Let (X_1, X_2) follows BGCh$(\alpha, \beta, \lambda_1, \lambda_2, \lambda_3)$, then the joint survival function of (X_1, X_2) for $x_1 > 0, x_2 > 0$, is

$$\begin{aligned} S_{X_1,X_2}(x_1, x_2) &= P(X_1 > x_1, X_2 > X_2) \\ &= P(T_1 > x_1, T_2 > x_2, T_3 > x_3) \\ &= \prod_{i=1}^{3} exp(\lambda_i\alpha(1 - e^{(x_i/\alpha)^\beta})), \end{aligned} \tag{3}$$

where $x_3 = \max\{x_1, x_2\}$.
Also, the joint survival function of (X_1, X_2) can be written as

$$\begin{aligned} S_{X_1,X_2}(x_1, x_2) &= \prod_{i=1}^{3} S_{GCh}(x_i; \alpha, \beta, \lambda_i) \\ &= \begin{cases} S_{GCh}(x_1; \alpha, \beta, \lambda_1) S_{GCh}(x_2; \alpha, \beta, \lambda_{23}) & \text{if } x_1 < x_2 \\ S_{GCh}(x_2; \alpha, \beta, \lambda_2) S_{GCh}(x_1; \alpha, \beta, \lambda_{13}) & \text{if } x_2 < x_1 \\ S_{GCh}(x; \alpha, \beta, \lambda_{123}) & \text{if } x_1 = x_2 = x. \end{cases} \end{aligned} \tag{4}$$

2.2. The joint probability density function

The following theorem gives the joint probability density function of the BGCh distribution.

Theorem 2: Let (X_1, X_2) follows BGCh$(\alpha, \beta, \lambda_1, \lambda_2, \lambda_3)$, then the joint pdf of (X_1, X_2) takes the form

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} f_1(x_1, x_2) & \text{if } 0 < x_1 < x_2 < \infty \\ f_2(x_1, x_2) & \text{if } 0 < x_2 < x_1 < \infty \\ f_3(x) & \text{if } 0 < x_1 = x_2 = x < \infty. \end{cases} \tag{5}$$

where

$$\begin{aligned} f_1(x_1, x_2) &= \lambda_1\lambda_{23}\beta^2(x_1/\alpha)^{(\beta-1)}(x_2/\alpha)^{(\beta-1)} e^{(x_1/\alpha)^\beta + (x_2/\alpha)^\beta} e^{\lambda_1\alpha(1-e^{(x_1/\alpha)^\beta}) + \lambda_{23}\alpha(1-e^{(x_2/\alpha)^\beta})} \\ &= f_{GCh}(x_1; \alpha, \beta, \lambda_1) f_{GCh}(x_2; \alpha, \beta, \lambda_{23}), \end{aligned}$$

$$f_2(x_1,x_2) = \lambda_{13}\lambda_2\beta^2(x_1/\alpha)^{(\beta-1)}(x_2/\alpha)^{(\beta-1)}e^{(x_1/\alpha)^\beta+(x_2/\alpha)^\beta}e^{\lambda_{13}\alpha(1-e^{(x_1/\alpha)^\beta})+\lambda_2\alpha(1-e^{(x_2/\alpha)^\beta})}$$
$$= f_{GCh}(x_1;\alpha,\beta,\lambda_{13})f_{GCh}(x_2;\alpha,\beta,\lambda_2),$$

and

$$f_3(x) = \lambda_3\beta(x/\alpha)^{(\beta-1)}e^{(x/\alpha)^\beta}e^{\lambda_{123}\alpha(1-e^{(x/\alpha)^\beta})}$$
$$= \frac{\lambda_3}{\lambda_{123}}f_{GCh}(x;\alpha,\beta,\lambda_{123}).$$

Proof: The forms of $f_1(.,.)$ and $f_2(.,.)$ can be obtained simply by differentiating $S_{X_1,X_2}(x_1,x_2)$ in (4) with respect to x_1 and x_2 for $x_1 < x_2$ and $x_2 < x_1$, respectively. The form of $f_3(x)$ can not obtained in the same way but it can be derived by using the following identity:

$$\int_0^\infty\int_0^{x_2} f_1(x_1,x_2)dx_1dx_2 + \int_0^\infty\int_0^{x_1} f_2(x_1,x_2)dx_2dx_1 + \int_0^\infty f_3(x)dx = 1$$

which completes the proof of the theorem.

Proposition 1: Let (X_1,X_2) follows BCh$(\beta,\lambda_1,\lambda_2,\lambda_3)$, then the joint pdf of (X_1,X_2) takes the form

$$g_{X_1,X_2}(x_1,x_2) = \begin{cases} g_1(x_1,x_2) & \text{if } 0 < x_1 < x_2 < \infty \\ g_2(x_1,x_2) & \text{if } 0 < x_2 < x_1 < \infty \\ g_3(x) & \text{if } 0 < x_1 = x_2 = x < \infty. \end{cases} \quad (6)$$

where

$$g_1(x_1,x_2) = \lambda_1\lambda_{23}\beta^2(x_1)^{(\beta-1)}(x_2)^{(\beta-1)}e^{(x_1)^\beta+(x_2)^\beta}e^{\lambda_1(1-e^{(x_1)^\beta})+\lambda_{23}(1-e^{(x_2)^\beta})}$$
$$= g_{Ch}(x_1;\beta,\lambda_1)g_{Ch}(x_2;\beta,\lambda_{23}),$$

$$g_2(x_1,x_2) = \lambda_{13}\lambda_2\beta^2(x_1)^{(\beta-1)}(x_2)^{(\beta-1)}e^{(x_1)^\beta+(x_2)^\beta}e^{\lambda_{13}(1-e^{(x_1)^\beta})+\lambda_2(1-e^{(x_2)^\beta})}$$
$$= g_{Ch}(x_1;\beta,\lambda_{13})g_{Ch}(x_2;\beta,\lambda_2),$$

and

$$g_3(x) = \lambda_3\beta(x)^{(\beta-1)}e^{(x)^\beta}e^{\lambda_{123}(1-e^{(x)^\beta})}$$
$$= \frac{\lambda_3}{\lambda_{123}}g_{Ch}(x;\beta,\lambda_{123}).$$

Proof: The result is obtained immediately from Theorem 2 upon setting $\alpha = 1$.

The BGCh distribution has both a singular part and an absolutely continuous part similar to Marshal-Olkin's bivariate exponential distribution, Sarhan and Balakrishnan bivariate distribution, the bivariate generalized exponential introduced by Kundu and Gupta (2009) and the bivariate generalized Rayleigh distribution provided by Sarhan (2019). The function $f_{X_1,X_2}(.,.)$ may be considered to be a density function for the BGCh distribution if it is understood that the first two terms are densities with respect to two-dimensional Lebesgue measure and the third term is a density function with respect to one dimensional Lebesgue measure, see Bemis *et al.* (1972). It is well known that although in one dimension the practical use of a distribution with this property is unusual, but they do arise quite naturally in higher dimensions, see Marshal and Olkin (1967).

In many practical situations it may happen that X_1 and X_2 both are continuous random variables, but $X_1 = X_2$ has a positive probability. The BGCh distribution may be used as a competing risk model or a shock model similar to the bivariate Marshall-Olkin model. Marshal and Olkin (1967) has examples in this connection.. The following theorem provides the explicit forms of the absolute continuous and the singular parts of the BGCh distribution.

Theorem 3: If (X_1, X_2) follows BGCh$(\alpha, \beta, \lambda_1, \lambda_2, \lambda_3)$, then

$$S_{X_1,X_2}(x_1, x_2) = \frac{\lambda_3}{\lambda_{123}} S_s(x_1, x_2) + \frac{\lambda_{12}}{\lambda_{123}} S_a(x_1, x_2).$$

For $x = max(x_1, x_2)$ we get,

$$S_s(x_1, x_2) = e^{\lambda_{123}\alpha(1-e^{(x)^\beta})},$$

and

$$S_a(x_1, x_2) = \frac{\lambda_{123}}{\lambda_{12}} \prod_{i=1}^{3} e^{\lambda_i\alpha(1-e^{(x_i/\alpha)^\beta})} - \frac{\lambda_3}{\lambda_1 2} e^{\lambda_{123}\alpha(1-e^{(x)^\beta})},$$

here $S_s(.,.)$ and $S_a(.,.)$ are the singular and the absolutely continuous parts, respectively.

Proof: The joint survival function $S_{X_1,X_2}(x_1, x_2)$ can be written as

$$S_{X_1,X_2}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2|A)P(A) + P(X_1 > x_1, X_2 > x_2|\acute{A})P(\acute{A})$$

Let $A = \{T_3 < T_1\} \cap \{T_3 < T_2\} \equiv \{X_1 = X_2\}$, therefore

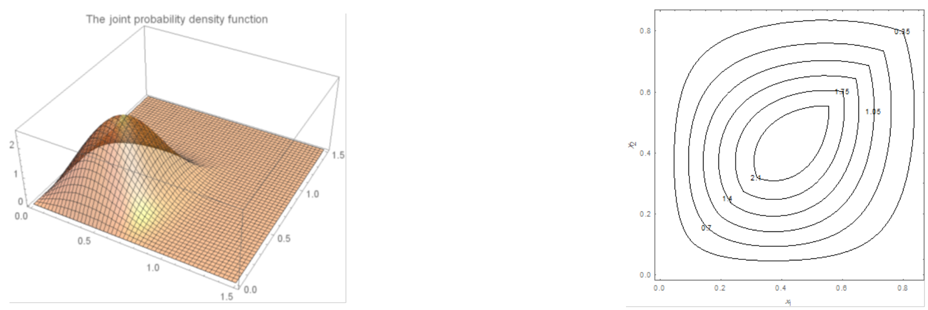
$$P(A) = \int_0^\infty \lambda_3\beta(x/\alpha)^{(\beta-1)} e^{(x/\alpha)^\beta} e^{\lambda_{123}\alpha(1-e^{(x/\alpha)^\beta})} dx = \frac{\lambda_3}{\lambda_{123}}$$

and

$$\begin{aligned} S_s(x_1, x_2) &= P(X_1 > x_1, X_2 > x_2|A) \\ &= \frac{\lambda_{123}}{\lambda_3} \int_0^\infty \lambda_3\beta(x/\alpha)^{(\beta-1)} e^{(x/\alpha)^\beta} e^{\lambda_{123}\alpha(1-e^{(x/\alpha)^\beta})} dx \\ &= e^{\lambda_{123}\alpha(1-e^{(x/\alpha)^\beta})}. \end{aligned}$$

Once $P(A)$ and $S_s(x_1, x_2)$ are obtained, the function $S_a(x_1, x_2)$ can be obtained by subtraction.

Different shapes of the joint pdf and corresponding contours for different sets of parameters values are provided in Figure 1.



Set (1): $(\lambda_1, \lambda_2, \lambda_3, \beta, \alpha)$=(2,2,2,2,1)

Set (2): $(\lambda_1, \lambda_2, \lambda_3, \beta, \alpha)$=(0.5,0.5,1,1.5,1)

Figure 1: The joint probability density function of the BGCh distribution and corresponding contour

2.3. The joint hazard rate function

Using the relation between the joint pdf of (X_1, X_2) and the joint survival function of (X_1, X_2), one can obtain the joint hazard rate function of (X_1, X_2) according to the relation

$$h_{X_1,X_2}(x_1, x_2) = \frac{f_{X_1,X_2}(x_1, x_2)}{S_{X_1,X_2}(x_1, x_2)}.$$

Here we use the forms (4) and (6) to obtain the joint hazard rate function. In Figure 2 we provide the surface plots of the joint hazard rate function and corresponding contours for different values of the parameters.

3. Statistical properties

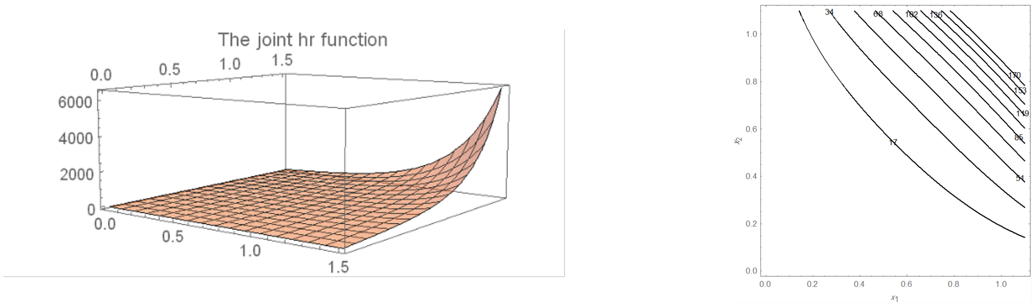
3.1. Marginal distributions

One can easily verify that the marginal distribution of $X_i, i = 1, 2,$ follows GCh$(\beta, \alpha, \lambda_i)$. For this, we first derive the marginal survival function of X_i, say $S_{X_i}(x)$, as follows

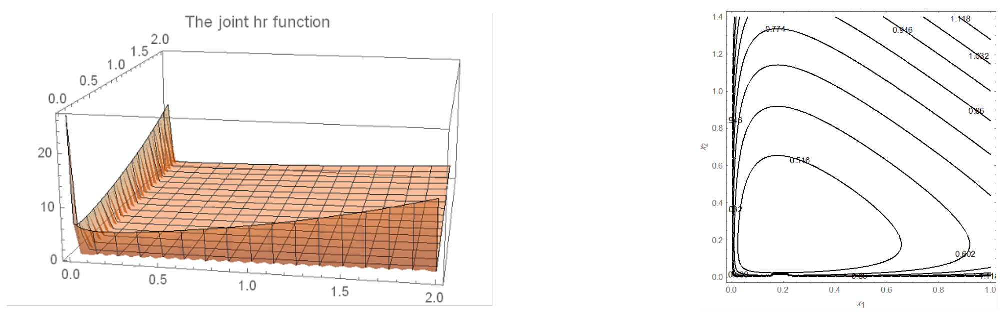
$$S_{X_i}(x) = P(X_i > x) = P(min(T_i, T_3) > x) = P(T_i > x, T_3 > x)$$

and since $T_i, i = 1, 2$ and T_3 are independent random variables, then

$$S_{X_i}(x) = P(T_i > x)P(T_3 > x) = S_{X_i}(x; \beta, \alpha, \lambda_{i3}) = e^{\lambda_{i3}\alpha(1-e^{(x/\alpha)^\beta})} \tag{7}$$



Set (1): $(\lambda_1, \lambda_2, \lambda_3, \beta, \alpha)$=(2,2,2,2,1)



Set (2): $(\lambda_1, \lambda_2, \lambda_3, \beta, \alpha)$=(0.2,0.2,0.8,0.8,1)

Figure 2: The joint hazard rate function of the BGCh distribution and corresponding contour

Using (7), the marginal pdf of X_i is

$$f_{X_i}(x) = \lambda_{i3}\beta(x/\alpha)^{(\beta-1)}e^{(x/\alpha)^\beta}e^{\lambda_{i3}\alpha(1-e^{(x/\alpha)^\beta})}, \tag{8}$$

and the marginal hazard rate function (hrf) of X_i is

$$h_{X_i}(x) = \lambda_{i3}\beta(x/\alpha)^{(\beta-1)}e^{(x/\alpha)^\beta}. \tag{9}$$

Xie *et al.* (2002) noted that the hrf depends only on the shape parameter β and they observed that: when $\beta > 1$, the hrf has an increasing shape and when $\beta < 1$, the hrf has a bathtub shape. Shapes of the pdf and hrf of X_i for different values of β, α and λ_{i3} are provided in Figure 3 . Also, Xie *et al.* (2002) showed that the GCh distribution can be used in modeling bathtub-shaped failure rate univariate lifetime data. Hence, we expect the BGCh distribution can be used in modeling bathtub-shaped failure rate bivariate lifetime data.

3.2. Conditional distributions

Having obtained the marginal pdf of X_1 and X_2, one can derive the conditional probability density function. The following theorem provides the conditional pdf of X_1 given $X_2 = x_2$, say $f_{X_1|X_2}(x_1|x_2)$.

Theorem 4: If (X_1, X_2) follows BGCh$(\beta, \alpha, \lambda_1, \lambda_2, \lambda_3)$, then the conditional pdf of X_1 given

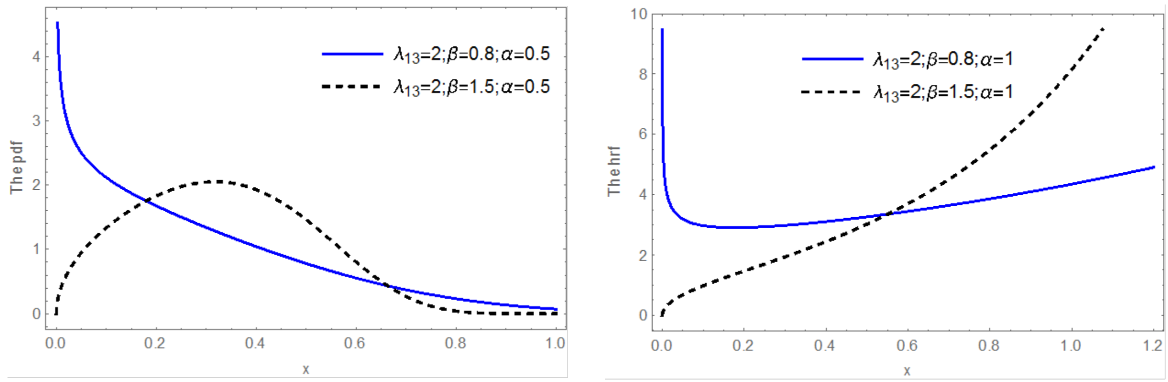


Figure 3: The probability density and hazard rate functions of the marginal distribution of X_1

$X_2 = x_2$ is

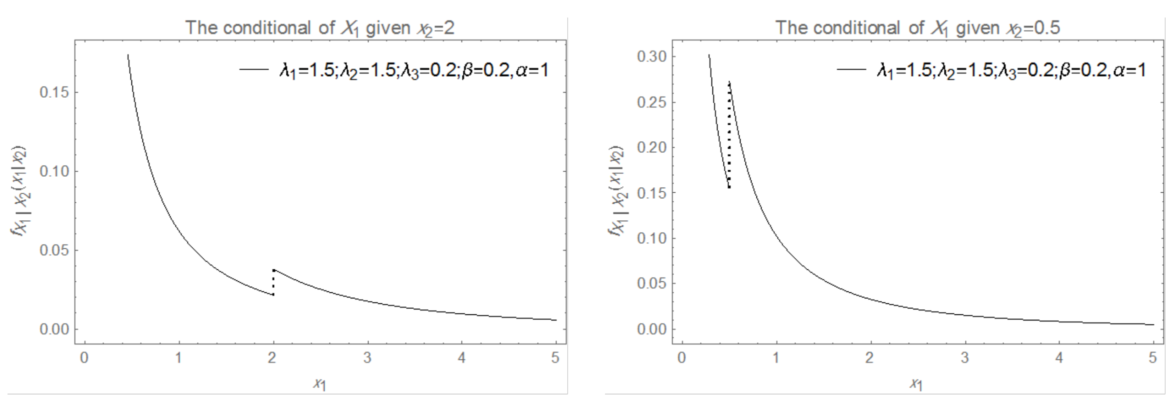
$$f_{X_1|X_2}(x_1, x_2) = \begin{cases} f_1(x_1|x_2) & \text{if } x_1 < x_2 \\ f_2(x_1|x_2) & \text{if } x_2 < x_1 \\ f_3(x_1|x_2) & \text{if } x_1 = x_2 = x, \end{cases} \tag{10}$$

where

$$f_1(x_1|x_2) = \lambda_1\beta(x_1/\alpha)^{(\beta-1)}e^{(x_1/\alpha)^\beta}e^{\lambda_1\alpha(1-e^{(x_1/\alpha)^\beta})},$$
$$f_2(x_1|x_2) = (\lambda_{12}\lambda_2)/\lambda_{23}\beta(x_1/\alpha)^{(\beta-1)}e^{(x_1/\alpha)^\beta}e^{\alpha(\lambda_{13}(1-e^{(x_1/\alpha)^\beta})-\lambda_3(1-e^{(x_2/\alpha)^\beta}))}, \text{ and}$$
$$f_3(x_1|x_2) = \lambda_3/\lambda_{23}e^{\lambda_1\alpha(1-e^{(x/\alpha)^\beta})}.$$

Proof: The results of this theorem are easily derived using the definition of conditional probability and the results of Theorem 2 and the form (8). Figure 4 shows some plots of the conditional pdf's of X_1 given $X_2 = x_2$ for different values of $x_2 (x_2 = 0.5, 1, 2)$ and different values of parameters.

Similarly, the conditional pdf of X_2 given $X_1 = x_1$ can be obtained in a similar manner as above. Also, one can note that if $\alpha = 1$, the conditional pdf in the case of BCh distribution can be obtained.



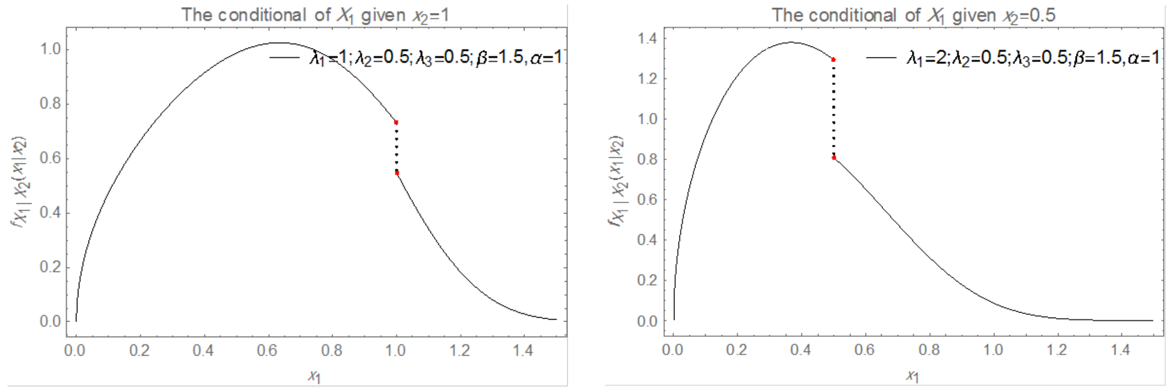


Figure 4: The conditional probability density function of X_1 given $X_2 = x_2$ at different sets of the parameters

4. Parameters' estimation

Suppose that $\{(X_{11}, X_{21}), (X_{12}, X_{22}), \ldots, (X_{1n}, X_{2n})\}$ is a random sample from BGCh $(\lambda_1, \lambda_2, \lambda_3, \beta, \alpha)$. The likelihood function for this sample is

$$L(\text{data}; \theta) = \prod_{i=1}^{n} f_1(x_{1i}, x_{2i})^{I(x_{1i}<x_{2i})} f_2(x_{1i}, x_{2i})^{I(x_{1i}>x_{2i})} f_3(x_{1i}, x_{2i})^{I(x_1 i=x_{2i})}, \tag{11}$$

where $I(A)$ is an indicator function that is equal to 1 if A is true and 0 otherwise and $\theta = (\lambda_1, \lambda_2, \lambda_3, \beta, \alpha)$. Substituting (5) in (11) and taking the natural logarithm, we obtain the log-likelihood function as

$$\begin{aligned}
\mathcal{LL} = \sum_{i=1}^{n} & I(x_{1i} < x_{2i})\{ln(\lambda_1) + ln(\lambda_{23}) + 2ln(\beta) + (\beta - 1)ln(x_{1i}/\alpha) + ln(x_{2i}/\alpha) + (x_{1i}/\alpha)^{\beta} \\
& + (x_{2i}/\alpha)^{\beta} + \lambda_1(1 - e^{(x_{1i}/\alpha)^{\beta}}) + (\lambda_{23})(1 - e^{(x_{2i}/\alpha)^{\beta}}))\} \\
& + I(x_{1i} > x_{2i})\{ln(\lambda_2) + ln(\lambda_{13}) + 2ln(\beta) + (\beta - 1)(ln(x_{1i}/\alpha) + ln(x_{2i}/\alpha) + (x_{1i}/\alpha)^{\beta} \\
& + (x_{2i}/\alpha)^{\beta} + \lambda_2(1 - e^{(x_{2i}/\alpha)^{\beta}}) + (\lambda_{13})(1 - e^{(x_{1i}/\alpha)^{\beta}})\} \\
& + I(x_{2i} = x_{1i})\{ln(\lambda_3) + ln(\beta) + (\beta - 1)ln(x_{1i}/\alpha) + (x_{1i}/\alpha)^{\beta} + (\lambda_{123})(1 - e^{(x_{1i}/\alpha)^{\beta}})\}.
\end{aligned} \tag{12}$$

4.1. Maximum likelihood estimation

Here we use maximum likelihood method to estimate the unknown parameters of the BGCh distribution. For fixed α, the likelihood equations are

$$\begin{aligned}
\frac{\partial \mathcal{LL}}{\partial \lambda_1} &= \frac{n_1}{\lambda_1} + \frac{n_2}{\lambda_{13}} + \sum_{i=1}^{n} I(x_{1i} < x_{2i})(1 - e^{(x_{1i}/\alpha)^{\beta}}) = 0,\\
\frac{\partial \mathcal{LL}}{\partial \lambda_2} &= \frac{n_1}{\lambda_{23}} + \frac{n_2}{\lambda_2} + \sum_{i=1}^{n} I(x_{1i} > x_{2i})(1 - e^{(x_{2i}/\alpha)^{\beta}}) = 0,\\
\frac{\partial \mathcal{LL}}{\partial \lambda_3} &= \frac{n_1}{\lambda_{23}} + \frac{n_2}{\lambda_{13}} + \frac{n_3}{\lambda_3} + \sum_{i=1}^{n} \{ I(x_{1i} < x_{2i})(1 - e^{(x_{2i}/\alpha)^{\beta}})\\
&\quad + \{I(x_{1i} > x_{2i}) + I(x_{1i} = x_{2i})\}(1 - e^{(x_{1i}/\alpha)^{\beta}})\} = 0,\\
\text{and}\\
\frac{\partial \mathcal{LL}}{\partial \beta} &= \sum_{i=1}^{n} I(x_{1i} < x_{2i})\{2/\beta + ln(x_{1i}/\alpha) + ln(x_{2i}/\alpha) + (x_{1i}/\alpha)^{\beta} ln(x_{1i}/\alpha)(1 - \lambda_1 e^{(x_{1i}/\alpha)^{\beta}})\\
&\quad + (x_{2i}/\alpha)^{\beta} ln(x_{2i}/\alpha)(1 - \lambda_{23}) e^{(x_{2i}/\alpha)^{\beta}}\}\\
&\quad + I(x_{1i} > x_{2i})\{2/\beta + ln(x_{1i}/\alpha) + ln(x_{2i}/\alpha) + (x_{1i}/\alpha)^{\beta} ln(x_{1i}/\alpha)(1 - \lambda_{13} e^{(x_{1i}/\alpha)^{\beta}})\\
&\quad + (x_{2i}/\alpha)^{\beta} ln(x_{2i}/\alpha)(1 - \lambda_2) e^{(x_{2i}/\alpha)^{\beta}})\}\\
&\quad + I(x_{2i} = x_{1i})\{1/\beta + ln(x_{1i}/\alpha) + (x_{1i}/\alpha)^{\beta} ln(x_{1i}/\alpha)(1 - \lambda_{123} e^{(x_{1i}/\alpha)^{\beta}})\} = 0,
\end{aligned} \tag{13}$$

where $n_1 = \sum_{i=1}^{n} I(x_{1i} < x_{2i})$, $n_2 = \sum_{i=1}^{n} I(x_{1i} > x_{2i})$, and $n_3 = \sum_{i=1}^{n} I(x_{1i} = x_{2i})$. The likelihood equations (13) do not have a closed-form solution, so a numerical technique must be used to find the maximum likelihood estimates (mles) of $\lambda_1, \lambda_2, \lambda_3,$ and β. The likelihood equations may have multiple roots, Small *et al.* (2000) discussed this problem using the Hessian matrix. They showed that the likelihood equations have a unique root when the Hessian matrix of the log-likelihood is negative definite for all value of $\boldsymbol{\theta}$. This relies on maximizing the log-likelihood function. The Hessian matrix is written as

$$T(\boldsymbol{\theta}) = \begin{pmatrix} \mathcal{LL}_{\lambda_1\lambda_1} & \mathcal{LL}_{\lambda_1\lambda_2} & \mathcal{LL}_{\lambda_1\lambda_3} & \mathcal{LL}_{\lambda_1\beta} \\ \mathcal{LL}_{\lambda_2\lambda_1} & \mathcal{LL}_{\lambda_2\lambda_2} & \mathcal{LL}_{\lambda_2\lambda_3} & \mathcal{LL}_{\lambda_2\beta} \\ \mathcal{LL}_{\lambda_3\lambda_1} & \mathcal{LL}_{\lambda_3\lambda_2} & \mathcal{LL}_{\lambda_3\lambda_3} & \mathcal{LL}_{\lambda_3\beta} \\ \mathcal{LL}_{\beta\lambda_1} & \mathcal{LL}_{\beta\lambda_2} & \mathcal{LL}_{\beta\lambda_3} & \mathcal{LL}_{\beta\beta} \end{pmatrix}$$

where $\mathcal{LL}_{\theta_i\theta_j} = \frac{\partial^2 \mathcal{LL}}{\partial\theta_i \partial\theta_j}$ is the second partial derivative of the log-likelihood function with respect to the components θ_i and θ_j of $\boldsymbol{\theta}$ and $T(\hat{\boldsymbol{\theta}})$ is the Hessian matrix computed at $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$.

Large-sample confidence intervals: Under regularity conditions, the mles of the parameters $\lambda_1, \lambda_2, \lambda_3,$ and β are asymptotically normally distributed with means equal to the true values of these parameters and variances given by the inverse of the information matrix. One can approximate the expected values of the second-order derivatives of logarithms of likelihood function with the maximum likelihood estimates of the parameters as given in Cohen (1965). That is, using normality property of mles, one can construct the asymptotic confidence interval for each parameter.

4.2. Bayes estimation

Now, we discuss the Bayesian estimation of the unknown parameters of the BGCh distribution. For fixed α, let the four parameters $\boldsymbol{\theta} = (\lambda_1, \lambda_2, \lambda_3, \beta)$ are independent random variables and follow the gamma prior distribution. That is, the joint prior pdf of $\boldsymbol{\theta}$ is

$$g_0(\boldsymbol{\theta}) \propto \lambda_1^{(a_1-1)}\lambda_2^{(a_2-1)}\lambda_3^{(a_3-1)}\beta^{(a_4-1)}e^{(-b_1\lambda_1-b_2\lambda_2-b_3\lambda_3-b_4\beta)}, \lambda_1, \lambda_2, \lambda_3, \beta > 0, \tag{14}$$

where all the hyperparameters a_i and $b_i, i = 1, 2, 3, 4$ are assumed to be positive and known. The log-prior density function is

$$\mathcal{g}_0(\boldsymbol{\theta}) \propto \sum_{i=1}^{3}(a_i - 1)ln(\lambda_i) + (a_4 - 1)ln(\beta) - \sum_{i=1}^{3} b_i\lambda_i - b_4\beta. \tag{15}$$

Using (12) and (15) and applying Bayes theorem, the joint posterior probability density function of $\boldsymbol{\theta}$, given data, is

$$g(\boldsymbol{\theta}|data) = \frac{1}{K}exp(\mathcal{LL} + \mathcal{g}_0(\boldsymbol{\theta})), \tag{16}$$

where K is the normalizing constant. Bayes estimators of the unknown parameters and/or of any function of the unknown parameters, say $w(\boldsymbol{\theta})$, can be obtained as follows

$$\hat{w}(data) = \frac{\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty w(\boldsymbol{\theta})exp(\mathcal{LL} + \mathcal{g}_0(\boldsymbol{\theta}))d\lambda_1 d\lambda_2 d\lambda_3 d\beta}{\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty exp(\mathcal{LL} + \mathcal{g}_0(\boldsymbol{\theta}))d\lambda_1 d\lambda_2 d\lambda_3 d\beta}. \tag{17}$$

Formula (17) involves a ratio of two multidimentional integrals and does not have analytical solution. Thus, some approximation methods were suggested to approximate these integrals and calculate the ratio of the integrals such as the methods discussed by Lindley (1980) and Tierney and Kadane (1986). These methods work well for low dimensions. In this paper we will use Markov Chain Monte Carlo (MCMC) method that work well in the case of high dimensions, see Gelman *et al.* (2003). MCMC method generates random draws from the joint posterior distribution by generating draws from an arbitrary distribution (proposal distribution) that easy to simulate from then apply an accept-reject method. Here, we use multivariate normal as a proposal distribution. The following steps can be followed to generate random draws from the joint posterior distribution (16):

1. Specify the size of the random draws we wish to generate, say m.
2. Choose an initial value of $\boldsymbol{\theta}$, say $\boldsymbol{\theta}^{(0)}$.
3. For $i = 1, 2, \ldots, m$, repeated the following steps:
   (a) Generate $\boldsymbol{\theta}^*$ from the multivariate normal with mean $\boldsymbol{\theta}^{(i-1)}$ and variance-covariance Σ.
   (b) Compute the ratio $\kappa = \min\{1, \frac{g(\boldsymbol{\theta}^*|\text{data})}{g(\boldsymbol{\theta}^{(i-1)}|\text{data})}\}$.
   (c) Generate a random value from uniform distribution on (0, 1).
   (d) If $\kappa \geq$ put $\boldsymbol{\theta}^{(i)} = \boldsymbol{\theta}^*$, otherwise put $\boldsymbol{\theta}^{(i)} = \boldsymbol{\theta}^{(i-1)}$.

Discarding the early m_0 number of burn-in draws and using the remaining $m - m_0$, $\boldsymbol{\theta}^{(m_0+1)}$, $\boldsymbol{\theta}^{(m_0+2)}, \ldots, \boldsymbol{\theta}^{(m)}$, as the chosen draws from the joint posterior distribution, the Bayes estimate of θ_j is

$$\hat{\theta}_j = \sum_{i=m_0+1}^{m-m_0} \frac{\theta_j^{(i)}}{m - m_0}, j = 1, 2, 3, 4.$$

Furthermore, for $0 < \nu < 1$, one can obtain the lower and upper bounds of the $100(1-\nu)\%$ Bayesian probability interval of θ_j via $(\nu/2)100th$ and $(1-\nu/2)100th$ percentiles of the sequence of the $m - m_0$ draws; $\boldsymbol{\theta}^{(m_0+1)}, \boldsymbol{\theta}^{(m_0+2)}, \ldots, \boldsymbol{\theta}^{(m)}$.

5. Simulation results and applications

In this section, some simulation results and the analysis of a data set are presented.

5.1. Simulation results

In this section, we provide the following steps to generate a random sample of the BGCh distribution:

1. Generate u_1, u_2 and u_3 from uniform (0, 1).
2. Compute $t_1 = \alpha(ln(1 - \frac{ln(1-u_1)}{\lambda_1 \alpha})^{1/\beta}, t_1 = \alpha(ln(1 - \frac{ln(1-u_2)}{\lambda_2 \alpha})^{1/\beta}$ and $t_3 = \alpha(ln(1 - \frac{ln(1-u_3)}{\lambda_3 \alpha})^{1/\beta}$.
3. Obtain $x_1 = min(t_1, t_3)$ and $x_2 = min(t_2, t_3)$.

To obtain some simulation results for samples size (n=100) and for different parameter values, we consider three different sets of parameter values namely: (i) $\lambda_1 = \lambda_2 = \lambda_3 = \beta = 1$, (ii) $\lambda_1 = \lambda_2 = \lambda_3 = 2, \beta = 1$, and (iii) $\lambda_1 = 0.5, \lambda_2 = 0.5, \lambda_3 = 1, \beta = 1.5$. We replicate the process 1000 times and report the average estimates and the root mean square errors (RMSEs) in Table 1. Also, we compute the Bayes estimates of the unknown parameters as mentioned in the previous section with assuming uniform priors. We simulate 10000 runs and replicate the process 1000 times. The average estimates and the RMSEs are also listed in Table 1 and one can note that results of Bayes estimates are better than mles.

5.2. Applications

In this section we present the analysis of a data set to discuss how the proposed distribution can be used in practice. This data represents the UEFA Champion's League Data and it was analyzed in Meintanis (2007) using the Marshall-Olkin exponential model (MO) and by Kundu and Gupta (2009) using the bivariate generalized exponential (BVGE) model, then by Sarhan (2019) using the bivariate generalized Rayleigh (BVGR) model. Kundu and Gupta (2009) reported that the BVGE model fits the data better than MO model and Sarhan (2019) reported that the BVGR model fits the data better than both the MO and the BVGE models. Here, we use the BCh model to reanalyze the same data and compare it with the three models; the MO, the BVGE and the BVGR but first we have fitted Ch(β, λ) model to the marginal and the minimum of the two marginals. The mles of

Table 1: The mles and the Bayes estimates and their RMSEs (in parentheses) of the parameters

Parameter value	Method	β	λ_1	λ_2	λ_3
	MLE	1.3834	2.6776	1.3080	1.5323
		(0.4243)	(1.6961)	(0.3732)	(0.5492)
(1.0, 1.0, 1.0, 1.0)	Bayes	1.3429	1.4783	1.2716	1.4297
		(0.3609)	(0.4788)	(0.2993)	(0.4316)
	MLE	2.9372	5.0508	2.8083	1.5604
		(1.9725)	(3.0754)	(0.8590)	(0.4518)
(1.0, 2.0, 2.0, 2.0)	Bayes	2.3606	2.4705	2.3693	1.3772
		(1.3661)	(0.4714)	(0.3867)	(0.6247)
	MLE	1.7625	4.3562	1.5935	1.4858
		(0.4276)	(3.8752)	(1.1251)	(0.4916)
(1.5, 0.5, 0.5, 1.0)	Bayes	0.9269	0.9891	1.3696	1.3696
		(0.5764)	(0.4892)	(0.8753)	(0.3736)

the unknown parameters, the Kolmogorov-Smirnov (K-S) distances between the empirical distribution function (EDF) and the fitted distribution function and the associated p values are reported in Table 2. Based on the p values, one can observe that Chen distribution may be used to fit X_1, X_2 and $min(X_1, X_2)$.

Table 2: The mles of the parameters, the K-S test statistics and associated p-values

Variable	mle	K-S	p-value
X_1	$\hat{\beta} = 0.403, \hat{\lambda} = 0.010$	0.013	0.572
X_2	$\hat{\beta} = 0.379, \hat{\lambda} = 0.184$	0.106	0.804
$\min(X_1, X_2)$	$\hat{\beta} = 0.389, \hat{\lambda} = 0.019$	0.094	0.899

Now, to test whether BCh distribution fits the data or not, we use the two-dimensional Kolomogorov-Sminrov test of goodness of fit as proposed by Peacock (1983). Using the computational environmental R peacock package, we obtain the value of test statistic as 0.2712 with p value 0.6482. Based on the p value, we cannot reject the null hypothesiss that the data came from the BCh distribution at 0.05 level of significance. For more details about multivariate Kolomogorov-Sminrov test of goodness of fit see Justel *et al.* (1997).

Hence, we have used the BCh model to analyze the bivariate data set. We use R to get mles of the unknown parameters. Table 3 shows the mles of the unknown parameters of the proposed distribution together with the values of the log-likelihood values and the Akaike information criterion (AIC=-2 LL+2k,k is the number of estimated parameters; see Akaike, 1974). The AIC suggests that the BCh distribution provides a better fit than the three models; the MO, the BVGE and the BVGR.

To indicate that a unique root for the likelihood equations exist. We use the estimates $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3$ and $\hat{\beta}$ obtained with respect to the given bivariate data set. These estimates

Table 3: The mles of the parameters, the log-likelihood values and AIC values

Model	mle	$\mathcal{L}$	AIC
MO	$\hat{\lambda}_1 = 0.012, \hat{\lambda}_2 = 0.014, \hat{\lambda}_3 = 0.022,$	-339.006	684.012
BVGE	$\hat{\alpha}_1 = 1.351, \hat{\alpha}_2 = 0.465, \hat{\alpha}_3 = 1.153, \hat{\beta} = 0.039$	-296.935	601.870
BVGR	$\hat{\alpha}_1 = 0.492, \hat{\alpha}_2 = 0.166, \hat{\alpha}_3 = 0.410, \hat{\lambda} = 0.020$	-293.357	594.714
BCh	$\hat{\lambda}_1 = 0.026, lam\hat{b}da_2 = 0.055, \hat{\lambda}_3 = 0.048, \hat{\beta} = 1.020$	0.094	0.899

are obtained using nlm R package which minimize the negative of the log-likelihood function. We obtain $T(\hat{\boldsymbol{\theta}})$ as follows:

$$\begin{pmatrix} 0.0424 & -1.63E-0710^{-7} & 9.8070 & 6.0055 \\ - & 33.114 & 2.0183 & 8.427 \\ - & - & 46.709 & 11.059 \\ - & - & - & 104.0074 \end{pmatrix}$$

The eigen values of this matrix are -103.6646, -42.8617, -31.9991 and -2.2724. This indicates that $T(\hat{\boldsymbol{\theta}})$ is negative definite. Then according to Small *et al.* (2000), the likelihood equations has a unique root. For more details see Thomas and Jose (2021).

For Bayesian computations, we obtain the Bayes estimates of the unknown parameters based on the uniform priors and the gamma priors. In the case of the gamma priors, we assume that all hyperparameters equal and equal to 0.5. For the two cases, the proposal distribution is multinormal with variance covariance matrix and the choice of its value depends on the acceptance rate which is assumed such that the acceptance rate (number of accepted runs out of total runs) increases. Here, we simulate 10000 runs from the joint posterior distribution of the four parameters and the early 20% of the runs were discarded. The trace plots of the draws are plotted in Figures 5 and 6 after discarding the early 2000 draws (burn-in period). Tables 4-5 list the posterior descriptive summaries of interest such as the posterior mean, median, standard deviation and the 95% Bayesian credible intervals.

Table 4: Summary results for the posterior parameters in the case of gamma priors (the acceptance rate is 38.18%)

Parameter	Mean	Median	Standard deviation	95% credible intervals
λ_1	0.0837	0.0754	0.0408	(0.0533, 0.1053)
λ_2	0.5379	0.5278	0.1352	(0.4347, 0.6332)
λ_3	0.1707	0.1604	0.0620	(0.1274, 0.2031)
β	0.2114	0.2114	0.0168	(0.1995, 0.2227)

6. Conclusion

In this paper, the bivariate generalized Chen distribution (BGCh) is proposed as a new bivariate lifetime distribution. The BGCh distribution is of Marshal-Olkin type whose marginal are generalized Chen distributions. One can observe that the BGCh distribution is

Table 5: Summary results for the posterior parameters in the case of uniform priors (the acceptance rate is 54.83%**)**

Parameter	Mean	Median	Standard deviation	95% credible intervals
λ_1	0.1137	0.1042	0.0533	(0.0756, 0.1401)
λ_2	0.6255	0.6134	0.1575	(0.5175, 0.7273)
λ_3	0.2227	0.2167	0.0779	(0.1657, 0.2697)
β	0.2019	0.2015	0.0171	(0.1900, 0.2127)

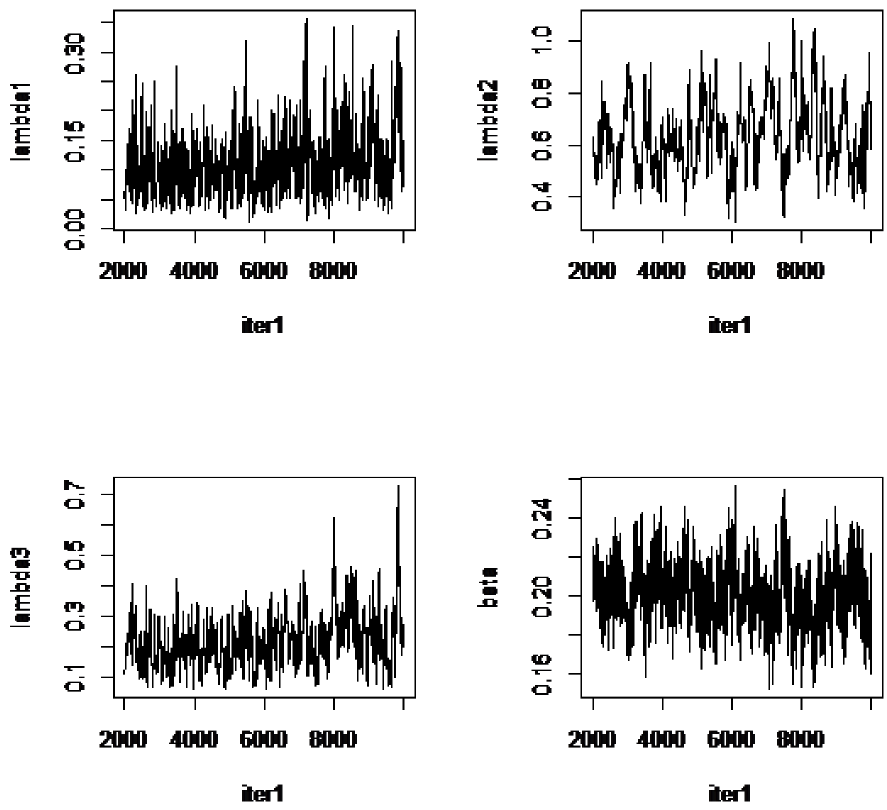


Figure 5: The trace plot of the random draws from the joint posterior distribution in the case of gamma priors

a singular distribution and has an absolute continuous and a singular part. Some statistical properties are investigated. The estimation of the parameters has been approached by maximum likelihood and Bayesian methods. For Bayesian method, we used the MCMC method. Numerical methods are required to calculate the desired estimates. One real data set is analyzed using the BCh distribution which showed a better fit than the MO, the BVGE and the BVGR distributions.

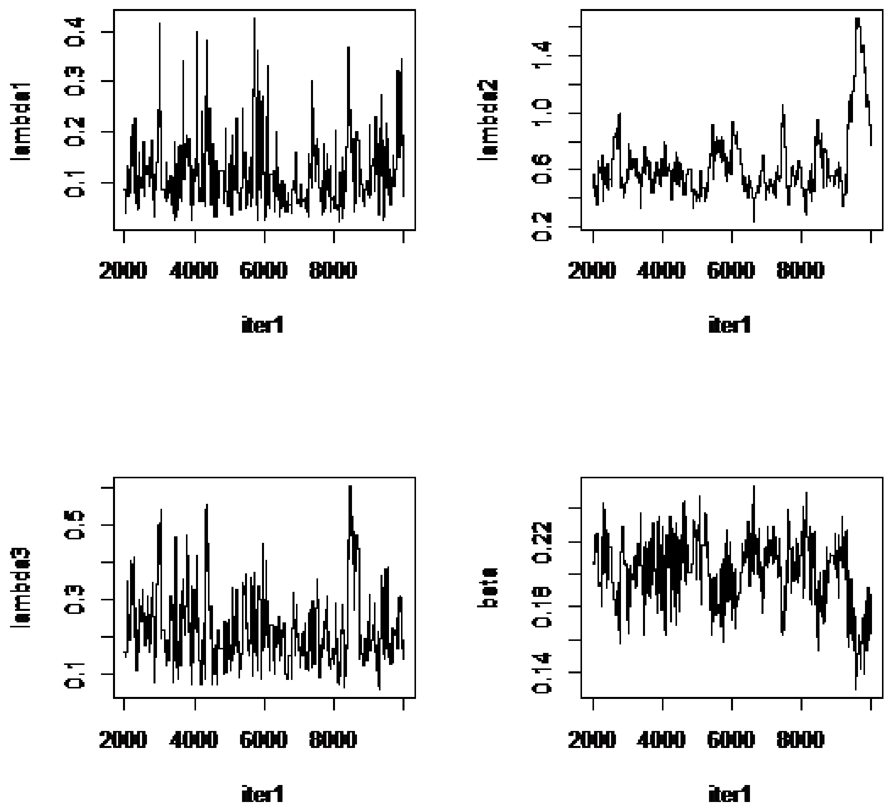


Figure 6: The trace plot of the random draws from the joint posterior distribution in the case of uniform priors

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