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Theory and Applicability of the Weighted Modified Lindley Distribution

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Abstract

As a bridge between the exponential and Lindley distributions, the modified Lindley distribution was created. It has been used successfully in a variety of fields related to survival analysis. In this study, we present a novel distribution that extends the modified Lindley distribution using the traditional weighted (or length/size-biased) approach. It is named as weighted modified Lindley distribution. This idea is mainly used to flexibilize the former modified Lindley distribution through the use of a one-parameter polynomial weight. This weight is intended to modulate the functionalities of the new distribution, well beyond those of the former modified Lindley distribution. The related probability density function, cumulative density function, hazard rate function, moments, moment generating function and characteristic function are analysed from a theoretical and practical point of view. Estimation of the parameters is done by the classical method of maximum likelihood and a simulation study is carried out to check the consistency of the maximum likelihood estimates. A data set is used to illustrate the application of the proposed distribution.

Key words: Data analysis; Lindley distribution; Estimation; Modified Lindley distribution; Moments; Weighted distributions.

AMS Subject Classifications: 60E05, 62E15

1. Introduction

Lindley is the inventor of the Lindley (L) distribution (see Lindley (1958)). For many statistical settings, the L distribution is established as an alternative to the exponential distribution. It is governed by the following one-parameter cumulative density function (cdf):

$$F_L(x;\upsilon) = 1 - \left[1 + \frac{\upsilon x}{1+\upsilon}\right] e^{-\upsilon x}, \quad x > 0,$$

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where $\upsilon > 0$, and $F_L(x;\upsilon) = 0$ for $x \leq 0$. Then its probability density function (pdf) is derived as

$$f_L(x;\upsilon) = \frac{\upsilon^2}{1+\upsilon}(1+x)e^{-\upsilon x}, \quad x > 0,$$

and $f_L(x;\upsilon) = 0$ for $x \leq 0$.

Several authors have researched and generalized this distribution during the last few decades. There is a vast literature in this area. Some examples of such distributions include the three-parameter L distribution by Zakerzadeh and Dolati (2009), generalized L distribution by Nadarajah *et al.* (2011), generalized Poisson-L distribution by Mahmoudi and Zakerzadeh (2010), power L distribution by Ghitany *et al.* (2013), two parameter-L distribution by Shanker and Mishra (2013a), quasi L distribution by Shanker and Mishra (2013b), transmuted L distribution by Merovci (2013), transmuted L-geometric distribution by Merovci and Elbatal (2014), beta-L distribution by Merovci and Sharma (2014), negative binomial-L distribution Zamani and Ismail (2010) and gamma-L distribution by Zeghdoudi and Nedjar (2016). For more details, see a comprehensive review study of the L distribution by Tomy (2018).

Among its generalizations, Ghitany *et al.* (2011) introduced the weighted L (WL) distribution, with pdf determined as

$$f_{WL}(x;\alpha,\upsilon) = \Psi_\alpha^{-1} x^{\alpha-1} f_L(x;\upsilon),$$

where $\alpha > 0$, Ψ_α represents the normalizing constant corresponding to the expectation of $X^{\alpha-1}$, X being a random variable with the L distribution with parameter υ. The pdf of the WL distribution can also be expressed as

$$f_{WL}(x;\alpha,\upsilon) = \frac{\upsilon^{\alpha+1}}{(\upsilon+\alpha)\Gamma(\alpha)} x^{\alpha-1}(1+x)e^{-\upsilon x}, \quad x > 0,$$

where $\Gamma(\alpha)$ denotes the Euler gamma function at α, and $f_{WL}(x;\alpha,\upsilon) = 0$ for $x \leq 0$. It is proved that the polynomial weight $x^{\alpha-1}$ modulates the shape properties of the functions of the former L distribution, increasing their capabilities in terms of modeling. As a consequence, the hazard rate function (hrf) of the WL distribution exhibits bathtub or increasing shapes. Furthermore, for some non-grouped or grouped survival data, the WL model is better than several well-known two-parameter survival models.

Recently, an intermediary distribution between the classical exponential and the L distribution has been proposed by Chesneau *et al.* (2019), called the modified L (ML) distribution. Its cdf is specified by

$$F_{ML}(x;\upsilon) = 1 - \left[1 + \frac{\upsilon x}{1+\upsilon} e^{-\upsilon x}\right] e^{-\upsilon x}, \quad x > 0,$$

with $\upsilon > 0$ and $F_{ML}(x;\upsilon) = 0$ for $x \leq 0$, and the related pdf is obtained as

$$f_{ML}(x;\upsilon) = \frac{\upsilon}{1+\upsilon}\left[(1+\upsilon)e^{\upsilon x} + 2\upsilon x - 1\right] e^{-2\upsilon x}, \quad x > 0,$$

and $f_{ML}(x;\upsilon) = 0$ for $x \leq 0$. In Chesneau *et al.* (2019), it is proved that a strong first order stochastic ordering property relates the exponential, L and ML distributions. In this precise mathematical sense, the ML distribution is "sandwiched" between the exponential and L distributions. Also, the hrf of the ML distribution is non-monotonic, contrary to the hrf of the exponential distribution, which is constant, and the one of the L distribution, which is increasing. In addition, an important structural property of the ML distribution is that $f_{ML}(x;\upsilon)$ can be expressed as a linear combination of exponential and gamma pdfs. Furthermore, in Chesneau *et al.* (2019), it is discussed the applicability of the ML model and illustrated its workability via several relevant practical data sets. More recently, Chesneau *et al.* (2020a,c) introduced two extensions of the ML distribution, namely the inverse ML distribution and the wrapped ML distribution, respectively.

The aim of this study is to offer an extension of the ML model that allows for more flexibility in modeling lifetime data. Following the idea of Ghitany *et al.* (2011), we propose the weighted ML (WML) distribution by considering the following weighted pdf:

$$f_{WML}(x;\alpha,\upsilon) = \Phi_\alpha^{-1} x^{\alpha-1} f_{ML}(x;\upsilon),$$

where $\alpha > 0$, Φ_α represents the normalizing constant corresponding to the expectation of $X^{\alpha-1}$, X being a random variable with the ML distribution with parameter υ. After simplifications, we arrive at the following analytical expression:

$$f_{WML}(x;\alpha,\upsilon) = \frac{(2\upsilon)^\alpha}{[(\upsilon+1)2^\alpha + \alpha - 1]\Gamma(\alpha)} x^{\alpha-1} \left[(1+\upsilon)e^{\upsilon x} + 2\upsilon x - 1\right] e^{-2\upsilon x}, \quad x > 0, \quad (1)$$

and $f_{WML}(x;\alpha,\upsilon) = 0$ for $x \leq 0$. Thus, the WML distribution is to the ML distribution, what the WL distribution is to the L distribution, with the hope of the same additional benefit from the statistical modelling point of view. This study develops all these aspects, respecting the rules of the art in the field.

The sections of this article are arranged as follows: Section 2 concerns some characteristics and properties of the WML distribution. Section 3 is devoted to the estimation of model parameters as well as real data applications. Section 4 ends the paper with conclusions.

2. Theoretical Work

Some relevant theoretical results on the WML distribution are presented in this section.

2.1. Analysis of the pdf

The pdf of the WML distribution as defined by Equation (1) satisfies the following asymptotic properties. In the case where x tends to be in the neighborhood of 0; an equivalent function is described below:

$$f_{WML}(x;\alpha,\upsilon) \sim \frac{(2\upsilon)^\alpha \upsilon}{[(\upsilon+1)2^\alpha + \alpha - 1]\Gamma(\alpha)} x^{\alpha-1}.$$

Hence, we see the importance of the new parameter α is the behavior of this function in 0; When $\alpha < 1$, $f_{WML}(x;\alpha,\upsilon)$ diverges to $+\infty$, when $\alpha = 1$, $f_{WML}(x;\alpha,\upsilon)$ tends to $\upsilon^2/(\upsilon+1)$, and when $\alpha > 1$, $f_{WML}(x;\alpha,\upsilon)$ tends to 0.

For the behavior at $x \to +\infty$, the following result holds:

$$f_{WML}(x;\alpha,\upsilon) \sim \frac{(2\upsilon)^{\alpha}(1+\upsilon)}{[(\upsilon+1)2^{\alpha}+\alpha-1]\Gamma(\alpha)} x^{\alpha-1}e^{-\upsilon x} \to 0.$$

In this case, the dominant term in the convergence is $e^{-\upsilon x}$; α plays a secondary role. The critical points of $f_{WML}(x;\alpha,\upsilon)$ are the solutions to the following equation: $d\log f_{WML}(x;\alpha,\upsilon)/dx = 0$, which is equivalent to the following analytical equation:

$$(\alpha-1)\frac{1}{x} + \upsilon\frac{(\upsilon+1)e^{\upsilon x}+2}{(1+\upsilon)e^{\upsilon x}+2\upsilon x-1} = 2\upsilon.$$

We see that α only modulates the term $(\alpha-1)/x$, which can be of great impact on the small values of x. The described critical points contain the possible mode of the WML distribution. They are not expressible in the strict mathematical sense, but can be determined numerically via any scientific software.

In order to provide a comprehensive study of the characteristics of $f_{WML}(x;\alpha,\upsilon)$, we end this part with a graphical analysis in Figure 1; it shows the panel of its possible shapes, depending on the conjoint values of the parameters α and υ. From Figure 1, one can observe various kinds of non-monotonic or monotonic shapes, such as reverse J-shaped, right-skewed and unimodal shapes.

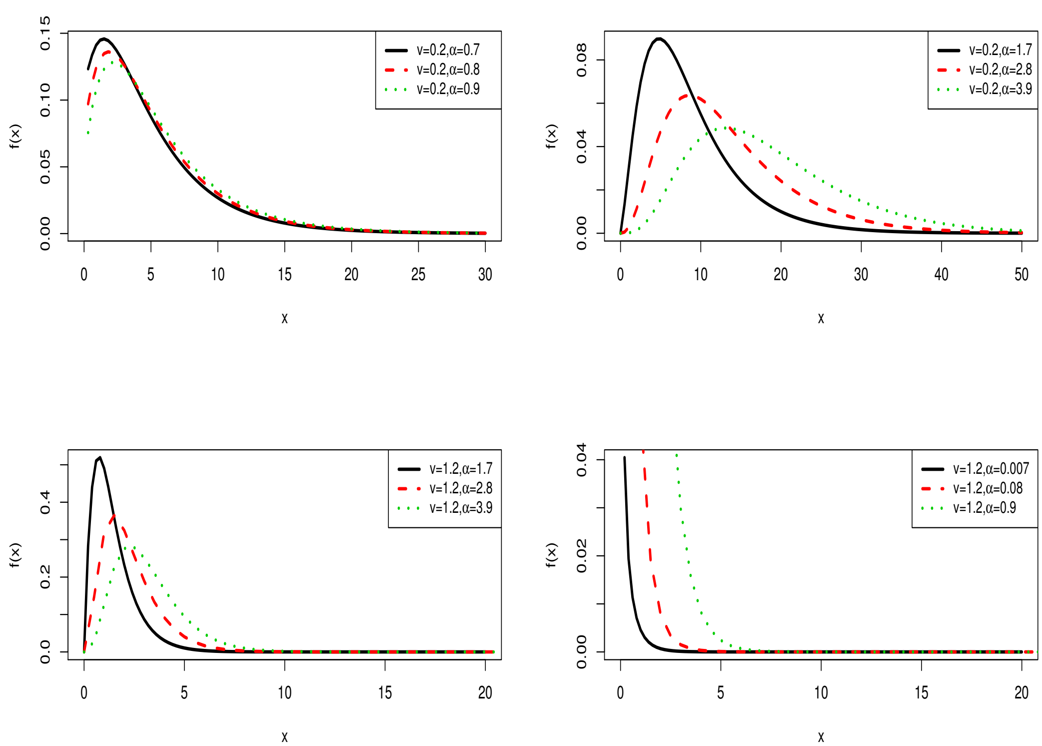

Figure 1: Examples of graphs of the pdf of the WML distribution

2.2. Expression of the cdf

Based on Equation (1), the cdf of the WML distribution can be determined; it can be expressed according to the lower incomplete Euler gamma function defined as $\gamma(s,x) = \int_0^x t^{s-1}e^{-t}dt$ with $s>0$ and $x>0$. Concretely, for any $x>0$, we have

$$\begin{aligned} F_{WML}(x;\alpha,\upsilon) &= \int_{-\infty}^{x} f_{WML}(t;\alpha,\upsilon)dt \\ &= \frac{(2\upsilon)^\alpha}{[(\upsilon+1)2^\alpha+\alpha-1]\Gamma(\alpha)}\left[(1+\upsilon)\int_0^x t^{\alpha-1}e^{-\upsilon t}dt + 2\upsilon\int_0^x t^\alpha e^{-2\upsilon t}dt - \int_0^x t^{\alpha-1}e^{-2\upsilon t}dt\right] \\ &= \frac{(2\upsilon)^\alpha}{[(\upsilon+1)2^\alpha+\alpha-1]\Gamma(\alpha)}\left[\frac{1+\upsilon}{\upsilon^\alpha}\gamma(\alpha,\upsilon x) + \frac{1}{(2\upsilon)^\alpha}\gamma(\alpha+1,2\upsilon x) - \frac{1}{(2\upsilon)^\alpha}\gamma(\alpha,2\upsilon x)\right]. \end{aligned}$$

By using the relation: $\gamma(s+1,x) = s\gamma(s,x) - x^s e^{-x}$, we arrive at the simple expression:

$$\begin{aligned} &F_{WML}(x;\alpha,\upsilon) = \\ &\frac{1}{[(\upsilon+1)2^\alpha+\alpha-1]\Gamma(\alpha)}\left[(1+\upsilon)2^\alpha\gamma(\alpha,\upsilon x) + (\alpha-1)\gamma(\alpha,2\upsilon x) - (2\upsilon x)^\alpha e^{-2\upsilon x}\right]. \end{aligned} \tag{2}$$

For $x \le 0$, we put $F_{WML}(x;\alpha,\upsilon) = 0$.

Some technical comments on this cdf are now given. As expected, by taking $\alpha = 1$, we get

$$F_{WML}(x;\alpha,\upsilon) = \frac{1}{2(\upsilon+1)}\left[2(1+\upsilon)(1-e^{-\upsilon x}) - 2\upsilon x e^{-2\upsilon x}\right] = F_{ML}(x;\upsilon).$$

Moreover, since $-(2\upsilon x)^\alpha e^{-2\upsilon x} < 0$, the following first-order stochastic dominance holds: $F_{WML}(x;\alpha,\upsilon) \le F_{MixG}(x;\alpha,\upsilon)$ for all $x \in \mathbb{R}$, where $F_{MixG}(x;\alpha,\upsilon)$ denotes the following generalized mixture cdf:

$$F_{MixG}(x;\alpha) = \lambda F_G(x;\alpha,\upsilon) + (1-\lambda)F_G(x;\alpha,2\upsilon),$$

where $\lambda = (1+\upsilon)2^\alpha/[(\upsilon+1)2^\alpha+\alpha-1]$ and $F_G(x;\alpha,\upsilon) = \gamma(\alpha,\upsilon x)/\Gamma(\alpha)$ corresponds to the cdf of the classical gamma distribution with parameters α and υ. Note that λ is always positive, but $1-\lambda$ can be negative if $\alpha < 1$. In this case, since $F_G(x;\alpha,\upsilon) \le F_G(x;\alpha,2\upsilon)$, for all $x \in \mathbb{R}$, we have

$$F_{WML}(x;\alpha,\upsilon) \le F_{MixG}(x;\alpha,\upsilon) \le F_G(x;\alpha,\upsilon).$$

Thus, in this case, the WML distribution first-order stochastically dominates the gamma distribution. For $\alpha > 1$, there is no such dominance; the situation is more complex. For illustrative purposes, Figure 2 shows the variation of $F_{WML}(x;\alpha,\upsilon)$ for varying α and υ.

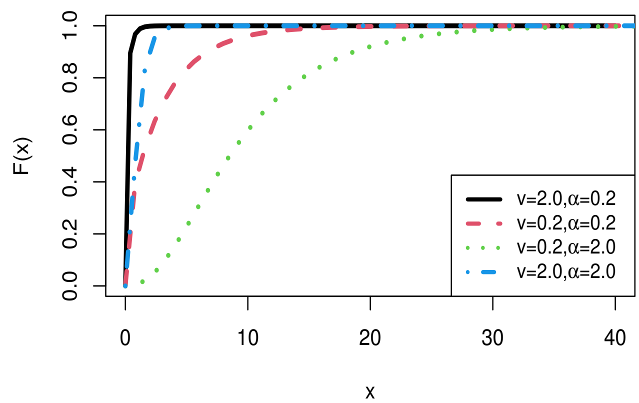

Figure 2: Examples of graphs of the cdf of the WML distribution

Last but not least, the cdf is essential for defining other distributional functions, such as the quantile function (qf) and hrf, which will be the subject of two coming subsections.

2.3. Quantile function

The qf is defined by the inverse function of $F_{WML}(x;\alpha,\upsilon)$, say $F^{-1}_{WML}(u;\alpha,\upsilon)$ with $u \in (0,1)$. In view of Equation (2), it is not possible to express it in an analytical way. However, it is always possible to do a numerical evaluation by giving values for the first quartile (when $u = 1/4$), the median (when $u = 1/2$) and the third quartile (when $u = 3/4$). In addition, this qf has a simple functional lower bound; the following inequality holds: For all $u \in (0,1)$ and $\alpha < 1$, $F^{-1}_{WML}(x;\alpha,\upsilon) \geq F^{-1}_{G}(x;\alpha)$, where $F^{-1}_{G}(x;\alpha,\upsilon)$ denotes the qf of the classical gamma distribution with parameters α and υ defined by $F^{-1}_{G}(x;\alpha,,\upsilon) = \upsilon^{-1}\gamma^{-1}(\alpha, u\Gamma(\alpha))$ with $u \in (0,1)$, $\gamma^{-1}(\alpha,y)$ being the inverse function of $\gamma(\alpha,x)$.

2.4. On the hrf

From $f_{WML}(x;\alpha,\upsilon)$ and $F_{WML}(x;\alpha,\upsilon)$ as given by Equations (1) and (2), respectively, we can present the hrf of the WML distribution by the following ratio function: $h_{WML}(x;\alpha,\upsilon) = f_{WML}(x;\alpha,\upsilon)/[1 - F_{WML}(x;\alpha,\upsilon)]$. When $x > 0$, it is given as

$$h_{WML}(x;\alpha,\upsilon) = \frac{(2\upsilon)^{\alpha}x^{\alpha-1}\left[(1+\upsilon)e^{\upsilon x} + 2\upsilon x - 1\right]e^{-2\upsilon x}}{[(\upsilon+1)2^{\alpha} + \alpha - 1]\Gamma(\alpha) - (1+\upsilon)2^{\alpha}\gamma(\alpha,\upsilon x) - (\alpha-1)\gamma(\alpha,2\upsilon x) + (2\upsilon x)^{\alpha}e^{-2\upsilon x}}.$$

and $h_{WML}(x;\alpha,\upsilon) = 0$ for $x \leq 0$. The possible shapes of $h_{WML}(x;\alpha,\upsilon)$ are of great interest in understanding the modelling capability of the WML model (see, Aarset (1987)). Since the expression of $h_{WML}(x;\alpha,\upsilon)$ is mathematically complex, we conduct a visual analysis in Figure 3, showing the diverse shapes possessed by this model. From Figure 3, it is clear that hrf has various kinds of non-monotonic shapes, such as reverse J-shaped, reversed N-shaped, right-skewed and unimodal shapes, which makes the proposed distribution more flexible to fit different data sets. As we know, the L and ML distributions have only unimodal hrf. Hence, the WML distribution is more flexible than its parent distributions, such as the L and ML distributions.

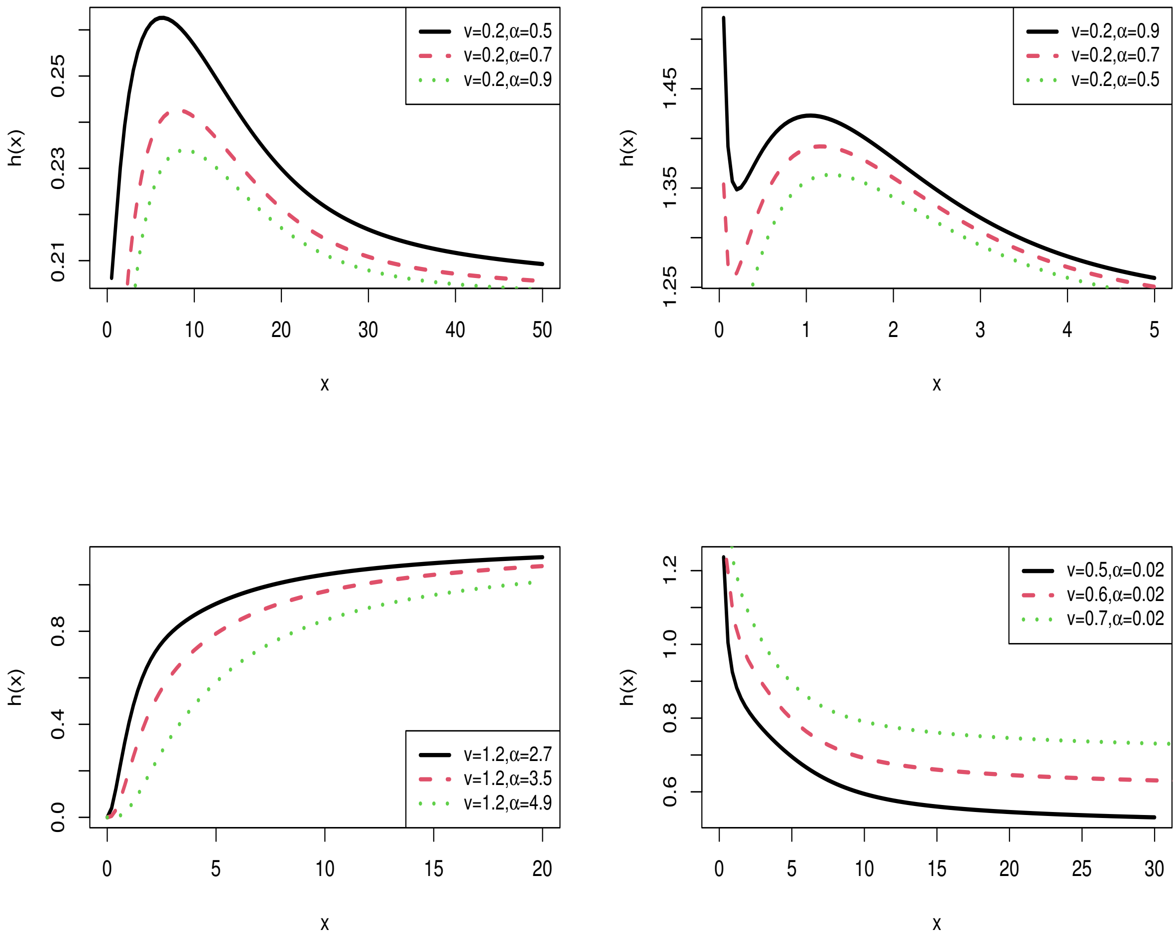

Figure 3: Examples of graphs of the hrf of the WML distribution

2.5. Mathematical moments

In this section, we study the useful moment characteristics and measures of the WML distribution. Let X be a random variable that follows the WML distribution. Besides, we discuss the incomplete moments of X, from which we derive moments and discuss some related quantities. The moment generating and characteristic functions are also expressed.

2.5.1. Incomplete moments

As a first information, the sth incomplete moment of X exists, and it is classically defined by $m_s(x) = E(X^s I(X \leq x))$, where E denotes the mathematical expectation operator.

Therefore, by taking into account the definition of $f_{WML}(x;\alpha,\upsilon)$, the integral definition of $m_s(x)$ becomes

$$m_s(x) = \int_0^x t^s f_{WML}(t;\alpha,\upsilon)dt$$
$$= \frac{(2\upsilon)^\alpha}{[(\upsilon+1)2^\alpha+\alpha-1]\Gamma(\alpha)}\left[(1+\upsilon)\int_0^x t^{s+\alpha-1}e^{-\upsilon t}dt + 2\upsilon\int_0^x t^{s+\alpha}e^{-2\upsilon t}dt - \int_0^x t^{s+\alpha-1}e^{-2\upsilon t}dt\right]$$
$$= \frac{(2\upsilon)^\alpha}{[(\upsilon+1)2^\alpha+\alpha-1]\Gamma(\alpha)}\times$$
$$\left[\frac{1+\upsilon}{\upsilon^{s+\alpha}}\gamma(s+\alpha,\upsilon x) + \frac{1}{(2\upsilon)^{s+\alpha}}\gamma(s+\alpha+1,2\upsilon x) - \frac{1}{(2\upsilon)^{s+\alpha}}\gamma(s+\alpha,2\upsilon x)\right].$$

Since $\gamma(s+\alpha+1,2\upsilon x) = (s+\alpha)\gamma(s+\alpha,2\upsilon x) - (2\upsilon x)^{s+\alpha}e^{-2\upsilon x}$, the sth incomplete moment is reduced to

$$m_s(x) = \frac{1}{(2\upsilon)^s[(\upsilon+1)2^\alpha+\alpha-1]\Gamma(\alpha)}\times$$
$$\left[(1+\upsilon)2^{s+\alpha}\gamma(s+\alpha,\upsilon x) + (s+\alpha-1)\gamma(s+\alpha,2\upsilon x) - (2\upsilon x)^{s+\alpha}e^{-2\upsilon x}\right]. \quad (3)$$

From this expression, by taking $s=0$, we logically obtain the expression of $F_{WML}(x;\alpha,\upsilon)$. Furthermore, some uses of this manageable expression are described below. Also, by taking $\alpha=1$, we rediscover the sth incomplete moment of a random variable with the former ML distribution.

2.5.2. Ordinary moments and related measures

Also, the ordinary moments of X can be easily obtained by applying $x\to+\infty$ in Equation (3). That is, the sth ordinary moment of X is given as

$$m_s = m_s(+\infty) = \frac{1}{(2\upsilon)^s[(\upsilon+1)2^\alpha+\alpha-1]\Gamma(\alpha)}\left[(1+\upsilon)2^{s+\alpha}+s+\alpha-1\right]\Gamma(s+\alpha).$$

In particular, by using the relation: $\Gamma(x+1) = x\Gamma(x)$ for $x>0$, the four first ordinary moments of X are

$$m_1 = \frac{\alpha\left[(1+\upsilon)2^{1+\alpha}+\alpha\right]}{2\upsilon[(\upsilon+1)2^\alpha+\alpha-1]}, \quad m_2 = \frac{\alpha(\alpha+1)\left[(1+\upsilon)2^{2+\alpha}+\alpha+1\right]}{(2\upsilon)^2[(\upsilon+1)2^\alpha+\alpha-1]},$$

$$m_3 = \frac{\alpha(\alpha+1)(\alpha+2)\left[(1+\upsilon)2^{3+\alpha}+\alpha+2\right]}{(2\upsilon)^3[(\upsilon+1)2^\alpha+\alpha-1]}$$

and

$$m_4 = \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)\left[(1+\upsilon)2^{4+\alpha}+\alpha+3\right]}{(2\upsilon)^4[(\upsilon+1)2^\alpha+\alpha-1]}.$$

The classical central and dispersion moment parameters of X follow immediately, including the mean given as $m = m_1$, variance given as $V = m_2 - m_1^2$, coefficient of variation $\sqrt{V}/m$, as well as the skewness and kurtosis coefficients obtained as

$$S = \frac{m_3 - 3m_2 m + 2m^3}{V^{3/2}}$$

and

$$K = \frac{m_4 - 4m_3 m + 6m_2 m^2 - 3m^4}{V^2},$$

respectively. The numerical pliancy of these important probabilistic measures is shown in Table 1. Since the skewness has positive values, the WML distribution is skewed to the right. In addition, the WML distribution can be platykurtic (when $K < 3$) and leptokurtic (when $K > 3$). Furthermore, the mean of the proposed distribution can be smaller or greater than its variance.

2.5.3. Some related functions

Incomplete and ordinary moments are the main ingredients of various functions or indexes that are useful in various applied areas. For instance, from the first incomplete and ordinary moments, one can express the mean residual function given as

$$\begin{aligned} M_{WML}(x) &= E(X - x \mid X > x) \\ &= \frac{m_1 - m_1(x)}{1 - F_{WML}(x;\alpha,\upsilon)} - x \\ &= \frac{[(1+\upsilon)2^{1+\alpha} + \alpha]\,\Gamma(\alpha+1) - (1+\upsilon)2^{1+\alpha}\gamma(\alpha+1,\upsilon x) - \alpha\gamma(1+\alpha,2\upsilon x) + (2\upsilon x)^{1+\alpha}e^{-2\upsilon x}}{2\upsilon[(\upsilon+1)2^{\alpha} + \alpha - 1]\Gamma(\alpha) - (1+\upsilon)2^{\alpha}\gamma(\alpha,\upsilon x) - (\alpha-1)\gamma(\alpha,2\upsilon x) + (2\upsilon x)^{\alpha}e^{-2\upsilon x}} - x \end{aligned}$$

and the mean reversed residual function defined as

$$\begin{aligned} M_{WML}^{rev}(x) &= E(x - X \mid X \le x) \\ &= x - \frac{m_1(x)}{F_{WML}(x;\alpha,\upsilon)} \\ &= x - \frac{1}{2\upsilon}\frac{(1+\upsilon)2^{1+\alpha}\gamma(1+\alpha,\upsilon x) + \alpha\gamma(1+\alpha,2\upsilon x) - (2\upsilon x)^{1+\alpha}e^{-2\upsilon x}}{(1+\upsilon)2^{\alpha}\gamma(\alpha,\upsilon x) + (\alpha-1)\gamma(\alpha,2\upsilon x) - (2\upsilon x)^{\alpha}e^{-2\upsilon x}}. \end{aligned}$$

In terms of reliability and life testing, these functions play a crucial role. See, for instance, Barlow and Proschan (1975) and Nanda *et al.* (2003). They can be used in the setting of the WML distribution for further purposes in this direction.

2.5.4. Moment functions

Based on Equation (1), the moment generating and characteristic functions of the WML distribution can be obtained using the lower incomplete Euler gamma function. Indeed, for any $x < \upsilon$, we have

$$
\begin{aligned}
&R_{WML}(x) = E(e^{xX}) = \int_0^{+\infty} e^{xt} f_{WML}(t;\alpha,\upsilon)dt \\
&= \frac{(2\upsilon)^\alpha}{[(\upsilon+1)2^\alpha+\alpha-1]\Gamma(\alpha)} \times \\
&\left[(1+\upsilon)\int_0^{+\infty} t^{\alpha-1}e^{-(\upsilon-x)t}dt + 2\upsilon\int_0^{+\infty} t^{\alpha}e^{-(2\upsilon-x)t}dt - \int_0^{+\infty} t^{\alpha-1}e^{-(2\upsilon-x)t}dt\right] \\
&= \frac{(2\upsilon)^\alpha}{[(\upsilon+1)2^\alpha+\alpha-1]\Gamma(\alpha)}\left[\frac{1+\upsilon}{(\upsilon-x)^\alpha}\Gamma(\alpha) + \frac{2\upsilon}{(2\upsilon-x)^{\alpha+1}}\Gamma(\alpha+1) - \frac{1}{(2\upsilon-x)^\alpha}\Gamma(\alpha)\right] \\
&= \frac{(2\upsilon)^\alpha}{(\upsilon+1)2^\alpha+\alpha-1}\left[\frac{1+\upsilon}{(\upsilon-x)^\alpha} + \frac{2\upsilon(\alpha-1)+x}{(2\upsilon-x)^{\alpha+1}}\right].
\end{aligned}
$$

By taking $\alpha = 1$, we rewrite the moment generating function of the former ML distribution. By using the standard formula, we have $m_s = R_{WML}(x)^{(s)} \mid_{x=0}$. Also, the rth cumulant of X can be obtained through the following equation: $\kappa_s = \{\log R_{WML}(x)\}^{(s)} \mid_{x=0}$. Also, the characteristic function of X is immediately deduced from $R_{WML}(x)$; it is given as

$$
\Psi_{WML}(x) = E(e^{ixX}) = \frac{(2\upsilon)^\alpha}{(\upsilon+1)2^\alpha+\alpha-1}\left[\frac{1+\upsilon}{(\upsilon-ix)^\alpha} + \frac{2\upsilon(\alpha-1)+ix}{(2\upsilon-ix)^{\alpha+1}}\right], \quad x \in \mathbb{R},
$$

where $i^2 = -1$. This function fully characterizes the WML distribution, and can be used for further results in distribution involving the WML distribution.

3. Estimation and Application

In this section, we will discuss estimation and its applicability in a concrete data analysis scenario.

3.1. Parametric estimation

The maximum likelihood (MaxLik) method can be applied to obtain efficient estimates of the WML model parameters. In this context, what is necessary is specified below. Let $x_1, \ldots, x_n$ be realizations of n independent random variables, all distributed following the WML distribution with parameters α and υ. Then, the estimates suggested by the MaxLik method are given by the arguments of the maxima of the likelihood function, or the log-likelihood function defined by

$$
\begin{aligned}
\ell(\alpha,\upsilon) = \sum_{i=1}^{n}\log f_{WML}(x_i;\alpha,\upsilon) &= n\alpha\log 2 + n\alpha\log\upsilon - n\log\Gamma(\alpha) - n\log[(\upsilon+1)2^\alpha+\alpha-1] \\
&+ (\alpha-1)\sum_{i=1}^{n}\log x_i - 2\upsilon\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\log\left[(1+\upsilon)e^{\upsilon x_i} + 2\upsilon x_i - 1\right].
\end{aligned}
$$

The maximum likelihood estimates (MaxLikEs) are denoted by $\hat{\alpha}$ and $\hat{\upsilon}$, satisfying $\ell(\alpha,\upsilon) \leq \ell(\hat{\alpha},\hat{\upsilon})$ for any $\alpha > 0$ and $\upsilon > 0$, by construction. They are also the solutions of the two following equations with respect to the parameters:

$$\frac{\partial}{\partial\alpha}\ell(\alpha,\upsilon)=n\log 2+n\log\upsilon-n\frac{\partial\Gamma(\alpha)/\partial\alpha}{\Gamma(\alpha)}-n\frac{(\upsilon+1)2^{\alpha}\log(2)+1}{(\upsilon+1)2^{\alpha}+\alpha-1}+\sum_{i=1}^{n}\log x_i=0$$

and

$$\frac{\partial}{\partial\upsilon}\ell(\alpha,\upsilon)=n\frac{\alpha}{\upsilon}-n\frac{2^{\alpha}}{(\upsilon+1)2^{\alpha}+\alpha-1}-2\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\frac{x_i[(\upsilon+1)e^{\upsilon x_i}+2]+e^{\upsilon x_i}}{(1+\upsilon)e^{\upsilon x_i}+2\upsilon x_i-1}=0.$$

Explicit formulations for $\hat{\alpha}$ and $\hat{\upsilon}$ are not possible due to the intricacy of these equations. As a result, numerical methods involving Newton-type algorithms must be used to solve them. Alternatively, one can investigate the maximization of $\ell(\alpha,\upsilon)$ numerically through specific functions in the `R` package, such as the `constrOptim` function, `optim` function or `maxLik` function.

The theory of MaxLikEs ensures that $\hat{\alpha}$ and $\hat{\upsilon}$ are efficient in several senses, including their fast numerical convergence to the underlying true values of the parameters. Other important properties are described in Casella and Berger (1990).

Using the asymptotic normal distribution of the MaxLikEs, we can evaluate the confidence intervals (CIs) of unknown parameters. In this regard, the observed Fisher information matrix $I(\alpha,\upsilon)$ formed of the negative second derivatives of the log-likelihood function must be determined. In this asymptotic framework, the $100(1-\gamma)\%$ CI for α is defined by the interval with the following lower bound (LB) and upper bound (UB):

$$LB=\hat{\alpha}\pm z_{\gamma/2}\sqrt{I_{\hat{\alpha}\hat{\alpha}}},\quad UB=\hat{\alpha}\pm z_{\gamma/2}\sqrt{I_{\hat{\alpha}\hat{\alpha}}},$$

where $z_{\gamma/2}$ is the percentile of the standard normal distribution with right tail probability $\gamma/2$, and $I_{\hat{\alpha}\hat{\alpha}}$ is the first diagonal component of $I^{-1}(\hat{\alpha},\hat{\upsilon})$. The same holds for the parameter υ by the consideration of $\hat{\upsilon}$ instead of $\hat{\alpha}$, and $I_{\hat{\upsilon}\hat{\upsilon}}$ instead of $I_{\hat{\alpha}\hat{\alpha}}$.

3.2. Simulation study

We are now conducting a simulation research to assess the performance of the MaxLikEs of the parameters of the WML distribution. The Newton formula is used because the qf of this distribution is not available in closed form. The simulation experiment was repeated 1000 times with sample sizes of 25, 80 and 150 from the WML distribution. The assessment was based on the following steps of simulation study:

1. Generate 1000 samples of size N.
2. Assign the sample size of n and the values of the parameters.
3. Assign the initial value for the random start y_0.
4. For $j=1,\ldots,n$, generate u_j from a random variable U_j following the unit uniform distribution.

5. Change y_0 by y^* by using the Newton formula as follows:

$$y^* = y_0 - \left\{ \frac{F_{WML}(y_0;\alpha,\upsilon) - u_j}{f_{WML}(y_0;\alpha,\upsilon)} \right\}.$$

6. If $|y_0 - y^*| \leq \epsilon$ for small $\epsilon > 0$, ϵ being considered as a tolerance limit, then $y = y^*$ is considered as a generated value from the WML distribution with parameters α and υ, else set $y_0 = y^*$ and go to the previous step.

7. Repeat steps 4 to 6 for $j = 1, \ldots, n$ to obtain n values $y_1, \ldots, y_n$.

8. Compute the MaxLikEs of α and υ from $y_1, \ldots, y_n$.

9. Repeat steps 2 to 8, N times.

10. Compute the Bias and mean square error (MSE) for each parameter, defined as

$$\text{Bias}(\alpha) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i - \alpha), \quad \text{MSE}(\alpha) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i - \alpha)^2,$$
$$\text{Bias}(\upsilon) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\upsilon}_i - \upsilon), \quad \text{MSE}(\upsilon) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\upsilon}_i - \upsilon)^2,$$

where $\hat{\alpha}_i$ and $\hat{\upsilon}_i$ are the MaxLikEs of α and υ, respectively, obtained at the i^{th} replication.

The parameter combinations are given below:

1. $\alpha = 1.5$, $\upsilon = 1.5$
2. $\alpha = 2$, $\upsilon = 3.5$
3. $\alpha = 3.5$, $\upsilon = 2.5$
4. $\alpha = 4$, $\upsilon = 2.5$

Table 2 presents the Bias, MSE, LB and UB related to the CIs of the parameters for different sample sizes. The Bias and MSE decrease as n increases. As a result, the MaxLik approach for estimating the parameters of the WML distribution using Bias and MSE works fairly well.

3.3. Application

This portion contains an application of the WML distribution to real lifetime data. To demonstrate the potential of the WML distribution, a comparison is made using two-parameter extensions or modifications of the L distribution, which are the WL distribution by (Ghitany *et al.*, 2011) and some other extended L distributions. Below is a list of the competing distributions.

1. The quasi L (QL) distribution (Shanker and Mishra, 2013b) with pdf

$$f(x;\alpha,\upsilon)=\frac{\upsilon(\alpha+x\upsilon)}{\alpha+1}e^{-\upsilon x},\quad x>0,$$

   and $f(x;\alpha,\upsilon)=0$ for $x\leq 0$.

2. The two-parameter L (SL) distribution (Shanker and Mishra, 2013a) with pdf,

$$f(x;\alpha,\upsilon)=\frac{\upsilon^2}{\alpha\upsilon+1}(\alpha+x)e^{-\upsilon x},\quad x>0,$$

   and $f(x;\alpha,\upsilon)=0$ for $x\leq 0$.

3. The exponentiated L (EL) distribution (see, Cordeiro *et al.*, 2013) with pdf,

$$f(x;\alpha,\upsilon)=\frac{\alpha\upsilon^2}{\upsilon+1}e^{-\upsilon x}(1+x)\left[1-\left(1+\frac{\upsilon x}{1+\upsilon}\right)e^{-\upsilon x}\right]^{\alpha-1},\quad x>0,$$

   and $f(x;\alpha,\upsilon)=0$ for $x\leq 0$.

4. The power L (PL) distribution (Ghitany *et al.*, 2013) with pdf,

$$f(x;\alpha,\upsilon)=\frac{\alpha\upsilon^2}{\upsilon+1}(1+x^\alpha)x^{\alpha-1}e^{-\upsilon x^\alpha},\quad x>0,$$

   and $f(x;\alpha,\upsilon)=0$ for $x\leq 0$.

For the pdfs above, it is supposed that $\upsilon>0$ and $\alpha>0$.

The MaxLik method is applied to estimate the unknown parameters, along with the determination of the related standard errors (SEs). The following criteria are used to choose the best-fitting distribution: negative maximized Log-likelihood value ($-\log L$), Akaike information criterion (AIC) and Bayesian information criterion (BIC). The value of the Kolmogorov-Smirnov (K-S) statistic and the p-value are also provided.

The real data set corresponds to the life of a fatigue fracture of Kevlar 373/epoxy that was subjected to steady pressure (at 90% stress) until it failed. Therefore, we have comprehensive data with accurate failure periods. The data set has been obtained from Barlow *et al.* (1984) and Andrews and Herzberg (1985). For previous studies on the data, see Chesneau *et al.* (2020a).

The values of this data set are: 0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960

The findings of a descriptive evaluation of the fitted models for the data set are shown in Table 3. The R program is used to perform the necessary calculations.

Based on the goodness-of-fit measures, the smallest −logL, AIC, BIC, K-S statistics and the highest p-values are obtained for the WML distribution. These observations indicate that the WML model provides the best fit for the data set. Moreover, from the study, the competing distributions can be ranked in the following order (best to the least): EL distribution, SL distribution, WL distribution, QL distribution, and PL distribution.

As a graphical approach, in Figure 4, we present the estimated pdfs against the fitted pdfs. In addition, the empirical cdf against the fitted cdfs is also given in Figure 5. From these figures, we see that the two fits of the estimated functions of the WML model have well captured the forms or curvatures of the empirical objects, confirming the previous numerical analysis.

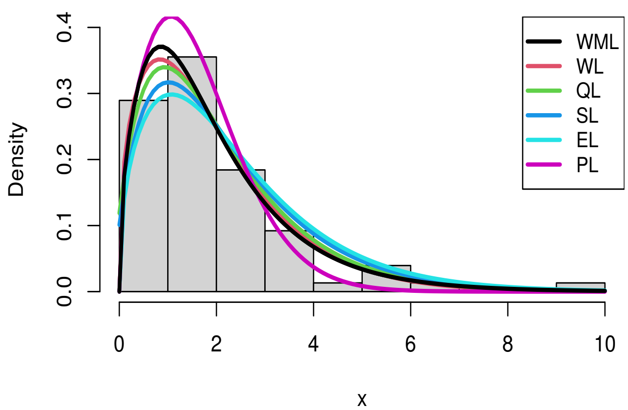

Figure 4: Graphs of the estimated pdfs of the considered distributions

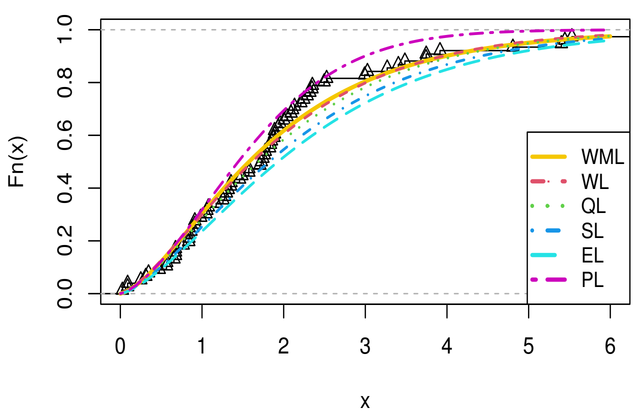

Figure 5: Graphs of the estimated cdfs of the considered distributions

4. Conclusions

In this paper, we introduced a weighted scheme for the modified L distribution, referred to as the weighted modified Lindley distribution. The main motivations for introducing this new distribution are provided. Various shapes of pdf and hrf, which are attractive for statistical modeling, are highlighted. In particular, we have exhibited that the pdf and hrf can be unimodal and monotonically decreasing. In addition, detailed and elegant discussions of incomplete moments, ordinary moments with their related measures, moment generating function and characteristic function are given. Parameter estimation is approached by the use of the maximum likelihood function in a simulation study. The usefulness of the new distribution is illustrated in an analysis of real data. Thus, the proposed model can be used quite effectively for analysing lifetime data.

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ANNEXURE

Table 1: Some numerical values of moment measures of the WML distribution

$(\alpha, v) \rightarrow$	$(2, 0.02)$	$(2, 2)$	$(0.2, 0.2)$	$(0.2, 2)$	$(0.75, 70)$
m	0.0015	1.0000	0.5182	9.9910	64.7160
m_2	0.0008	1.4711	3.1704	143.5341	5751.618
m_3	0.0008	2.8846	31.9664	2741.3372	514672.4
m_4	0.0012	7.1034	461.7491	66105.4505	46346481
V	7.9357	0.4712	2.9019	43.5341	1563.458
S	11.1390	1.4569	5.5259	1.5155	0.9691
K	185.9511	6.2699	47.5460	6.6337	2.0563

Table 2: Simulation results related to the parameters of the WML distribution

Combinations	n		Bias	MSE	LB	UB
$\alpha = 1.5,\ \upsilon = 1.5$	100	α	0.0585	0.0751	1.4990	1.6110
		υ	0.0644	0.0780	1.4981	1.6177
	200	α	0.0295	0.0346	1.4941	1.5550
		υ	0.0336	0.0348	1.4962	1.5591
	500	α	0.0169	0.0138	1.49067	1.5271
		υ	0.0173	0.0132	1.4973	1.5273
$\alpha = 2,\ \upsilon = 3.5$	100	α	0.0635	0.1055	1.9999	2.1259
		υ	0.1309	0.3794	3.4981	3.7489
	200	α	0.0352	0.0477	1.9988	2.0651
		υ	0.0677	0.1700	3.4911	3.6241
	500	α	0.0120	0.0207	1.9994	2.0246
		υ	0.0215	0.0704	3.4913	3.5447
$\alpha = 3.5,\ \upsilon = 2.5$	100	α	0.1240	0.3002	3.4993	3.7287
		υ	0.0947	0.1818	2.4992	2.6762
	200	α	0.0453	0.1397	3.4930	3.5968
		υ	0.0450	0.0887	2.4988	2.5858
	500	α	0.0103	0.0555	3.4897	3.5310
		υ	0.0146	0.0325	2.4942	2.5303
$\alpha = 4,\ \upsilon = 2.5$	100	α	0.1163	0.3703	3.9992	4.2334
		υ	0.0826	0.1625	2.4905	2.6600
	200	α	0.0556	0.1778	3.9976	4.1135
		υ	0.0368	0.0839	2.4970	2.5767
	500	α	0.0162	0.0667	3.9936	4.0388
		υ	0.0135	0.0307	2.4982	2.5288

Table 3: Descriptive evaluation of the fitted models for the data set

Model	MaxLikE (SE)	$-\log$L	AIC	BIC	K-S	p-value
WML	$\hat{\upsilon} = 0.7020$ (0.1303) $\hat{\alpha} = 1.2723$ (0.2657)	121.4213	246.8426	251.5041	0.0931	0.4965
WL	$\hat{\upsilon} = 1.0007$ (0.1469) $\hat{\alpha} = 1.3809$ (0.2339)	122.0275	248.055	252.7164	0.10413	0.3573
QL	$\hat{\upsilon} = 0.9543$ (0.0954) $\hat{\alpha} = 0.1498$ (0.1437)	121.6503	247.3006	251.962	0.13049	0.1374
SL	$\hat{\upsilon} = 0.9544$ (0.0954) $\hat{\alpha} = 6.3676$ (6.4571)	121.6503	247.3006	251.962	0.10247	0.3765
EL	$\hat{\upsilon} = 9364$ (0.1047) $\hat{\alpha} = 1.3905$ (0.2376)	121.8991	247.7981	252.4596	0.10221	0.3796
PL	$\hat{\upsilon} = 0.7046$ (0.0819) $\hat{\alpha} = 1.1425$ (0.0908)	122.4001	248.8001	253.4616	0.11233	0.2719