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Joint Importance Measures for Repairable Multistate Systems

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Abstract

In order to identify the vulnerable components whose joint effect would have changed system performance and ensure the required reliability of various multistate systems, joint importance measures of relevant components are used in the early design of systems. Due to the complexity of multistate systems that have the properties of non-linearity, uncertainty, and randomness, which make it difficult to analyze the reasons of failure mechanisms, model the system, estimate its reliability, and evaluate the joint importance measures of its components. This paper discussed measures of joint importance of three components for repairable multistate systems based on the classical Birnbaum measure. Eight importance measures are studied in detail. These joint importance measures provide a time-dependent analysis of the relevancy of components, thus adding insights on the contributions of the joint effect of three components on the system reliability or performance over time. An illustrative example is given. The results of the study show that joint importance measures can be a valuable decision-support tool for designers and engineers in the design of systems.

Key words: Birnbaum importance; Multistate systems; Repairable components; Joint importance measures.

1. Introduction

The evaluation of joint importance measures for identifying group of relevant components in complex systems is a major concern of reliability engineers and designers. Importance and joint importance measures are widely used to identify the impact and locate the vulnerable spots at the early design stages. The identification of most important component or group of components in a repairable multistate complex system by investigating the improvement resulted in performance measures like reliability or availability or unreliability/risk or unavailability etc with the improvement in corresponding component performance measures, is to be addressed in detail in situation where minor repair or replacement after complete failure of components admits. The concept of the joint importance measures of components or subsystems or modules is crucial, in order to ensure and improve the product

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quality, reliability, and safety. This is also essential for allocating the limited resources at the design stage, to reduce the cost and providing maintenance to take proper care of crucial components at the operation stage. The information provided by the joint importance measures can be used to give proper repair/replacement activities to the components. Thereby one can ensure the system performance in high level always for the continuous supply of service for the allocated mission. Quantifying the joint importance of components using an efficient method becomes essential in multistate systems, at the early design stage. However, the determination of the relevant components or subsystems at the early stage is challenging, because it is usually difficult to analyze and describe non-linear dependent relationships of components in complex systems, and obtain sufficient reliability information from the joint operational condition of components or system, Borgonovo and Apostolakis (2001).

The development of importance measures and its use can be seen in Birnbaum (1969), Fussell and Vesely (1972), Barlow and Proschan (1975) and Natvig (1985), see also Natvig (1979) and Natvig and Gåsemyr (2009). Since these measures solely depend on the probabilistic characteristics of the system's components and its structure, these traditional measures of importance can be characterized as generic. In power generation system, communication systems, network systems, the multistate approach can be adopted. Fundamental results on multistate system(MSS)s is available in Griffith (1980). The extensions of the Birnbaum measure for binary state systems to the multistate case can be seen in Dui *et al.* (2019). Natvig *et al.* (2011) and Natvig *et al.* (2009) studied on Importance measures for repairable systems. Algorithm for solution of a problem of maximum flow in a network with power estimation is given by Dinic (1970). Dui *et al.* (2015) has given semi-Markov process-based integrated importance measures for multi-state systems. Borgonovo and Apostolakis (2001) discussed a new importance measure for risk-informed decision making. Optimization of linear consecutive-k-out-of-n system with a Birnbaum importance-based genetic algorithm is given by Cai *et al.* (2016). Cai *et al.* (2017) discussed maintenance optimization of continuous state systems. Huseby and Natvig (2009) has introduced advanced discrete simulation methods applied to repairable multi-state systems. Huseby and Natvig (2013) has given discrete event simulation methods applied to advanced importance measures of repairable components in multistate network flow systems. Importance and sensitivity analysis of multistate systems using the universal generating function is carried out by Levitin and Lisnianski (1999). Generalized importance measures for multistate elements based on performance level restrictions can be seen in Levitin *et al.* (2003). Natvig (2011) has given a detailed description of multistate systems reliability theory with applications. Natvig *et al.* (2009) has given application of Natvig measures of component importance in repairable systems. Ramirez-Marquez and Coit (2005) introduced new composite importance measures for multi-state systems with multistate components. Ramirez-Marquez and Coit (2007) explained Multi-state component relevancy analysis for reliability improvement in multi-state systems. Ramirez-Marquez *et al.* (2006) has given new ideas on multi-state component relevancy and importance. Si *et al.* (2012b) proposed Integrated importance measure of component states based on loss of system performance. Si *et al.* (2012a) discussed the integrated importance measure of multistate coherent systems for maintenance processes. Si *et al.* (2013) has introduced component state-based integrated importance measure for multi-state systems. Si *et al.* (2019) proposed system reliability allocation and optimization based on generalized Birnbaum importance measure. Wu and Coolen (2013) has given a cost-based importance measure for system components: an extension of the Birnbaum importance. Wu *et al.* (2016) used component importance to optimization of preventive maintenance policy.Zhu *et al.* (2017) discussed Birnbaum importance based heuristics for multi-type component assignment problems. Monte-Carlo simulation analysis of the effects on deferent system performance levels on the importance on multistate components is given by Zio and Podofillini (2003). Zio and Podofillini (2006) discussed components interactions in the differential importance measure. Zio *et al.* (2004) described estimation of the importance measures of multistate elements by Monte-Carlo simulation. Zio *et al.* (2007) has given an example in railway industry of importance measures-based prioritization for improving the performance of multi-state systems. Dui *et al.* (2019) proposed system performance-based joint importance analysis guided maintenance for repairable systems. Dui *et al.* (2020) introduced component joint importance measures for maintenances in submarine blowout preventer system. A detailed study on joint importance measures for unrepairable systems can be seen in Chacko and Manoharan (2008, 2011), Chacko (2020, 2023a) and Chacko (2023b).

The investigation of component joint performance with regard to the variation in system performance is crucial for the repair or replacement activities (Chacko (2022)). Existing Joint importance measures for components in multistate systems are used to identify group of components for unrepairable components and systems (Chacko (2022, 2021)). But, sometimes, systems are repairable or its components can be repaired/replaced as a cost effective strategy. The main objective of this paper is to study on joint importance measures for three components of repairable systems which are defined in the Birnbaum sense, a method of observing change in system performance with respect to change in component performance. Moreover, a multistate behavior to the components is assumed.

In the present paper, generic joint importance measures for three components of a repairable systems are studied in detail, which measure the interaction effect of three repairable components. Each component is assumed to follows periodic life cycles, starting out in the top state, say $M_i, i = 1, 2, \dots, n$ and then moving through the intermediate states $k, M_i > k > 0$, until they reaches down state 0. Then, they are repaired or replaced, and a new life cycle starts. Moreover, repair at intermediate states is also assumed. Component i is allowed to have minor repair at state $k, M_i > k > 0$, to reach to state $k + 1$. If the component reaches the state 0, it will undergo corrective maintenance or replacement to bring the component to as good as new condition.

The present paper includes four sections. In section 2, the new joint importance measures are discussed. Applications are given in section 3. An illustrative example is given in section 4. Conclusions are given in final section.

2. Relevancy and importance in multistate systems

In a binary system setup, the Birnbaum-importance(B-importance) of component i (Birnbaum (1969)) is the probability that i^{th} component is relevant for the system. That is

$$I_B(i; p) = Pr\{\phi(X) = 1|X_i = 1\} - Pr\{\phi(X) = 1|X_i = 0\} \quad (1)$$

This measure is generic since it is defined based on probability and system structure function. Here we consider joint importance measures for three repairable components in MSS setup based on B –importance. For that, a multistate system of n components is considered. Joint importance measures for three components of the multistate system are discussed.

Let $X(t) = (X_1(t), X_2(t), ..., X_n(t))$ be the state vector of n components and $\phi(X(t))$ represent state of the system, where $X_i(t)$ represent state of the component i at time $t, X_i(t)$ takes the values in $S_i = \{0, 1, \ldots, M_i\}, i \in \{1, 2, \ldots, n\}$. That is,

$$\phi(X(t)) = \phi(X_1(t), ..., X_n(t)) = k, k \in \{0, 1, 2, \ldots, M\}, M = \text{Max}_{\{1 \leq i \leq n\}}\{M_i\}.$$

where $\phi(x_i, X(t)) = \phi(X_1(t), ..., X_{i-1}(t), x_i, X_{i+1}(t), ..., X_n(t))$,
i.e,$(x_i, X(t)) = (X_1(t), ..., X_{i-1}(t), x_i, X_{i+1}(t), ..., X_n(t))$.

We also define a function $f_i : S_i \to R$ that represents the component's physical state at time t given by $f_i(X_i(t)) = f_i(x_i)$ if $X_i(t) = x_i \in Si, i \in \{1, 2, \ldots, n\}$. For example f_i represents the flow capacity of the component in a network system. It is important to understand that the functions $f_i, i \in \{1, 2, \ldots, n\}$, do not necessarily have to be non-decreasing and hence provide modeling flexibility by avoiding this restriction.

To define the relevancy of the repairable components in system functioning, let us define two functions $X_i^+(t)$ and $X_i^-(t)$, for $i = 1, 2, \ldots, n$.

$X_i^+(t)$ denotes the next state of component i and is defined by

$$X_i^+(t) = X_i(t) - 1, \ if \ X_i(t) > 0 \ \ and \ \ = M_i, \ if \ X_i(t) = 0 \tag{2}$$

Since it's a repairable periodic cycle component, on reaching state 0, the component is repaired. Hence its next state at time t from 0 will be M_i.

Similarly, we define $X_i^-(t)$ as the previous state of component i and is given by

$$X_i^-(t) = X_i(t) + 1, \ if \ X_i(t) < M_i \ \ and \ \ = 0, \ if \ X_i(t) = M_i \tag{3}$$

Here when the component at time t is in the highest possible state M_i, it implies that the previous state will be 0, since the component was repaired. A component is said to be in n-relevant or p-relevant at time t, if there is change in system state when component move either to next state by gradual degradation or to previous state at time t by minor maintenance. That is, component is said to be n-relevant, while component moves to its next state at time t if

$$\phi(X_i(t), \mathbf{X}(t)) \neq \phi(X_i^+(t), \mathbf{X}(t)) \ \ or \ \ \phi(X_i(t), \mathbf{X}(t)) - \phi(X_i^+(t), \mathbf{X}(t)) \neq 0. \tag{4}$$

Similarly, we say that component i is p-relevant while component i moves back to its previous state at time t if

$$\phi(X_i^-(t), \mathbf{X}(t)) \neq \phi(X_i(t), \mathbf{X}(t)) \ \ or \ \ \phi(X_i^-(t), \mathbf{X}(t)) - \phi(X_i(t), \mathbf{X}(t)) \neq 0. \tag{5}$$

We define $\phi_i(t) = \phi(X_i(t), X(t)), \phi_i^+(t) = \phi(X_i^+(t), X(t))$ and $\phi_i^-(t) = \phi(X_i^-(t), X(t))$.

Then component i is n-relevant if, $nREL(i) = \phi_i^+(t) - \phi_i(t)$ is not equal to zero. Hence, component i is n-relevant at time t if it would result in a system state change, while changing the component i to its next state.

Then we denote component i is p-relevant if say, $pREL(i) = \phi_i(t) - \phi_i^-(t)$ is not equal to zero. Hence, component i is p-relevant at time t if it would result in a system state change, while changing the component i back to its previous state.

For an easier representation, define the following functions,

$$\phi_{ij}^{**}(t) = \phi(X_i(t), X_j(t), X(t)), \phi_{ij}^{+*}(t) = \phi(X_i^+(t), X_j(t), X(t)),$$
$$\phi_{ij}^{*+}(t) = \phi(X_i(t), X_j^+(t), X(t)), \phi_{ij}^{++}(t) = \phi(X_i^+(t), X_j^+(t), X(t)),$$
$$\phi_{ij}^{+-}(t) = \phi(X_i^+(t), X_j^-(t), X(t)), \phi_{ij}^{*-}(t) = \phi(X_i(t), X_j^-(t), X(t))$$
$$\phi_{ij}^{-+}(t) = \phi(X_i^-(t), X_j^+(t), X(t)), \phi_{ij}^{-*}(t) = \phi(X_i^-(t), X_j(t), X(t))$$
$$\phi_{ij}^{--}(t) = \phi(X_i^-(t), X_j^-(t), X(t)), \phi_{ijk}^{***}(t) = \phi(X_i(t), X_j(t), X_k(t), X(t))$$
$$\phi_{ijk}^{+**}(t) = \phi(X_i^+(t), X_j(t), X_k(t), X(t)), \phi_{ijk}^{*+*}(t) = \phi(X_i(t), X_j^+(t), X_k(t), X(t))$$
$$\phi_{ijk}^{++*}(t) = \phi(X_i^+(t), X_j^+(t), X_k(t), X(t)), \phi_{ijk}^{**+}(t) = \phi(X_i(t), X_j(t), X_k^+(t), X(t))$$
$$\phi_{ijk}^{+*+}(t) = \phi(X_i^+(t), X_j(t), X_k^+(t), X(t)), \phi_{ijk}^{*++}(t) = \phi(X_i(t), X_j^+(t), X_k^+(t), X(t))$$
$$\phi_{ijk}^{+++}(t) = \phi(X_i^+(t), X_j^+(t), X_k^+(t), X(t)), \phi_{ijk}^{-**}(t) = \phi(X_i^-(t), X_j(t), X_k(t), X(t))$$
$$\phi_{ijk}^{*-*}(t) = \phi(X_i(t), X_j^-(t), X_k(t), X(t)), \phi_{ijk}^{--*}(t) = \phi(X_i^-(t), X_j^-(t), X_k(t), X(t))$$
$$\phi_{ijk}^{**-}(t) = \phi(X_i(t), X_j(t), X_k^-(t), X(t)), \phi_{ijk}^{-*-}(t) = \phi(X_i^-(t), X_j(t), X_k^-(t), X(t))$$
$$\phi_{ijk}^{*--}(t) = \phi(X_i(t), X_j^-(t), X_k^-(t), X(t)), \phi_{ijk}^{---}(t) = \phi(X_i^-(t), X_j^-(t), X_k^-(t), X(t))$$

Suppose, at time t, simultaneously i^{th} component is changing to its next state and j^{th} component is also changing to its next state. Then their states are jointly nn-relevant, if

$$nnREL(i,j) = \phi_{ij}^{**}(t) - \phi_{ij}^{+*}(t) - \phi_{ij}^{*+}(t) + \phi_{ij}^{++}(t) \neq 0 \tag{6}$$

Suppose i^{th} component is changing to its next state and j^{th} component is changing back to its previous state. Then components i and j are jointly np-relevant, if

$$npREL(i,j) = \phi_{ij}^{*-}(t) - \phi_{ij}^{+-}(t) - \phi_{ij}^{**}(t) + \phi_{ij}^{+*}(t) \neq 0 \tag{7}$$

Suppose i^{th} component is changing back to its previous state and j^{th} component is changing to its next state. Then components i and j are jointly pn-relevant, if

$$pnREL(i,j) = \phi_{ij}^{-*}(t) - \phi_{ij}^{**}(t) - \phi_{ij}^{-+}(t) + \phi_{ij}^{*+}(t) \neq 0 \tag{8}$$

Suppose i^{th} component is changing back to its previous state and j^{th} component is also changing back to its previous state. Then components i and j are jointly pp-relevant, if

$$ppREL(i,j) = \phi_{ij}^{--}(t) - \phi_{ij}^{*+}(t) - \phi_{ij}^{-*}(t) + \phi_{ij}^{**}(t) \neq 0 \tag{9}$$

Now, to measure the effect of joint movement of three components, in either direction, let us consider the following statements.

Suppose, at time t, i^{th} component is changing to its next state, j^{th} component is also changing to its next state and k^{th} component is also changing to its next state. Then components i, j and k are jointly *nnn*-relevant, if

$$nnnREL(i,j,k) = \phi_{ijk}^{***}(t) - \phi_{ijk}^{+**}(t) - \phi_{ijk}^{*+*}(t) + \phi_{ijk}^{++*}(t) - \phi_{ijk}^{**+}(t) + \phi_{ijk}^{+*+}(t) + \phi_{ijk}^{*++}(t) - \phi_{ijk}^{+++}(t) \neq 0 \quad (10)$$

Suppose at time t, i^{th} component is changing to its next state, j^{th} component is changing back to its previous state and k^{th} component is changing to its next state. Then components i, j and k are jointly *npn*-relevant, if

$$npnREL(i,j,k) = \phi_{ijk}^{*-*}(t) - \phi_{ijk}^{+-*}(t) - \phi_{ijk}^{***}(t) + \phi_{ijk}^{+**}(t) - \phi_{ijk}^{*-+}(t) + \phi_{ijk}^{+-+}(t) + \phi_{ijk}^{**+}(t) - \phi_{ijk}^{+*+}(t) \neq 0 \quad (11)$$

Suppose i^{th} component is changing back to its previous state, j^{th} component is changing to its next state and k^{th} component is changing to its next state. Then components i, j and k are jointly *pnn*-relevant, if

$$pnnREL(i,j,k) = \phi_{ijk}^{-**}(t) - \phi_{ijk}^{***}(t) - \phi_{ijk}^{-+*}(t) + \phi_{ijk}^{*+*}(t) - \phi_{ijk}^{-*+}(t) + \phi_{ijk}^{**+}(t) + \phi_{ijk}^{-++}(t) - \phi_{ijk}^{*++}(t) \neq 0 \quad (12)$$

Suppose i^{th} component is changing back to its previous state, j^{th} component is also changing back to its previous state and k^{th} component is changing to its next state.Then components i, j and k are jointly *ppn*-relevant, if

$$ppnREL(i,j,k) = \phi_{ijk}^{--*}(t) - \phi_{ijk}^{*-*}(t) - \phi_{ijk}^{-**}(t) + \phi_{ijk}^{***}(t) - \phi_{ijk}^{--+}(t) + \phi_{ijk}^{*-+}(t) + \phi_{ijk}^{-*+}(t) - \phi_{ijk}^{**+}(t) \neq 0 \quad (13)$$

Suppose, at time t, i^{th} component is changing to its next state, j^{th} component is also changing to its next state and k^{th} component is changing back to its previous state. Then components i, j and k are jointly *nnp*-relevant, if

$$nnpREL(i,j,k) = \phi_{ijk}^{**-}(t) - \phi_{ijk}^{+*-}(t) - \phi_{ijk}^{*+-}(t) + \phi_{ijk}^{++-}(t) - \phi_{ijk}^{***}(t) + \phi_{ijk}^{+**}(t) + \phi_{ijk}^{*+*}(t) - \phi_{ijk}^{++*}(t) \neq 0 \quad (14)$$

Suppose, at time t, i^{th} component is changing to its next state, j^{th} component is changing back to its previous state and k^{th} component is changing back to its previous state. Then components i, j and k are jointly *npp*-relevant, if

$$nppREL(i,j,k) = \phi_{ijk}^{*--}(t) - \phi_{ijk}^{+--}(t) - \phi_{ijk}^{**-}(t) + \phi_{ijk}^{+*-}(t) - \phi_{ijk}^{*-*}(t) + \phi_{ijk}^{+-*}(t) + \phi_{ijk}^{***}(t) - \phi_{ijk}^{+**}(t) \neq 0 \quad (15)$$

Suppose i^{th} component is changing back to its previous state, j^{th} component is changing to its next state and k^{th} component is changing back to its previous state. Then components i, j and k are jointly *pnp*-relevant, if

$$pnpREL(i,j,k) = \phi_{ijk}^{-*-}(t) - \phi_{ijk}^{**-}(t) - \phi_{ijk}^{-+-}(t) + \phi_{ijk}^{*+-}(t) - \phi_{ijk}^{-**}(t) + \phi_{ijk}^{***}(t) + \phi_{ijk}^{-+*}(t) - \phi_{ijk}^{*+*}(t) \neq 0 \quad (16)$$

Suppose i^{th} component is changing back to its previous state, j^{th} component is changing back to its previous state and k^{th} component is also changing back to its previous state. Then components i, j and k are jointly *ppp*-relevant, if

$$pppREL(i,j,k) = \phi_{ijk}^{---}(t) - \phi_{ijk}^{*--}(t) - \phi_{ijk}^{-*-}(t) + \phi_{ijk}^{**-}(t) - \phi_{ijk}^{--*}(t) + \phi_{ijk}^{*-*}(t) + \phi_{ijk}^{-**}(t) - \phi_{ijk}^{***}(t) \neq 0 \quad (17)$$

To find the joint importance of three repairable components, in Birnbaum sense, the following measures are proposed, by considering three components, i, j and k.

$$I_{NNNB}^{ijk}(t) = P\{nnnREL(i,j,k) \neq 0\} = \sum_{w=0}^{M_k}\sum_{v=0}^{M_j}\sum_{u=0}^{M_i} P[(\phi(X_i(t)=u, X_j(t)=v, X_k(t)=w, \mathbf{X}(t)) - \phi(X_i(t)=u-1, X_j(t)=v, X_k(t)=w, \mathbf{X}(t))) - (\phi(X_i(t)=u, X_j(t)=v-1, X_k(t)=w, \mathbf{X}(t)) - \phi(X_i(t)=u-1, X_j(t)=v-1, X_k(t)=w, \mathbf{X}(t)))] - [(\phi(X_i(t)=u, X_j(t)=v, X_k(t)=w-1, \mathbf{X}(t)) - \phi(X_i(t)=u-1, X_j(t)=v, X_k(t)=w-1, \mathbf{X}(t))) - (\phi(X_i(t)=u, X_j(t)=v-1, X_k(t)=w-1, \mathbf{X}(t)) - \phi(X_i(t)=u-1, X_j(t)=v-1, X_k(t)=w-1, \mathbf{X}(t))) \neq 0] \quad (18)$$

$$I_{\{NPNB\}}^{ijk}(t) = P\{npnREL(i,j,k) \neq 0\} = \sum_{w=0}^{M_k}\sum_{v=0}^{M_j}\sum_{u=0}^{M_i} P[(\phi(X_i(t)=u, X_j(t)=v+1, X_k(t)=w, \mathbf{X}(t)) - \phi(X_i(t)=u-1, X_j(t)=v+1, X_k(t)=w, \mathbf{X}(t))) - (\phi(X_i(t)=u, X_j(t)=v, X_k(t)=w, \mathbf{X}(t)) - \phi(X_i(t)=u-1, X_j(t)=v, X_k(t)=w, \mathbf{X}(t)))] - [(\phi(X_i(t)=u, X_j(t)=v+1, X_k(t)=w-1, \mathbf{X}(t)) - \phi(X_i(t)=u-1, X_j(t)=v+1, X_k(t)=w-1, \mathbf{X}(t))) - (\phi(X_i(t)=u, X_j(t)=v, X_k(t)=w-1, \mathbf{X}(t)) - \phi(X_i(t)=u-1, X_j(t)=v, X_k(t)=w-1, \mathbf{X}(t))) \neq 0] \quad (19)$$

$$I_{\{PNNB\}}^{ijk}(t) = P\{pnnREL(i,j,k) \neq 0\} = \sum_{w=0}^{M_k}\sum_{v=0}^{M_j}\sum_{u=0}^{M_i} P[(\phi(X_i(t)=u+1, X_j(t)=v, X_k(t)=w, \mathbf{X}(t)) - \phi(X_i(t)=u, X_j(t)=v, X_k(t)=w, \mathbf{X}(t))) - (\phi(X_i(t)=u+1, X_j(t)=v-1, X_k(t)=w, \mathbf{X}(t)) - \phi(X_i(t)=u, X_j(t)=v-1, X_k(t)=w, \mathbf{X}(t)))] - [(\phi(X_i(t)=u+1, X_j(t)=v, X_k(t)=w-1, \mathbf{X}(t)) - \phi(X_i(t)=u, X_j(t)=v, X_k(t)=w-1, \mathbf{X}(t))) - (\phi(X_i(t)=u+1, X_j(t)=v-1, X_k(t)=w-1, \mathbf{X}(t)) - \phi(X_i(t)=u, X_j(t)=v-1, X_k(t)=w-1, \mathbf{X}(t))) \neq 0] \quad (20)$$

$$I_{\{PPNB\}}^{ijk}(t) = P\{ppnREL(i,j,k) \neq 0\} = \sum_{w=0}^{M_k}\sum_{v=0}^{M_j}\sum_{u=0}^{M_i} P[(\phi(X_i(t)=u+1, X_j(t)=v+1, X_k(t)=w, \mathbf{X}(t)) - \phi(X_i(t)=u, X_j(t)=v+1, X_k(t)=w, \mathbf{X}(t))) - (\phi(X_i(t)=u+1, X_j(t)=v, X_k(t)=w, \mathbf{X}(t)) - \phi(X_i(t)=u, X_j(t)=v, X_k(t)=w, \mathbf{X}(t)))] - [(\phi(X_i(t)=u+1, X_j(t)=v+1, X_k(t)=w-1, \mathbf{X}(t)) - \phi(X_i(t)=u, X_j(t)=v+1, X_k(t)=w-1, \mathbf{X}(t))) - (\phi(X_i(t)=u+1, X_j(t)=v, X_k(t)=w-1, \mathbf{X}(t)) - \phi(X_i(t)=u, X_j(t)=v, X_k(t)=w-1, \mathbf{X}(t))) \neq 0] \quad (21)$$

$$I^{ijk}_{NNPB}(t) = P\{nnpREL(i,j,k) \neq 0\} = \sum_{w=0}^{M_k}\sum_{v=0}^{M_j}\sum_{u=0}^{M_i} P[(\phi(X_i(t)=u, X_j(t)=v, X_k(t)=w+1, \mathbf{X}(t)) - \phi(X_i(t)=u-1, X_j(t)=v, X_k(t)=w+1, \mathbf{X}(t))) - (\phi(X_i(t)=u, X_j(t) = v-1, X_k(t)=w+1, \mathbf{X}(t)) - \phi(X_i(t)=u-1, X_j(t)=v-1, X_k(t)=w+1, \mathbf{X}(t)))] - [(\phi(X_i(t)=u, X_j(t)=v, X_k(t)=w, \mathbf{X}(t)) - \phi(X_i(t)=u-1, X_j(t)= v, X_k(t)=w, \mathbf{X}(t))) - (\phi(X_i(t)=u, X_j(t)=v-1, X_k(t)=w, \ \mathbf{X}(t)) - \phi(X_i(t) = u-1, X_j(t)=v-1, X_k(t)=w, \mathbf{X}(t))) \neq 0] \tag{22}$$

$$I^{ijk}_{NPPB}(t) = P\{nppREL(i,j,k) \neq 0\} = \sum_{w=0}^{M_k}\sum_{v=0}^{M_j}\sum_{u=0}^{M_i} P[\ (\phi(X_i(t)=u, X_j(t)=v+1, X_k(t) \ = w+1, \mathbf{X}(t)) - \phi(X_i(t)=u-1, X_j(t)=v+1, X_k(t)=w+1, \mathbf{X}(t))) - (\phi(X_i(t) = u, X_j(t)=v, X_k(t)=w+1, \mathbf{X}(t)) - \phi(X_i(t)=u-1, X_j(t)=v, X_k(t)=w+ 1, \ \mathbf{X}(t)))] - [(\phi(X_i(t)=u, X_j(t)=v+1, X_k(t)=w, \ \mathbf{X}(t)) - \phi(X_i(t)=u-1, X_j(t)=v+1, X_k(t)=w, \mathbf{X}(t))) - (\phi(X_i(t)=u, X_j(t)=v, X_k(t)=w, \mathbf{X}(t)) - \phi(X_i(t)=u-1, X_j(t)=v, X_k(t)=w, \mathbf{X}(t))) \neq 0] \tag{23}$$

$$I^{ijk}_{PNPB}(t) = P\{pnpREL(i,j,k) \neq 0\} = \sum_{w=0}^{M_k}\sum_{v=0}^{M_j}\sum_{u=0}^{M_i} P[(\phi(X_i(t)=u+1, X_j(t)=v, X_k(t)=w +1, \mathbf{X}(t)) - \phi(X_i(t)=u, X_j(t)=v, X_k(t)=w+1, \mathbf{X}(t))) - (\phi(X_i(t)=u+1, X_j(t)=v-1, X_k(t)=w+1, \mathbf{X}(t)) - \phi(X_i(t)=u, X_j(t)=v-1, X_k(t)=w+ 1, \mathbf{X}(t)))] - [(\phi(X_i(t)=u+1, X_j(t)=v, X_k(t)=w, \mathbf{X}(t)) - \phi(X_i(t)=u, X_j(t) = v, X_k(t)=w, \ \mathbf{X}(t))) - (\phi(X_i(t)=u+1, X_j(t)=v-1, X_k(t)=w, \ \mathbf{X}(t)) - \phi(X_i(t)=u, X_j(t)=v-1, X_k(t)=w, \mathbf{X}(t))) \neq 0] \tag{24}$$

$$I^{ijk}_{PPPB}(t) = P\{pppREL(i,j,k) \neq 0\} = \sum_{w=0}^{M_k}\sum_{v=0}^{M_j}\sum_{u=0}^{M_i} P[(\phi(X_i(t)=u+1, X_j(t)=v+1, X_k(t) = w+1, \mathbf{X}(t)) - \phi(X_i(t)=u, X_j(t)=v+1, X_k(t)=w+1, \mathbf{X}(t))) - (\phi(X_i(t)= u+1, X_j(t)=v, X_k(t)=w+1, \mathbf{X}(t)) - \phi(X_i(t)=u, X_j(t)=v, X_k(t)=w+1, \mathbf{X}(t)))] - [(\phi(X_i(t)=u+1, X_j(t)=v+1, X_k(t)=w, \mathbf{X}(t)) - \phi(X_i(t)=u, X_j(t) = v+1, X_k(t)=w, \mathbf{X}(t))) - (\phi(X_i(t)=u+1, X_j(t)=v, X_k(t)=w, \mathbf{X}(t)) - \phi(X_i(t)=u, X_j(t)=v, X_k(t)=w, \mathbf{X}(t))) \neq 0] \tag{25}$$

Clearly, $I^{ijk}_{NNNB}(t)$ is the joint importance measure of three components i, j and k at time t when the three components i, j and k enters its next state, $I^{ijk}_{NPNB}(t)$ is the joint importance measure of three components i, j and k at time t when the components i and k enters its next state and the component j enters its previous state, $I^{ijk}_{PNNB}(t)$ is the joint importance measure of three components i, j and k at time t when the component i enters its previous state and the components j and k enters its next state and $I^{ijk}_{PPNB}(t)$ is the joint importance measure of three components i, j and k at time t when both components i and j enters its previous state and the component k enters the next state, $I^{ijk}_{NNPB}(t)$ is the joint importance measure of three components i, j and k at time t when the two components i and j enters its next state and the component k enters the previous state, $I_{NPPB}^{ijk}(t)$ is the joint importance measure of three components i, j and k at time t when the component i enters its next state and components j and k enters its previous state, $I_{PNPB}^{ijk}(t)$ is the joint importance measure of three components i, j and k at time t when the components i and k enters its previous state and the component j enters its next state and $I_{PPPB}^{ijk}(t)$ is the joint importance measure of three components i, j and k at time t when the three components i, j and k enters its previous state.

3. Application

In multistate system reliability engineering, the problem of identification of most important component of group of component is required for giving proper repair or maintenance activities to provide the system active for the completion of assigned mission. Most of the existing measures are useful for this purpose if repair or maintenance is not considered. In the proposed measures, the major advantage is that, one can measure importance and joint importance measures when repair or maintenance is applied to the components. Adoption of proper maintenance activity is unavoidable in system engineering. The proposed results are useful to the multistate and binary state systems.

4. Illustration

To illustrate the joint importance of components, we consider a network flow system which is given in Figure 1. In this example, there is a directed network flow system consisting of 6 components represented by edges of the network.

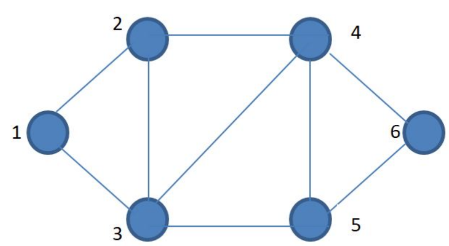

Figure 1: Network flow system

The state functions of the components f_1, f_2, f_3, f_4, f_5 and f_6 represent the flow capacity functions of the components given by

$$f_1(u) = f_6(u) = 2.5\ u,\ u = 0, 1, 2,$$

$$f_2(u) = 1.5\ u,\ u = 0, 1, 2,$$

$$f_3(u) = f_5(u) = 5.0\ u,\ u = 0, 1,$$

$$f_4(u) = 1.0\ I(u = 1) + 2.5\ I(u = 2),\ u = 0, 1, 2,$$

where I is the indicator function. The physical state of the system is the amount of flow that can be sent through the network from the source node 1 to the terminal node 6. In order to express the system state as a function of the component states, we identify the minimal cut sets in the network. These are $K_1 = \{1\}, K_2 = \{2,3\}, K_3 = \{3,4\}, K_4 = \{4,5\}, K_5 = \{6\}$. According to the well-known max-flow-min cut theorem, we then have

$$\phi(X(t)) = \min_{1\leq j\leq 5} \sum_{i\in K_j} f_i(X_i(t))$$

The probabilities of each component in its states are given by

$$p_1(u) = p_2(u) = p_4(u) = p_6(u) = \begin{cases} \frac{25}{105} & ,u=0 \\ \frac{35}{105} & ,u=1 \\ \frac{45}{105} & ,u=2 \end{cases}$$

$$p_3(u) = p_5(u) = \begin{cases} \frac{45}{100} & ,u=0 \\ \frac{55}{100} & ,u=1 \end{cases}$$

Here we have computed the physical joint importance $I^{ijk}(t)$ for all the possible combinations of three components. The results are given in Table 1.

It is clear from the example that the component group (3,4,5) is the most important set in any case considered. But the ranking of the rest of the three sets of components keeps changing. The proposed measures give the investigator the ability to look at relevancy from several angles, which is useful in a diagnostic environment as well as when the investigation is done to support decisions for system improvement.

5. Conclusions

In the present paper, a repairable multistate system is considered. The single component Birnbaum importance measure is generalized to three component joint importance measure for multistate systems in eight different ways. The measures gives an insight regarding change in system performance to support decisions regarding improvement of the system, through the movement of components in same/opposite directions. Since the proposed measures are investigating the behavior of components on system performance, they are useful in a diagnostic checking. These joint importance measures are highly appropriate while considering repairable components. In order to locate the weakest group or more consistent group, the proposed measures will be helpful. So more repair activities can be ensured weakest group.

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Table 1: Joint importance of three components i, j and k

i,j,k	$I^{(ijk)}_{NNNB}$	$I^{(ijk)}_{NNPB}$	$I^{(ijk)}_{NPNB}$	$I^{(ijk)}_{NPPB}$	$I^{(ijk)}_{PNNB}$	$I^{(ijk)}_{PNPB}$	$I^{(ijk)}_{PPNB}$	$I^{(ijk)}_{PPPB}$
1,2,3	0.1895	0.1895	0.2211	0.2211	0.2211	0.2211	0.2579	0.2579
1,2,4	0.1119	0.1119	0.1306	0.1306	0.1306	0.1306	0.1523	0.1523
1,2,5	0	0	0	0	0	0	0	0
1,2,6	0.0639	0.0746	0.0746	0.0870	0.0746	0.0870	0.0870	0.1015
1,3,4	0.3286	0.3482	0.3286	0.3482	0.3833	0.3999	0.3833	0.3999
1,3,5	0.2487	0.2487	0.2487	0.2487	0.2902	0.2902	0.2902	0.2902
1,3,6	0.2542	0.2965	0.2542	0.2965	0.2965	0.3460	0.2965	0.3460
1,4,5	0.2394	0.3183	0.2394	0.2394	0.2793	0.2793	0.2793	0.2793
1,4,6	0.1927	0.2248	0.1927	0.2248	0.2248	0.2623	0.2248	0.2623
1,5,6	0.1026	0.3912	0.1026	0.1197	0.1197	0.1396	0.0490	0.1396
2,3,4	0.3317	0.3317	0.3317	0.3317	0.3869	0.3869	0.3869	0.3869
2,3,5	0	0	0	0	0	0	0	0
2,3,6	0.1895	0.2211	0.1895	0.2211	0.2211	0.2579	0.2211	0.2579
2,4,5	0	0	0	0	0	0	0	0
2,4,6	0.1119	0.1306	0.1119	0.1306	0.1306	0.1523	0.1306	0.1523
2,5,6	0	0	0	0	0	0	0	0
3,4,5	0.5804	0.5804	0.5804	0.5804	0.5804	0.5804	0.5804	0.5804
3,4,6	0.3286	0.3833	0.3428	0.3999	0.3286	0.3833	0.3428	0.3999
3,5,6	0.2487	0.2902	0.2487	0.2902	0.2487	0.2902	0.2487	0.2902
4,5,6	0.2394	0.2793	0.2394	0.2793	0.2394	0.2793	0.2394	0.2793

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