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Shrinkage Testimators of Exponential Scale Parameter Using General Entropy Loss Function

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Abstract

This paper proposes some shrinkage testimator(s) for the scale parameter of components subjected to life testing experiment when the failure rate remains constant. Different choices of shrinkage factor(s) have been considered by making it dependent on test statistic. Properties of proposed testimator(s) have been studied under General Entropy Loss Function (GELF). Relative Risk analysis has been carried out and it has been observed that the proposed testimator(s) perform better than the available best estimator for different degrees of asymmetry and level of significance. In particular it is recommended to use the square of the shrinkage factor for the best performance of these testimators. Further it is observed that a lower value of level of significance, α = 1% provides better control over the relative risk.

Key words: Exponential model; Scale parameter; Shrinkage testimator; General entropy loss function; Level of significance; Euler's psi function; Relative risks.

1 Introduction

1.1 The Model

The aim of systems reliability is to forecast various system performance measures such as mean life time, guarantee period and reliability etc. In general, the type of failure distribution depends on the failure mechanism of components. If the failure rate is constant, which is mostly true for electronic components during the major part of their useful life, the failure time follows an exponential distribution with the probability density function

$$f(x;\theta) = \frac{1}{\theta}\, e^{-x/\theta} \quad ; \quad x,\theta > 0$$

$$0, \text{ otherwise} \tag{1.1.1}$$

In the context of life testing and reliability estimation, numerous data have been examined and it has been found that exponential distribution fits well for most of the cases. Several authors have proposed estimators, testimators for the mean life time θ with different choices of shrinkage

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factors (S.F.) under different loss functions. The choice of an appropriate loss function is guided by financial consideration apart from other considerations such as over estimation being more serious than under-estimation or vice-versa. Many types of loss functions are available in Bayesian literature with Canfield (1970) an idea of asymmetric loss function came into existence and now broadly the loss functions may be classified as ‘Symmetric’ and ‘Asymmetric’. Zellner (1986) proposed a very useful asymmetric loss function which was later modified by Basu and Ebrahimi (1991).Later Calabria and Pulcini (1996) proposed the General Entropy Loss Function (GELF).

1.2 Incorporating a Point Guess and $\boldsymbol{\widehat{\theta}_{ST}}$

In many real life situations the experimenter may have some prior information regarding the parameter being estimated due to some past experience or similar kind of studies and it is thought to apply this information to the inference procedures of the original model. If the prior information is available only in the form of a point (a single) value (say) θ_0 for θ. For example a Reliability engineer knows that in how many days the item under test may fail (say) 100 days due to his past experience of performance. Here we may take $\theta_0 = 100$ days. For such situations it is suggested to start with the current (sample) information, construct an estimator $\widehat{\theta}$ (MVUE or UMVUE) and modify it by incorporating the guess θ_0 (sometimes called natural origin) so that the resulting estimator or testimator though perhaps biased, has smaller risk than that of $\widehat{\theta}$ in some interval around θ_0. This method constructing an estimator of θ that utilizes the prior information θ_0 leads to what is known is shrinkage estimator or shrinkage testimator.

1.3 Asymmetric Loss Functions

We know in many real life situations the overestimation or underestimation are not of equal consequences. Several authors such as Canfield (1970), Zellner (1986), Basu and Ebrahimi (1991), Srivastava (1996), Srivastava and Tanna (2001, 2007), Srivastava and Shah (2010) and others have shown that the estimators or testimators of the parameters of interest under the asymmetric loss function demonstrate their superiority over the estimators obtained under squared error loss function (SELF).

The loss function $L\left(\hat{\theta},\theta\right)$ provides a measure of the financial consequences arising from a wrong estimate of the unknown quantity θ. Basu and Ebrahimi (1991) defined a modified LINEX loss function useful for the scale parameter of a distribution but sometimes it did not give the Bayes estimators of parameter of interest in the closed forms.

A suitable alternative to modified LINEX loss is the General Entropy Loss Function (GELF) proposed by Calabria and Pulcini (1996) and is given by:

$$L_E\left(\hat{\theta},\theta\right) \propto \left\{\left(\frac{\hat{\theta}}{\theta}\right)^p - p\,ln\left(\frac{\hat{\theta}}{\theta}\right) - 1\right\} \tag{1.2.1}$$

whose minimum occurs at $\hat{\theta} = \theta$.

This loss function is a generalization of the entropy loss used by several authors (for example, Dey and Liu, 1992), where the shape parameter 'p' is equal to unity (1). The more general version of (1.2.1) allows different shapes of the loss function to be considered when $p > 0$, a positive error ($\hat{\theta} > \theta$) causes more serious error than a negative error) and $p < 0$ for the reverse situations. If we are considering prior distributions, then the Bayes estimate of θ under GELF is in a closed form and is given by

$$\hat{\theta}^E = [E_\theta(\theta^{-p})]^{-1/p} \tag{1.2.2}$$

provided that $E_\theta(\theta^{-p})$ exists and is finite. When $p = -1$, the Bayes estimate (1.2.2) coincides with the Bayes estimate under the squared error loss function.

1.4 Background

Srivastava and Shah (2010) have studied the properties of shrinkage testimators for shrinkage factor(s) dependent on test statistics under the asymmetric loss function and have reported that the proposed testimator(s) perform better than the usual best estimator. In this paper an attempt has been made to demonstrate that how **shrinkage testimation procedure works under GELF.**

We have proposed the shrinkage testimators in section 2. The risks of the proposed testimators have been derived in section 3. The section 4 deals with risk comparisons for different shrinkage testmators and the paper concludes with section 5.

2 Shrinkage Testimator(s)

Using the sample and prior guess information a shrinkage testimator for the scale parameter can be proposed as follows

$$\hat{\theta}_{ST1} = \begin{cases} k\bar{x} + (1-k)\theta_0 & ; \; if \;\; x_1^2 \le \frac{2n\bar{x}}{\theta_0} \le x_2^2 \\ \bar{x} & ; \quad otherwise \end{cases} \tag{2.1}$$

where k being dependent on test statistic is given by $k = \frac{2n\bar{x}}{\theta_0\, x^2}$ and $x^2 = (x_2^2 - x_1^2)$.

This choice of shrinkage factor ensures that the shrinkage factor lies exactly between zero and one. Now, taking the 'square' of k (*i.e.* k$=$ k^2), another testimator is defined as

$$\hat{\theta}_{ST2} = \begin{cases} \left(\frac{2n\bar{x}}{\theta_0\, x^2}\right)^2 \bar{x} + \left[1 - \left(\frac{2n\bar{x}}{\theta_0\, x^2}\right)^2\right]\theta_0 & ; \; if \;\; H_0 \; is \; accepted \\ \bar{x} & ; \quad otherwise \end{cases} \tag{2.2}$$

Now, we derive the risk(s) of these testimator(s) in the next section.

3 Risk(s) of the Testimator(s)

The risk of $\hat{\theta}_{ST_1}$ under $L_E\left(\hat{\theta},\theta\right)$ defined by

$$R(\hat{\theta}_{ST_1}) = E[\,\hat{\theta}_{ST_1} \,|\, L_E\left(\hat{\theta},\theta\right)]$$

$$= E\left[k\,\bar{x} + (1-k)\theta_0 \Big/ \chi_1^2 < \frac{2n\bar{x}}{\theta_0} < \chi_2^2\right] \cdot p\left[\chi_1^2 < \frac{2n\bar{x}}{\theta_0} < \chi_2^2\right]$$
$$+ E\left[\bar{x} \,\middle|\, \frac{2n\bar{x}}{\theta_0} < \chi_1^2 \,\cup\, \frac{2n\bar{x}}{\theta_0} > \chi_2^2\right] \cdot p\left[\frac{2n\bar{x}}{\theta_0} < \chi_1^2 \cup \frac{2n\bar{x}}{\theta_0} > \chi_2^2\right] \tag{3.1}$$

$$= \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} \left[\frac{\frac{2n\bar{x}}{\theta_0\ x^2}\left(\bar{x}-\theta_0\right)+\theta_0}{\theta}\right]^p f(\bar{x})\ d\bar{x}$$
$$- \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} p\ln\left[\frac{\frac{2n\bar{x}}{\theta_0\ x^2}\left(\bar{x}-\theta_0\right)+\theta_0}{\theta}\right] f(\bar{x})\ d\bar{x}$$
$$- \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} f(\bar{x})\ d\bar{x} + \int_0^{\frac{\chi_1^2\theta_0}{2n}} \left[\left(\frac{\bar{x}}{\theta}\right)^p - p\ln\left(\frac{\bar{x}}{\theta}\right) - 1\right] f(\bar{x})\,d\bar{x}$$
$$+ \int_{\frac{\chi_2^2\theta_0}{2n}}^{\infty} \left[\left(\frac{\bar{x}}{\theta}\right)^p - p\ln\left(\frac{\bar{x}}{\theta}\right) - 1\right] f(\bar{x})\,d\bar{x} \tag{3.2}$$

Where $f(\bar{x}) = \frac{1}{\Gamma n}\left(\frac{n}{\theta}\right)^n (\bar{x})^{n-1} e^{\frac{-n\bar{x}}{\theta}}$

A straight forward integration of (3.2) gives

$$
\begin{aligned}
R(\hat{\theta}_{ST_1}) \;=\; & I_1 - I_2 - \left\{ I\left(\frac{x_2^2\,\phi}{2}, n\right) - I\left(\frac{x_1^2\,\phi}{2}, n\right)\right\} + \\
& \left(\frac{1}{n}\right)^p \frac{\Gamma(p+n)}{\Gamma n}\left\{ I\left(\frac{x_1^2\,\phi}{2}, n+p\right) - I\left(\frac{x_2^2\,\phi}{2}, n+p\right) + 1\right\} - \\
& \left\{ I\left(\frac{x_1^2\,\phi}{2}, n\right) - I\left(\frac{x_2^2\,\phi}{2}, n\right) + 1\right\} - I_3 - I_4
\end{aligned}
\tag{3.3}
$$

where

$$
I_1 = \int_{\frac{x_1^2\phi}{2}}^{\frac{x_2^2\phi}{2}} \left[\frac{2t^2}{n\phi x^2} - \frac{2t\phi}{\phi x^2} + \phi\right]^p \frac{1}{\Gamma n}\, e^{-t}\, t^{n-1}\, dt
$$

$$
I_2 = \int_{\frac{x_1^2\phi}{2}}^{\frac{x_2^2\phi}{2}} p \ln\left[\frac{2t^2}{n\phi x^2} - \frac{2t\phi}{\phi x^2} + \phi\right] \frac{1}{\Gamma n}\, e^{-t}\, t^{n-1}\, dt
$$

$$
I_3 = \int_{0}^{\frac{x_1^2\phi}{2}} p \ln\left(\frac{t}{n}\right) \frac{1}{\Gamma n}\, e^{-t}\, t^{n-1}\, dt
$$

$$
I_4 = \int_{\frac{x_2^2\phi}{2}}^{\infty} p \ln\left(\frac{t}{n}\right) \frac{1}{\Gamma n}\, e^{-t}\, t^{n-1}\, dt
$$

Again, we obtain the risk of $\hat{\theta}_{ST_2}$ under $L_E(\hat{\theta}, \theta)$ as

$$
R(\hat{\theta}_{ST_2}) = E\,[\,\hat{\theta}_{ST_2} \,|\, L_E(\hat{\theta}, \theta)]
$$

$$= E\left[\left(\frac{2n\bar{x}}{\theta_0\,\chi^2}\right)^2(\bar{x}-\theta_0)+\theta_0 \Big/ \chi_1^2 < \frac{2n\bar{x}}{\theta_0} < \chi_2^2\right]\cdot p\left[\chi_1^2 < \frac{2n\bar{x}}{\theta_0} < \chi_2^2\right]$$

$$+ E\left[\bar{x} \;\middle|\; \frac{2n\bar{x}}{\theta_0} < \chi_1^2 \;\cup\; \frac{2n\bar{x}}{\theta_0} > \chi_2^2\right]\cdot p\left[\frac{2n\bar{x}}{\theta_0} < \chi_1^2 \cup \frac{2n\bar{x}}{\theta_0} > \chi_2^2\right] \tag{3.4}$$

$$= \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} \left[\frac{\left(\frac{2n\bar{x}}{\theta_0\,x^2}\right)^2(\bar{x}-\theta_0)+\theta_0}{\theta}\right]^p f(\bar{x})\;d\bar{x}$$

$$- \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} p\ln\left[\frac{\left(\frac{2n\bar{x}}{\theta_0\,x^2}\right)^2(\bar{x}-\theta_0)+\theta_0}{\theta}\right] f(\bar{x})\;d\bar{x}$$

$$- \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} f(\bar{x})\;d\bar{x} \;+\; \int_0^{\frac{\chi_1^2\theta_0}{2n}} \left[\left(\frac{\bar{x}}{\theta}\right)^p - p\ln\left(\frac{\bar{x}}{\theta}\right)-1\right] f(\bar{x})\,d\bar{x}$$

$$+ \int_{\frac{\chi_2^2\theta_0}{2n}}^{\infty} \left[\left(\frac{\bar{x}}{\theta}\right)^p - p\ln\left(\frac{\bar{x}}{\theta}\right)-1\right] f(\bar{x})\,d\bar{x} \tag{3.5}$$

Where $f(\bar{x}) = \frac{1}{\Gamma n}\left(\frac{n}{\theta}\right)^n(\bar{x})^{n-1}e^{\frac{-n\bar{x}}{\theta}}$

A straight forward integration of (3.5) gives:

$$R(\hat{\theta}_{ST_2}) = I_1 - I_2 - \left\{ I\left(\frac{x_2^2\,\phi}{2}, n\right) - I\left(\frac{x_1^2\,\phi}{2}, n\right) \right\} +$$
$$\left(\frac{1}{n}\right)^p \frac{\Gamma(p+n)}{\Gamma n} \left\{ I\left(\frac{x_1^2\,\phi}{2}, n+p\right) - I\left(\frac{x_2^2\,\phi}{2}, n+p\right) + 1 \right\} -$$
$$\left\{ I\left(\frac{x_1^2\,\phi}{2}, n\right) - I\left(\frac{x_2^2\,\phi}{2}, n\right) + 1 \right\} - I_3 - I_4 \tag{3.6}$$

where

$$I_1 = \int_{\frac{x_1^2\,\phi}{2}}^{\frac{x_2^2\,\phi}{2}} \left[\frac{4\,t^3}{n\,\phi^2\,(x^2)^2} - \frac{4t^2}{\phi\,(x^2)^2} + \phi \right]^p \frac{1}{\Gamma n}\, e^{-t}\, t^{n-1}\, dt$$

$$I_2 = \int_{\frac{x_1^2\,\phi}{2}}^{\frac{x_2^2\,\phi}{2}} p \ln \left[\frac{4\,t^3}{n\,\phi^2\,(x^2)^2} - \frac{4t^2}{\phi\,(x^2)^2} + \phi \right] \frac{1}{\Gamma n}\, e^{-t}\, t^{n-1}\, dt$$

$$I_3 = \int_{0}^{\frac{x_1^2\,\phi}{2}} p \ln \left(\frac{t}{n}\right) \frac{1}{\Gamma n}\, e^{-t}\, t^{n-1}\, dt$$

$$I_4 = \int_{\frac{x_2^2\,\phi}{2}}^{\infty} p \ln \left(\frac{t}{n}\right) \frac{1}{\Gamma n}\, e^{-t}\, t^{n-1}\, dt$$

4 Relative Risk(s)

A natural way of comparing the risk of the proposed testimators, is to study its performance with respect to the best available estimator i.e. $\bar{x}$ in this case. For this purpose, we obtain the risk of $\bar{x}$ under $L_E\left(\hat{\theta}, \theta\right)$ as:

$$R_E(\bar{x}) = E[\,\bar{x} \mid L_E(\hat{\theta},\theta)]$$

$$= \int_0^\infty \left[\left(\frac{\bar{x}}{\theta}\right)^p - p \ln\left(\frac{\bar{x}}{\theta}\right) - 1 \right] f(\bar{x})\, d\bar{x} \tag{4.1}$$

A straightforward integration of (4.1) gives

$$R_E(\bar{x}) = \left[\frac{\Gamma(n+P)}{\Gamma n\,(n^p)} - \frac{p}{\Gamma n}\{\psi(n) - \ln(n)\} \right] - 1 \tag{4.2}$$

where ψ (n) is the Euler's Psi function.

Now, we define the Relative Risk of $\hat{\theta}_{ST1}$ with respect to $\bar{x}$ under $L_E(\hat{\theta},\theta)$ as follows:

$$RR_1 = \frac{R_E(\bar{x})}{R(\hat{\theta}_{ST_1})} \tag{4.3}$$

Using (4.2) and (3.3) the expression for RR_1 given in (4.3) can be obtained; it is observed that RR_1 is a function of ϕ, n, α and 'p '. To observe the behavior of $\hat{\theta}_{ST_1}$, we have taken several values of these viz α = 1%, 5%, 10%, n_1 = 5, 8, 10, 12 and p = -3, -2, -1, 2, 3, 4; 'p' is the prime important factor which decides about the seriousness of over/under estimation in the real life situation. Similarly, we define the Relative Risk of $\hat{\theta}_{ST_2}$ with respect to $\bar{x}$ under $L_E(\hat{\theta},\theta)$ as follows

$$RR_2 = \frac{R_E(\bar{x})}{R(\hat{\theta}_{ST_2})} \tag{4.4}$$

Using (4.2) and (3.6) the expression for RR_2 given in (4.4) can be obtained, again it is observed that RR_2 is a function of ϕ, n, α and 'p', to observe the behavior of $\hat{\theta}_{ST_2}$, we have taken several values of these, same as in the case of $\hat{\theta}_{ST_1}$.

4 Conclusions

- We have computed the values of Relative Risk (RR_1) for the data set mentioned above and some of the graphs have been assembled in the appendix by (i) keeping 'α' to be fixed and varying 'p' (ii) keeping 'p' to be fixed and varying 'α' as we wish to recommend for these two values. All the graphs are not shown however our recommendations are based on all the graphs.

- For n = 5, α = 1% and for different values of 'p' (positive as well as negative) $\hat{\theta}_{ST_1}$ performs better than the conventional estimator for all the values of 'p' with its best performance for p = –3 and p = 2 for the whole range of φ. Considered here *i.e*. $0.2 \le \varphi \le 1.6$.
- Next we have changed to α = 5%. Similar pattern of behavior is observed for the relative risk and p = –3 and p = 2 provide the best results. However, the magnitude of RR is small compared to α = 1% values.
- We have also considered α = 10% in order to observe the behavior for still higher level of significance just to confirm whether under different loss function the value of 'α' gets changed or not. We found that $\hat{\theta}_{ST_1}$ performs still better than the conventional estimator but the magnitude of relative risk values becomes smaller but mostly still above unity.
- So by comparing the magnitudes of these relative risk values a small value of α = 1% is recommended. Also by varying 'n' it is observed that relative risk values are higher for n = 5 compared to its other values of 8, 10 and 12. Hence a smaller 'n' is suggested. A higher RR_1 value indicates a 'better' control over the risk. So, by choosing appropriate value of 'p' and 'α' a better gain in terms of performance of $\hat{\theta}_{ST_1}$ can be achieved.
- $\hat{\theta}_{ST_2}$, is another testimator proposed by taking the 'SQUARE' of shrinkage factor. We have again prepared the relevant graphs of Relative Risk (RR_2) to observe the performance of $\hat{\theta}_{ST_2}$ with respect to the conventional estimator for the same set of values as we have considered studying the behavior of $\hat{\theta}_{ST_1}$. We observe the following:
- For almost the entire range of φ *i.e.* $0.2 \le \varphi \le 1.4$ the values of relative risks (in terms of magnitude) are higher than those for shrinkage factor taken itself indicating that square of shrinkage factor could be a better choice.
- The performance of $\hat{\theta}_{ST_1}$ is best for n = 5 and α = 1 % compared to the other values considered of these two quantities.
- Finally it can be concluded that the shrinkage testimator with square of shrinkage factor performs better for small sample sizes and small level of significance. The testimators perform better under GELF.

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Appendix

Graphs of Relative Risk for $\hat{\theta}_{ST_1}$ and $\hat{\theta}_{ST_2}$ with respect to conventional estimator

Graphs of Relative Risk for $\hat{\theta}_{ST_1}$

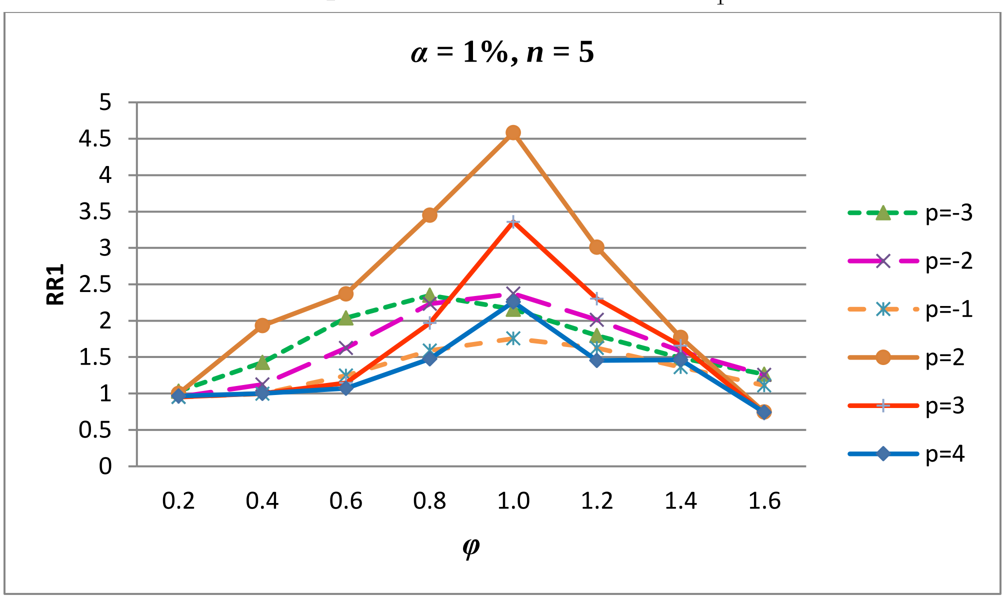

Figure 1

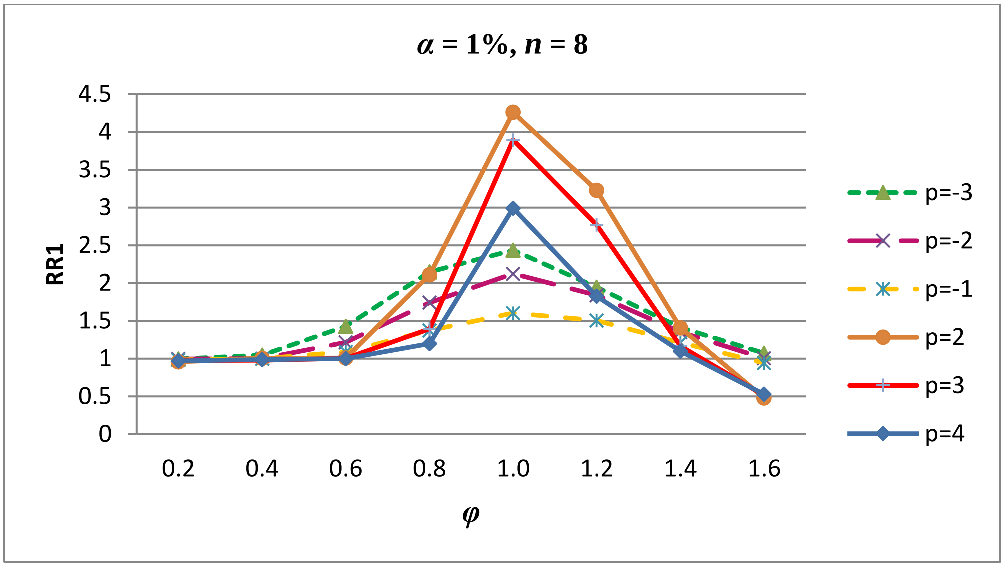

Figure 2

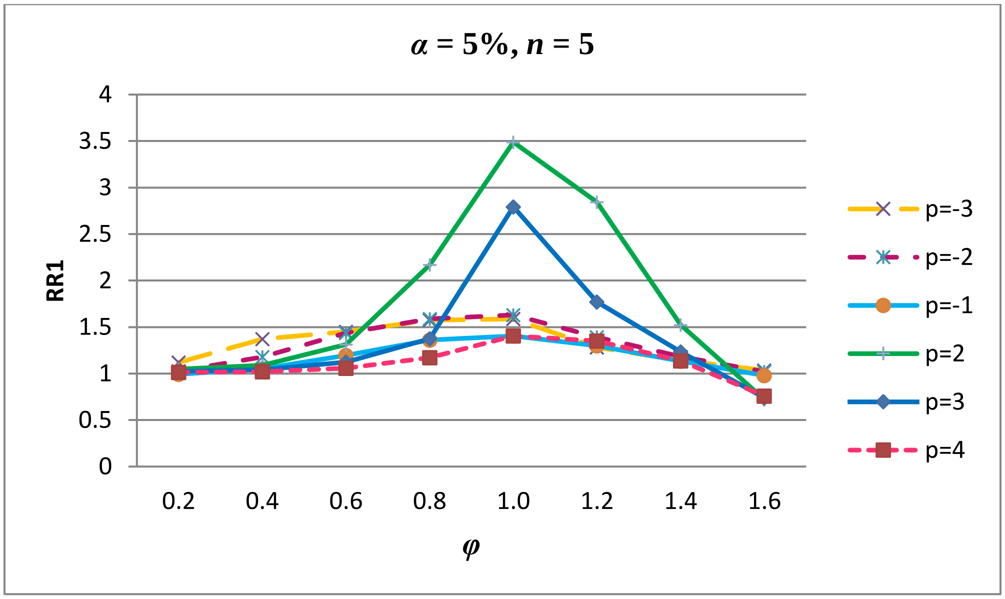

Figure 3

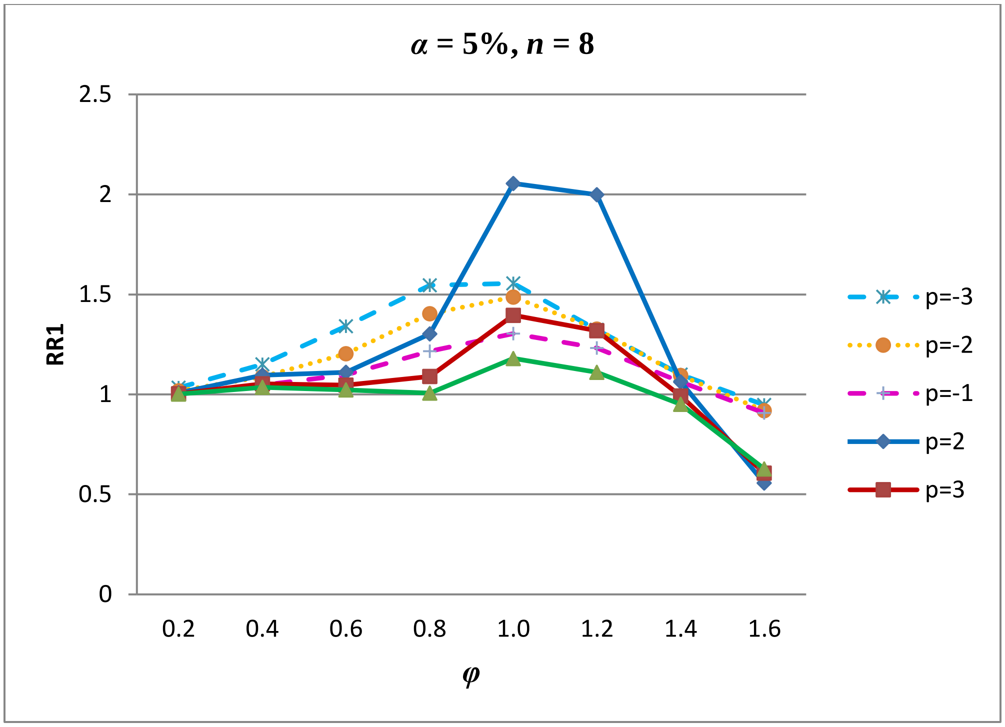

Figure 4

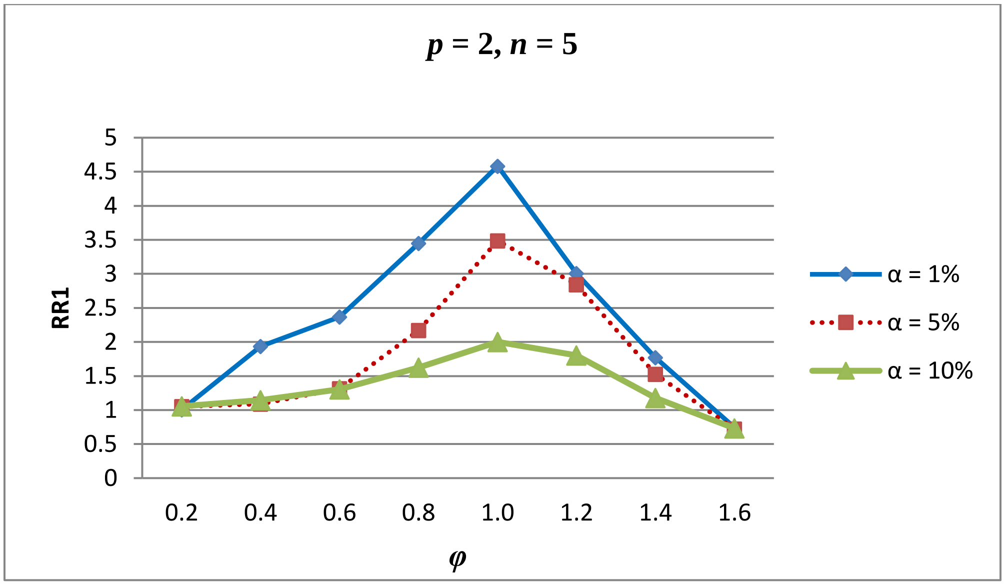

Figure 5

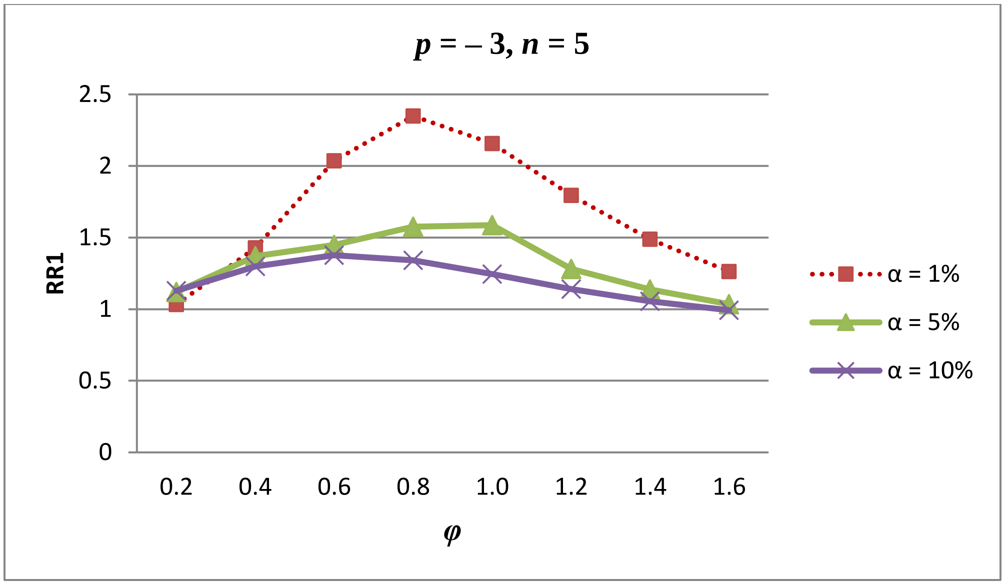

Figure 6

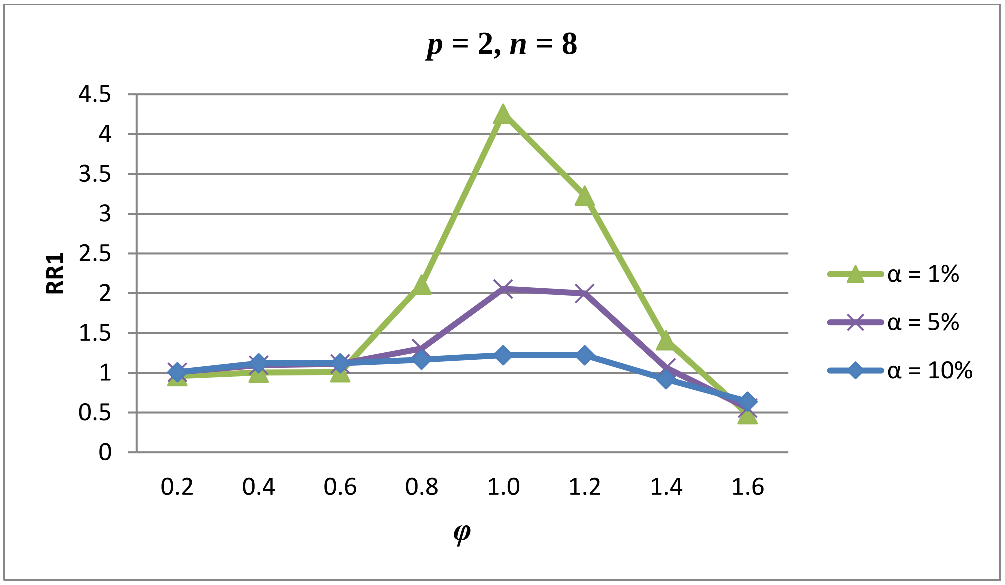

Figure 7

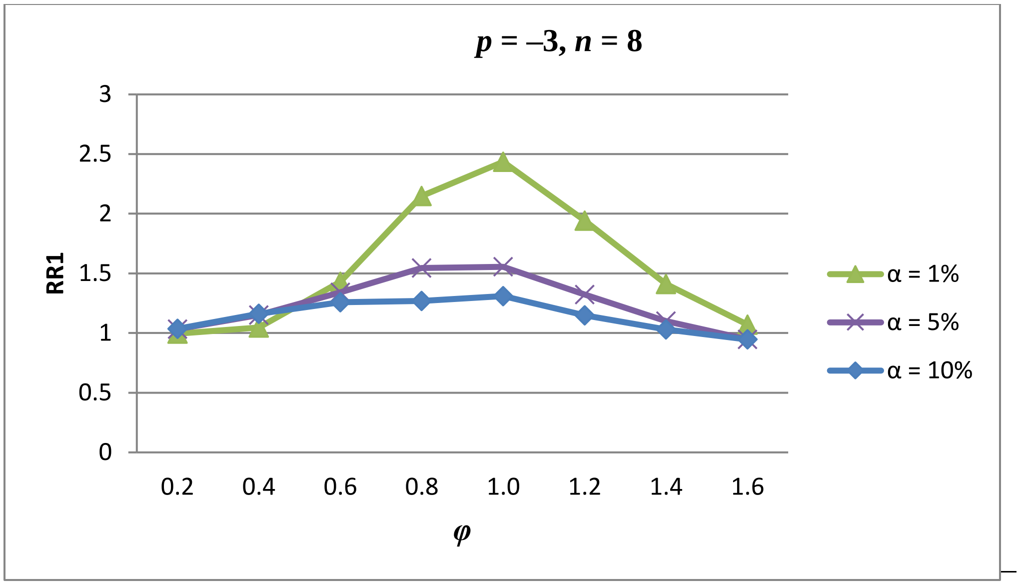

Figure 8

Graphs of Relative Risk for $\hat{\theta}_{DST_2}$

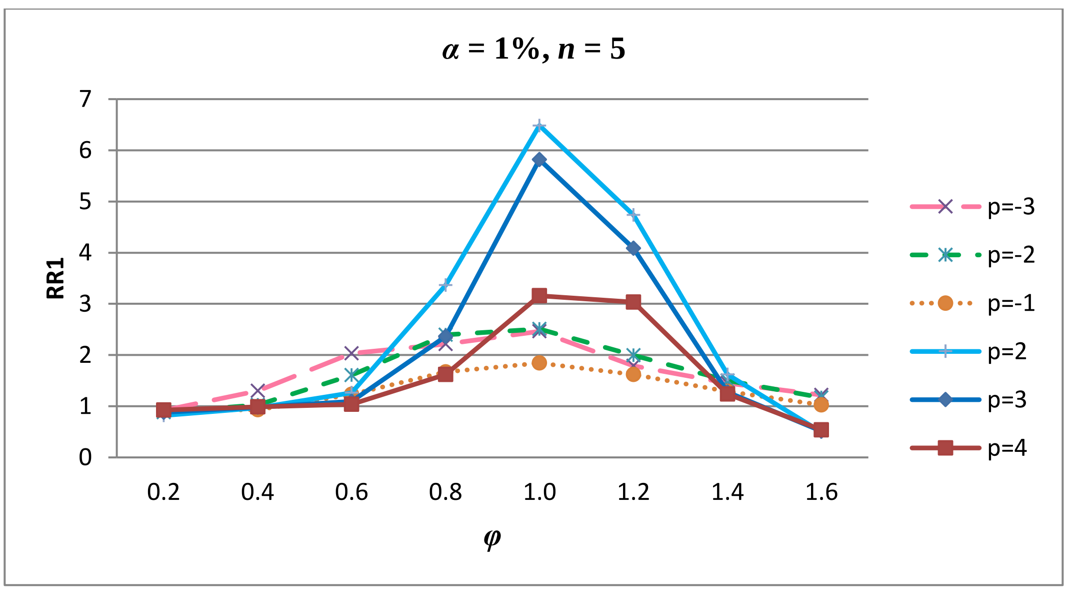

Figure 9

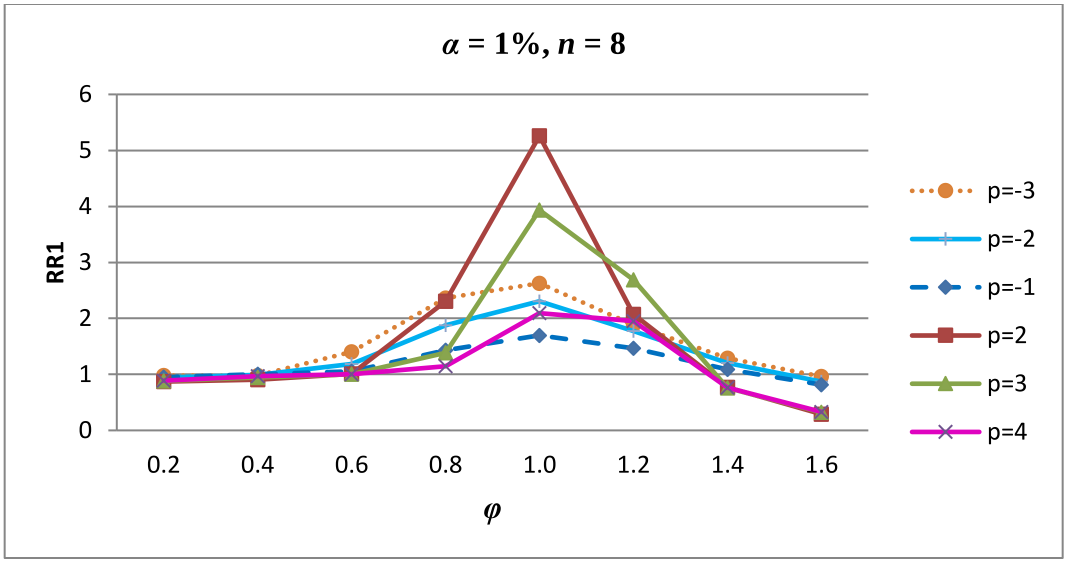

Figure 10

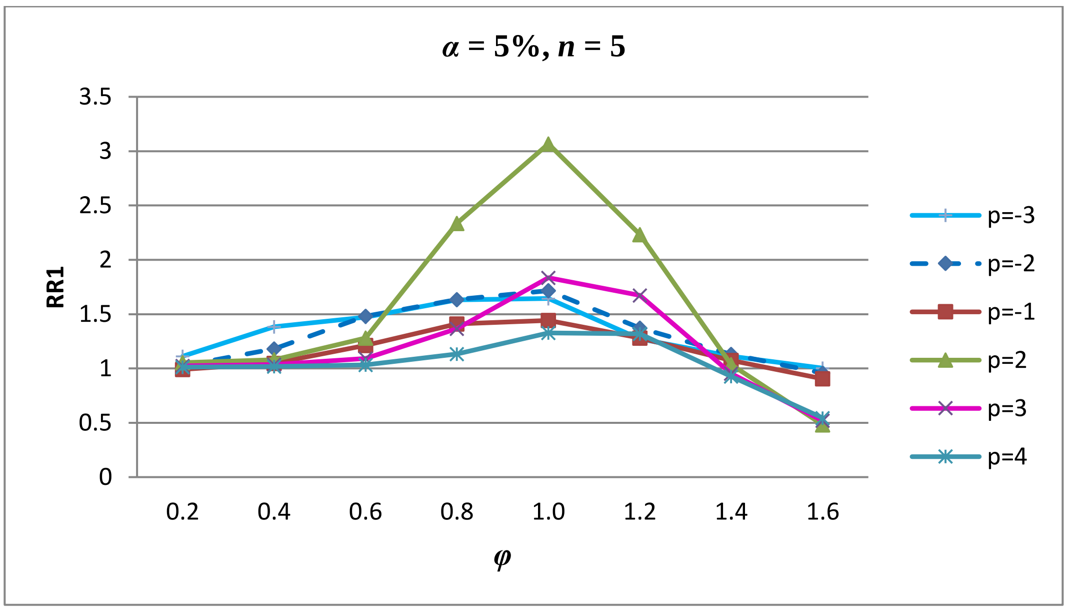

Figure 11

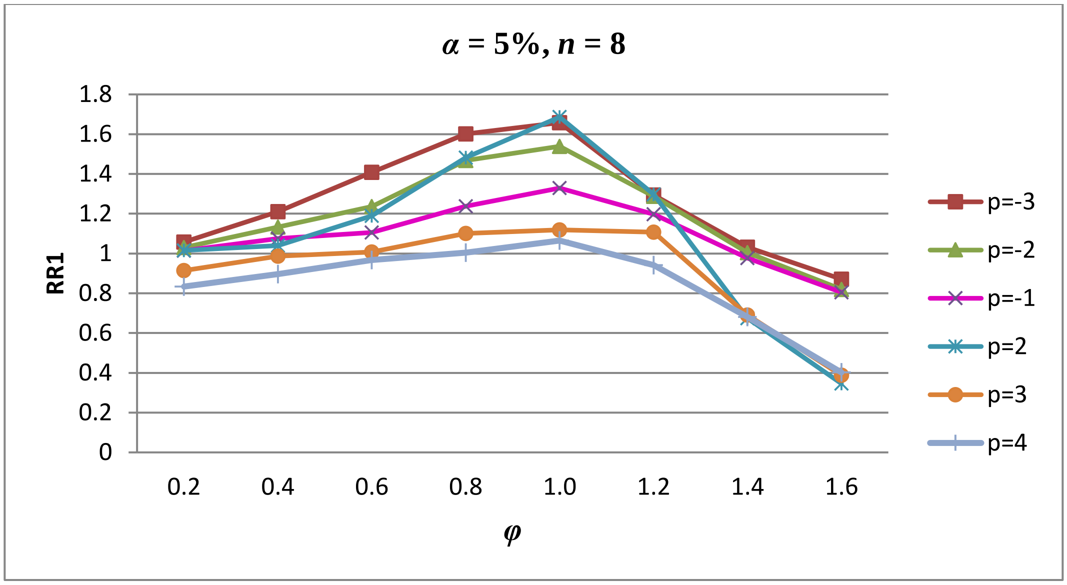

Figure 12

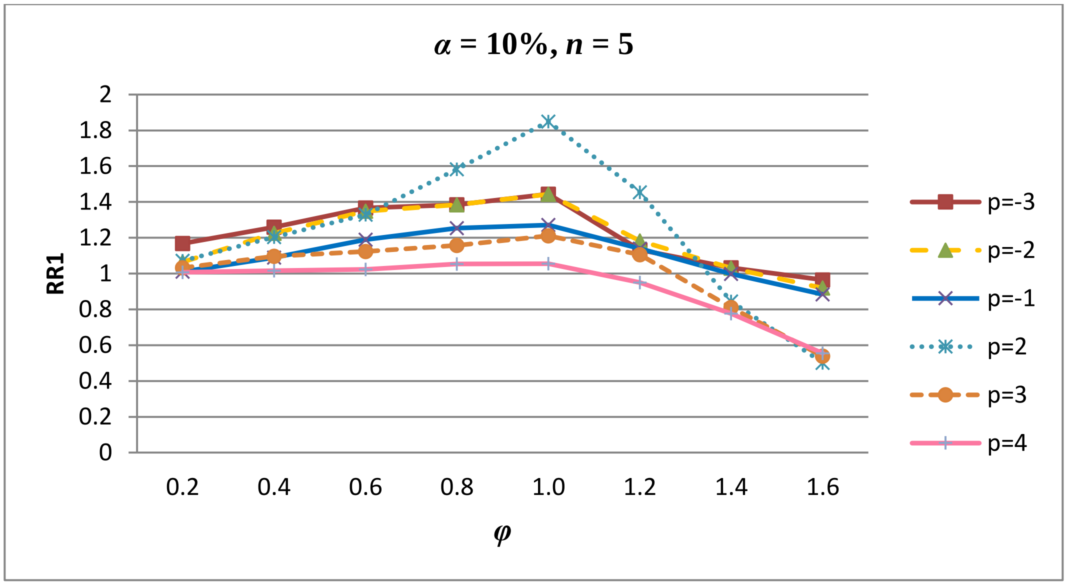

Figure 13

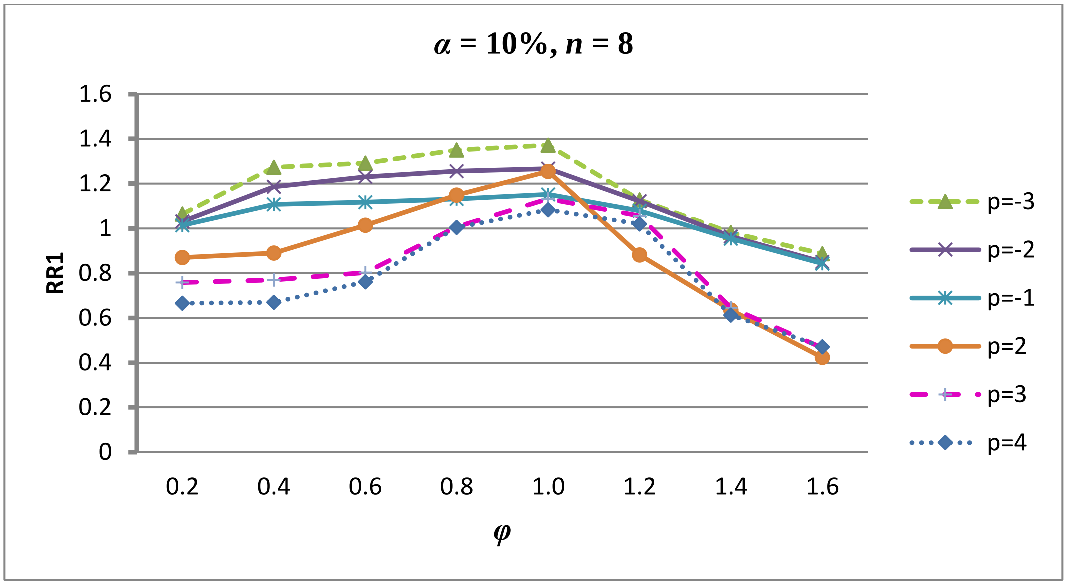

Figure 14

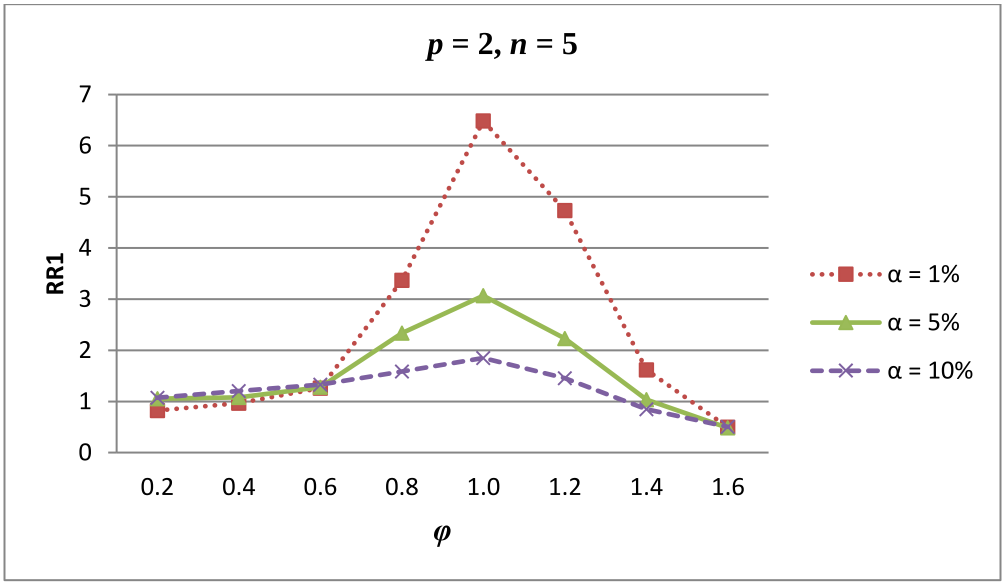

Figure 15

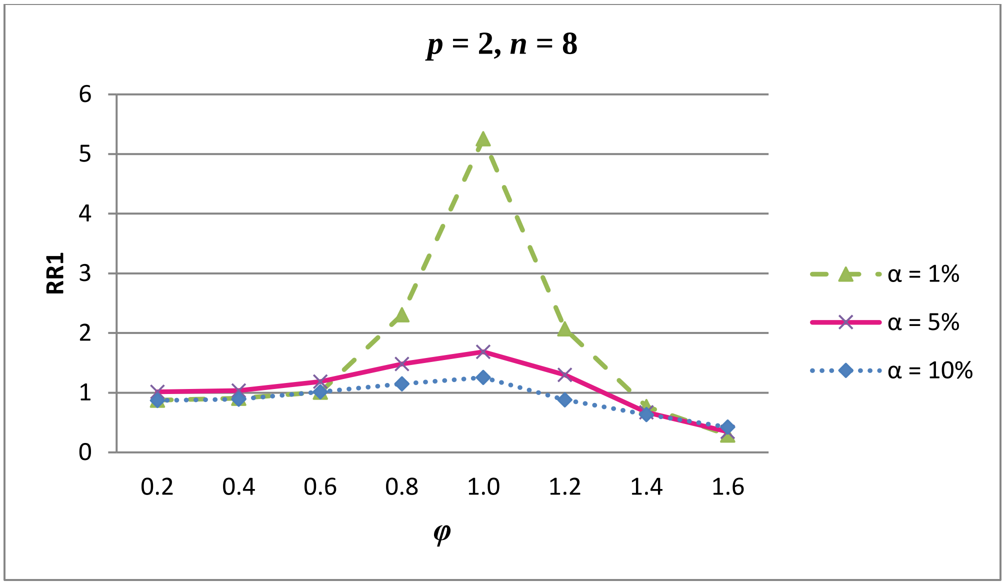

Figure 16

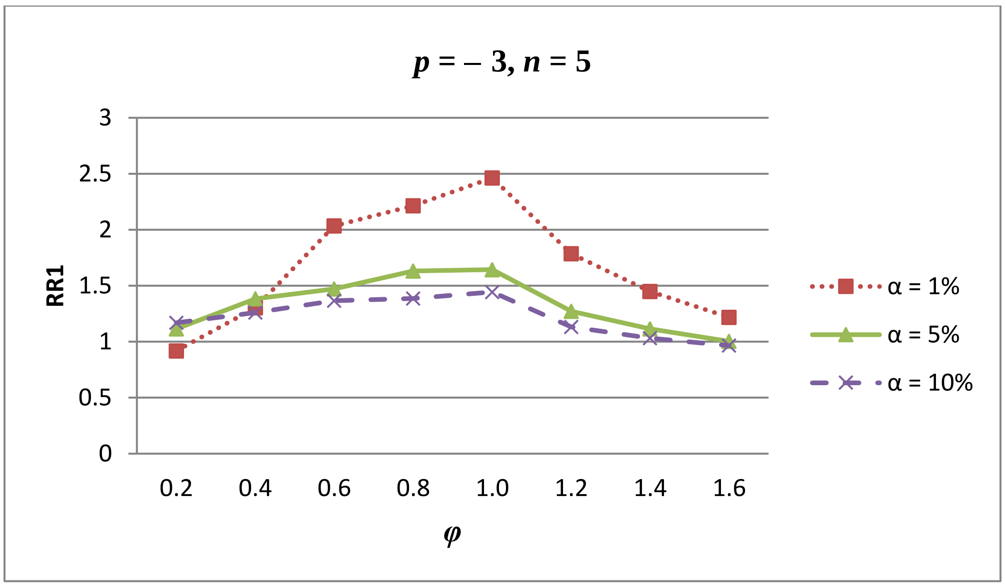

Figure 17

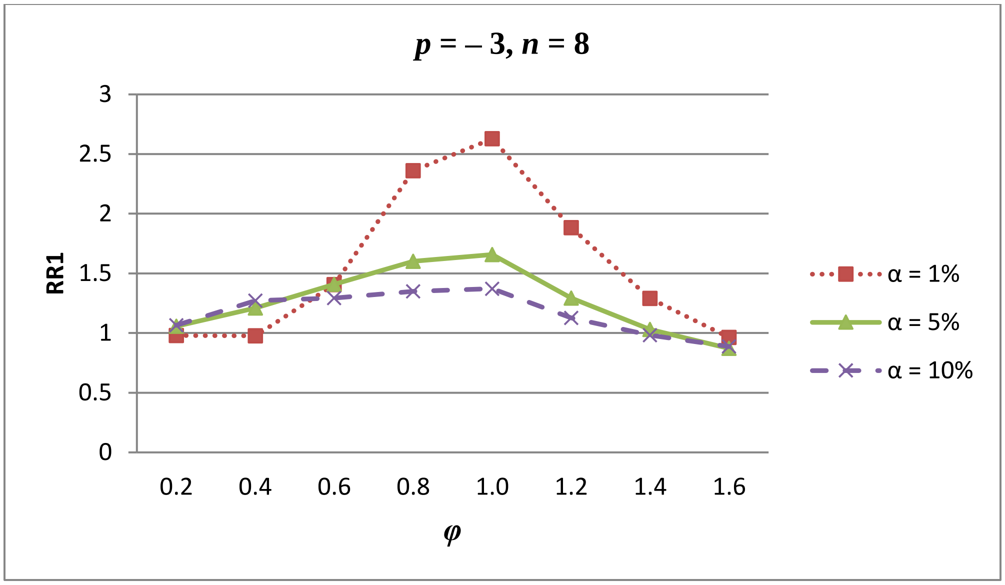

Figure 18