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# Chain Ratio-Type and Exponential Chain Ratio-Ratio-Type Estimators in Double Sampling for Stratification

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**Abstract**

In this paper, we have suggested chain ratio-type estimator, exponential chain ratio-ratio-type estimator, improved estimator and a general class of estimators in Double sampling for stratification for finite population mean. Different conditions were obtained under which the proposed estimators perform better than unbiased estimator, ratio-type and product-type estimators and ratio and product-type exponential estimators. An empirical study is carried out to demonstrate the performance of the proposed estimators over existing estimators.

*Keywords*: Auxiliary variable; Bias; Mean squared error.

**Mathematics Subject Classification Code:** 62D05.

## 1. Introduction and Notations

When the population is homogeneous, for selecting a representative sample from the population, the practitioner usually uses the simple random sampling scheme. However, in practice, heterogeneous population are also encountered. In such situation, stratification is one of the most widely used procedures in sample survey to provide samples that are representatives of major sub-groups of a population and improve precision of estimators, see Holt and Smith (1979). In stratified random sampling, it is assumed that strata weights as well as sampling frame are available in advance. But there are several situations of practical importance where strata weights are known and the frame within strata is not available. For example, in a household survey in a city, number of households in different colonies may be available, but list of households may not be available, see Tailor *et al.* (2014). In such a situation the technique of post stratification is effectively employed. However, in other situations strata weights may not be known exactly as they become outdated with the passage of time. Further the information on the stratification variable may not be readily available but could be made available by diverting a part of the survey budget, see Tripathi and Bahl (1991, p. 2590). Under these situations that procedure of double sampling for stratification (*DSS*) can be employed.

Let $U=\{U_1,U_2,..U_N\}$ be a finite population of size *N*. let (*y*, *x*) be the (study, auxiliary) variates respectively. It is desired to estimate the population mean $\bar{Y}$ of study variable *y* and consider it desirable to stratify the population based on the values of an auxiliary character *x* but the frequency distribution of *x* is not known. The sampling frame for different strata, the

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strata weights $W_h = \dfrac{N_h}{N}; h = 1,2,...L,$ ($N_h$ being the size of the $h^{th}$ stratum) are unknown although the strata may be fixed in advance, see Ige and Tripathi (1987, p. 192). In such situations *DSS* scheme is used. In *DSS* scheme we draw a first phase sample $S^{(1)}$ of size $n'$ from the population *U* using simple random sampling without replacement (*SRSWOR*) scheme and observe auxiliary variable *x*. let $x_j, j = 1,2,...,n'$ be the *x*-observations and $\bar{x}' = \dfrac{1}{n'}\sum_{j=1}^{n'} x_j$, the sample mean. The sample $S^{(1)}$ is then divided into *L* strata on the basis of information gathered for auxiliary variable *x* through $S^{(1)}$. Let $n'_h$ be the number of units in $S^{(1)}$ falling into stratum $h\left(h = 1,2,...,L; \sum_{h=1}^{L} n'_h = n'\right)$, $n' = \{n'_1, n'_2, ..., n'_L\}$ yielding the representation $\bar{x}' = \sum_{h=1}^{L} w_h \bar{x}'_h$, where $\bar{x}'_h = \sum_{j=1}^{n'_h} \dfrac{x'_{hj}}{n'_h}$ and $w_h = \dfrac{n'_h}{n'}$ such that $E(w_h) = W_h = \dfrac{N_h}{N}$. Subsamples of sizes $n_h = \upsilon_h n'_h, 0 < \upsilon_h < 1 (h = 1,2,...,L)$, $\upsilon_h$ is known in advance for each *h*, are then drawn independently, using *SRSWOR* within each stratum and *y*, the study variable is measured.

Let $n = \sum_{h=1}^{L} n_h$, $n = \{n_1, n_2, ..., n_L\}$ and $y_{hj}, j = 1,2,...,n_h; h = 1,2,...,L$ denote *y* observations, $\bar{y}_h = \sum_{j=1}^{n_h} \dfrac{y_{hj}}{n_h}$. It is assumed throughout the paper that $n'_h$ is large enough so that $\Pr(n'_h = 0) = 0$ for all *h*.

Further we denote
First degree of approximation: *fda*,
$f = \dfrac{n'}{N}$: Sampling fraction,

$$\bar{Y}_h = \frac{1}{N_h}\sum_{j=1}^{N_h} y_{hj}, \bar{X}_h = \frac{1}{N_h}\sum_{j=1}^{N_h} x_{hj}, \bar{Y} = \frac{1}{N}\sum_{h=1}^{L}\sum_{j=1}^{N_h} y_{hj}, \bar{X} = \frac{1}{N}\sum_{h=1}^{L}\sum_{j=1}^{N_h} x_{hj},$$

$$S_{yh}^2 = \frac{1}{N_h - 1}\sum_{j=1}^{N_h}\left(y_{hj} - \bar{Y}_h\right)^2, S_{xh}^2 = \frac{1}{N_h - 1}\sum_{j=1}^{N_h}\left(x_{hj} - \bar{X}_h\right)^2, S_{yxh} = \frac{1}{N_h - 1}\sum_{j=1}^{N_h}\left(y_{hj} - \bar{Y}_h\right)\left(x_{hj} - \bar{X}_h\right),$$

$$S_y^2 = \frac{1}{N-1}\sum_{h=1}^{L}\sum_{j=1}^{N_h}\left(y_{hj} - \bar{Y}\right)^2, S_x^2 = \frac{1}{N-1}\sum_{h=1}^{L}\sum_{j=1}^{N_h}\left(x_{hj} - \bar{X}\right)^2, S_{yx} = \frac{1}{N_h - 1}\sum_{j=1}^{N_h}\left(y_{hj} - \bar{Y}\right)\left(x_{hj} - \bar{X}\right),$$

$$\gamma = \left(\frac{1-f}{n'}\right), R = \frac{\bar{Y}}{\bar{X}}, \varphi_h = \left(\frac{1}{\upsilon_h} - 1\right), k = \frac{\theta_{yx}}{R\theta_x}, \ \theta_y = \sum_{h=1}^{L} W_h \varphi_h S_{yh}^2, \ \theta_x = \sum_{h=1}^{L} W_h \varphi_h S_{xh}^2,$$

$$\theta_{yx} = \sum_{h=1}^{L} W_h \varphi_h S_{yxh}, \ \rho = \frac{\theta_{yx}}{\sqrt{\theta_y \theta_x}}.$$

We note that

$\bar{y}_{ds} = \sum_{h=1}^{L} w_h \bar{y}_h$, $\bar{x}_{ds} = \sum_{h=1}^{L} w_h \bar{x}_h$ are unbiased estimators of the population means $\bar{Y}$ and $\bar{X}$ respectively, where $\bar{y}_h = \frac{1}{n_h}\sum_{j=1}^{n_h} y_{hj}$ and $\bar{x}_h = \frac{1}{n_h}\sum_{j=1}^{n_h} x_{hj}$.

Now, to obtain the biases and mean squared errors (*MSE*s) of various estimators of population mean $\bar{Y}$, we write

$$\bar{y}_{ds} = \bar{Y}(1+\varepsilon_0), \bar{x}_{ds} = \bar{X}(1+\varepsilon_1), \bar{x}' = \bar{X}(1+\varepsilon'_1)$$

such that $E(\varepsilon_0) = E(\varepsilon_1) = E(\varepsilon'_1) = 0$ and

$$E(\varepsilon_0^2) = \frac{1}{\bar{Y}^2}\left[\gamma S_y^2 + \frac{1}{n'}\theta_y\right],$$

$$E(\varepsilon_1^2) = \frac{1}{\bar{X}^2}\left[\gamma S_x^2 + \frac{1}{n'}\theta_x\right],$$

$$E(\varepsilon'^2_1) = E(\varepsilon_1 \varepsilon'_1) = \frac{1}{\bar{X}^2}\gamma S_x^2,$$

$$E(\varepsilon_0 \varepsilon_1) = \frac{1}{\bar{Y}\bar{X}}\left[\gamma S_{yx} + \frac{1}{n'}\theta_{yx}\right],$$

$$E(\varepsilon_0 \varepsilon'_1) = \frac{1}{\bar{Y}\bar{X}}\gamma S_{yx}.$$

### 1.1. Reviewing some existing estimators

The conventional unbiased estimator for population mean $\bar{Y}$ [which does not utilize the entire information gathered on the first-phase (preliminary large) sample and the stratified sub samples] is defined by

$$\bar{y}_{ds} = \sum_{h=1}^{L} w_h \bar{y}_h \tag{1}$$

with mean squared error /variance

$$V(\bar{y}_{ds}) = MSE(\bar{y}_{ds}) = \gamma S_y^2 + \frac{1}{n'}\theta_y, \tag{2}$$

is well known [see Rao (1973); Cochran (1977)].

Based on *DSS*, utilizing the auxiliary information obtained on the first phase sample both at the designing as well as at estimation stages, Ige and Tripathi (1987) proposed the ratio-type (*RT*) and product-type (*PT*) estimators for $\bar{Y}$ respectively as

$$\hat{\bar{Y}}_{R(dss)} = \bar{y}_{ds}\left(\frac{\bar{x}'}{\bar{x}_{ds}}\right), \tag{3}$$

$$\hat{\bar{Y}}_{P(dss)} = \bar{y}_{ds}\left(\frac{\bar{x}_{ds}}{\bar{x}'}\right). \tag{4}$$

To the *fda*, the *MSE*s of $\hat{\bar{Y}}_{R(dss)}$ and $\hat{\bar{Y}}_{P(dss)}$ are respectively given by

$$MSE\left(\hat{\bar{Y}}_{R(dss)}\right) = \gamma S_y^2 + \frac{1}{n'}\left[\theta_y + R^2\theta_x(1-2k)\right], \tag{5}$$

$$MSE\left(\hat{\bar{Y}}_{P(dss)}\right) = \gamma S_y^2 + \frac{1}{n'}\left[\theta_y + R^2\theta_x(1+2k)\right]. \tag{6}$$

Further, motivated by Bahl and Tuteja (1991), Tailor *et al*. (2014) suggested *RT* and *PT* exponential estimators respectively as

$$\hat{\bar{Y}}_{\mathrm{Re}(dss)} = \bar{y}_{ds} \exp\left\{\frac{\bar{x}' - \bar{x}_{ds}}{\bar{x}' + \bar{x}_{ds}}\right\}, \tag{7}$$

$$\hat{\bar{Y}}_{Pe(dss)} = \bar{y}_{ds} \exp\left\{\frac{\bar{x}_{ds} - \bar{x}'}{\bar{x}' + \bar{x}_{ds}}\right\}. \tag{8}$$

The *MSE*s of *RT* and *PT* exponential estimators to the *fda* are respectively given by

$$MSE\left(\hat{\bar{Y}}_{\mathrm{Re}(dss)}\right) = \gamma S_y^2 + \frac{1}{n'}\left[\theta_y + \frac{R^2\theta_x}{4}(1-4k)\right], \tag{9}$$

$$MSE\left(\hat{\bar{Y}}_{Pe(dss)}\right) = \gamma S_y^2 + \frac{1}{n'}\left[\theta_y + \frac{R^2\theta_x}{4}(1+4k)\right]. \tag{10}$$

From (2), (5), (6), (9) and (10), it is observed that the *RT* estimator $\hat{\bar{Y}}_{R(dss)}$, the *PT* estimator $\hat{\bar{Y}}_{P(dss)}$, the *RT* exponential estimator $\hat{\bar{Y}}_{\mathrm{Re}(dss)}$ and the *PT* exponential estimator $\hat{\bar{Y}}_{Pe(dss)}$ are better than $\bar{y}_{ds}$ if the conditions $k > \frac{1}{2}$, $k < -\frac{1}{2}$, $k > \frac{1}{4}$ and $k < -\frac{1}{4}$ respectively hold good.

## 2. Proposed Chain-Type Estimators in DSS

### 2.1. Chain RT estimator

On replacing $\bar{y}_{ds}$ by $\hat{\bar{Y}}_{R(dss)}$ in (3), we get chain *RT* estimator in *DSS* for population mean $\bar{Y}$ as

$$\hat{\bar{Y}}^{C}_{R(dss)} = \hat{\bar{Y}}_{R(dss)}\left(\frac{\bar{x}'}{\bar{x}_{ds}}\right) = \bar{y}_{ds}\left(\frac{\bar{x}'}{\bar{x}_{ds}}\right)^2. \tag{11}$$

Putting $\bar{y}_{ds} = \bar{Y}(1+\varepsilon_0)$, $\bar{x}_{ds} = \bar{X}(1+\varepsilon_1)$ and $\bar{x}' = \bar{X}(1+\varepsilon'_1)$ in (11) we have

$$=\bar{Y}(1+\varepsilon_0)(1+\varepsilon'_1)^2(1+\varepsilon_1)^{-2}. \tag{12}$$

We assume that $|\varepsilon_1|<1$ so that the term $(1+\varepsilon_1)^{-2}$ is expandable. Now, expanding the right hand side (*RHS*) of (12) multiplying out and neglecting terms of $\varepsilon$ 's having power greater than two, we have

$$\left(\hat{\hat{\bar{Y}}}{}^{C}_{R(dss)}-\bar{Y}\right)\cong\bar{Y}\left[\varepsilon_0-2\varepsilon_1+2\varepsilon'_1+3\varepsilon_1^2+\varepsilon'^2_1-2\varepsilon_0\varepsilon_1+2\varepsilon_0\varepsilon'_1-4\varepsilon_1\varepsilon'_1\right]. \tag{13}$$

To obtain the bias of $\hat{\hat{\bar{Y}}}{}^{C}_{R(dss)}$ to the *fda*, we take the expectation of both sides of (13) and thus

$$B\left(\hat{\hat{\bar{Y}}}{}^{C}_{R(dss)}\right)=\frac{R\theta_x}{n'\bar{X}}(3-k). \tag{14}$$

which is negligible if sample size $n'$ is sufficiently large.

Squaring both sides of (13), neglecting terms of $\varepsilon$ 's having power greater than two, we have

$$=\bar{Y}^2\left[\varepsilon_0^2+4\varepsilon_1^2+4\varepsilon'^2_1-4\varepsilon_0\varepsilon_1+4\varepsilon_0\varepsilon'_1-8\varepsilon_1\varepsilon'_1\right]. \tag{15}$$

Taking expectation of both sides of (15) we get the *MSE* of $\hat{\hat{\bar{Y}}}{}^{C}_{R(dss)}$ to the *fda* as

$$MSE\left(\hat{\hat{\bar{Y}}}{}^{C}_{R(dss)}\right)=\gamma S_y^2+\frac{1}{n'}\left[\theta_y+4R^2\theta_x(1-k)\right]. \tag{16}$$

From (2), (5), (9) and (16) it can be shown that

$$MSE\left(\hat{\hat{\bar{Y}}}{}^{C}_{R(dss)}\right)<MSE\left(\bar{y}_{ds}\right)\text{if } k>1, \tag{17}$$

$$MSE\left(\hat{\hat{\bar{Y}}}{}^{C}_{R(dss)}\right)<MSE\left(\hat{\hat{\bar{Y}}}_{R(dss)}\right)\text{ if } k>\frac{3}{2}, \tag{18}$$

and

$$MSE\left(\hat{\hat{\bar{Y}}}{}^{C}_{R(dss)}\right)<MSE\left(\hat{\hat{\bar{Y}}}_{\mathrm{Re}(dss)}\right)\text{ if } k>\frac{5}{4}. \tag{19}$$

Thus, the proposed chain *RT* estimator $\hat{\hat{\bar{Y}}}{}^{C}_{R(dss)}$ is more efficient than the estimators $\bar{y}_{ds}$, $\hat{\hat{\bar{Y}}}_{R(dss)}$ and $\hat{\hat{\bar{Y}}}_{\mathrm{Re}(dss)}$ if the conditions (17), (18) and (19) are satisfied respectively. It is also observed from (17), (18) and (19) that the condition $k>\frac{3}{2}$ is sufficient for the proposed estimator $\hat{\hat{\bar{Y}}}{}^{C}_{R(dss)}$ to be more efficient than the estimators $\bar{y}_{ds}$, $\hat{\hat{\bar{Y}}}_{R(dss)}$ and $\hat{\hat{\bar{Y}}}_{\mathrm{Re}(dss)}$.

## 2.2. Chain PT estimator

On replacing $\bar{y}_{ds}$ by $\hat{\bar{Y}}_{P(dss)}$ in (4), we get a chain *PT* estimator in *DSS* for population mean $\bar{Y}$ as

$$\hat{\bar{Y}}^{C}_{P(dss)} = \hat{\bar{Y}}_{P(dss)}\left(\frac{\bar{x}_{ds}}{\bar{x}'}\right) = \bar{y}_{ds}\left(\frac{\bar{x}_{ds}}{\bar{x}'}\right)^2. \tag{20}$$

Inserting $\bar{y}_{ds} = \bar{Y}(1+\varepsilon_0), \bar{x}_{ds} = \bar{X}(1+\varepsilon_1)$ and $\bar{x}' = \bar{X}(1+\varepsilon'_1)$ in (20) we have

$$\hat{\bar{Y}}^{C}_{P(dss)} = \bar{Y}(1+\varepsilon_0)\left\{\frac{\bar{X}(1+\varepsilon_1)}{\bar{X}(1+\varepsilon'_1)}\right\}^2 = \bar{Y}(1+\varepsilon_0)(1+\varepsilon_1)^2(1+\varepsilon'_1)^{-2}. \tag{21}$$

We assume that $|\varepsilon'_1| < 1$ so that the term $(1+\varepsilon'_1)^{-2}$ is expandable. Now, expanding the *RHS* of (21), multiplying out, neglecting terms of $\varepsilon$'s having power greater than two and then subtracting $\bar{Y}$ from both sides of (21), we have

$$\left(\hat{\bar{Y}}^{C}_{P(dss)} - \bar{Y}\right) \cong \bar{Y}\left[\varepsilon_0 + 2\varepsilon_1 - 2\varepsilon'_1 + 3\varepsilon'^2_1 + \varepsilon_1^2 + 2\varepsilon_0\varepsilon_1 - 2\varepsilon_0\varepsilon'_1 - 4\varepsilon_1\varepsilon'_1\right]. \tag{22}$$

Taking expectation of both sides of (22) we get the bias of $\hat{\bar{Y}}^{C}_{P(dss)}$ to the *fda* as

$$B\left(\hat{\bar{Y}}^{C}_{P(dss)}\right) = \frac{R\theta_x}{n'\bar{X}}(1+2k). \tag{23}$$

which is negligible if sample size $n'$ is large enough.

Now, squaring both sides of (22) and neglecting terms of $\varepsilon$'s having power greater than two we have

$$= \bar{Y}^2\left[\varepsilon_0^2 + 4\varepsilon_1^2 + 4\varepsilon'^2_1 + 4\varepsilon_0\varepsilon_1 - 4\varepsilon_0\varepsilon'_1 - 8\varepsilon_1\varepsilon'_1\right]. \tag{24}$$

Taking expectation of both sides of (24) we get the *MSE* of $\hat{\bar{Y}}^{C}_{P(dss)}$ to the *fda* as

$$MSE\left(\hat{\bar{Y}}^{C}_{P(dss)}\right) = \gamma S_y^2 + \frac{1}{n'}\left[\theta_y + 4R^2\theta_x(1+k)\right]. \tag{25}$$

It can be easily observed from (2), (6), (10) and (25) that the suggested chain *PT* estimator $\hat{\bar{Y}}^{C}_{P(dss)}$ is more efficient than the estimators $\bar{y}_{ds}$, $\hat{\bar{Y}}_{P(dss)}$ and $\hat{\bar{Y}}_{Pe(dss)}$ respectively if the conditions $k < -1$, $k < -\frac{3}{2}$ and $k < -\frac{5}{4}$ holds good. It is further observed that the condition $k < -1$ is sufficient for the proposed chain *PT* estimator $\hat{\bar{Y}}^{C}_{P(dss)}$ to be more efficient than the estimators $\bar{y}_{ds}$, $\hat{\bar{Y}}_{P(dss)}$ and $\hat{\bar{Y}}_{PExp(dss)}$.

## 2.3. Chain RT exponential estimator

Inserting $\hat{\hat{Y}}_{\mathrm{Re}(dss)}$ in place of $\bar{y}_{ds}$ in (7), we obtain the chain *RT* exponential estimator in *DSS* for population mean $\bar{Y}$ as

$$\hat{\hat{Y}}^{C}_{\mathrm{Re}(dss)} = \hat{\hat{Y}}_{\mathrm{Re}(dss)} \exp\left\{\frac{(\bar{x}'-\bar{x}_{ds})}{(\bar{x}'+\bar{x}_{ds})}\right\} = \bar{y}_{ds} \exp\left\{\frac{2(\bar{x}'-\bar{x}_{ds})}{(\bar{x}'+\bar{x}_{ds})}\right\}. \quad (26)$$

Expressing (26) in terms of $\varepsilon$ 's we have

$$\hat{\hat{Y}}^{C}_{\mathrm{Re}(dss)} = \bar{Y}(1+\varepsilon_0)\exp\left\{(\varepsilon'_1-\varepsilon_1)\left(1+\frac{(\varepsilon'_1+\varepsilon_1)}{2}\right)^{-1}\right\}. \quad (27)$$

Expanding the *RHS* of (27), multiplying out, neglecting terms of $\varepsilon$ 's having power greater than two and then subtracting $\bar{Y}$ from both sides, we have

$$\left(\hat{\hat{Y}}^{C}_{\mathrm{Re}(dss)} - \bar{Y}\right) = \bar{Y}\left[\varepsilon_0 - \varepsilon_1 + \varepsilon'_1 + \varepsilon_1^2 + \varepsilon_0\varepsilon'_1 - \varepsilon_0\varepsilon_1 - \varepsilon_1\varepsilon'_1\right]. \quad (28)$$

Taking expectation of both sides of (28) we get bias of $\hat{\hat{Y}}^{C}_{\mathrm{Re}(dss)}$ to the *fda* as

$$B\left(\hat{\hat{Y}}^{C}_{\mathrm{Re}(dss)}\right) = \frac{R\theta_x}{n'\bar{X}}(1-k) \quad . \quad (29)$$

For sufficiently large $n'$, the bias of $\hat{\hat{Y}}^{C}_{\mathrm{Re}(dss)}$ at (29) is negligible.

Squaring both sides of (28), neglecting terms of $\varepsilon$ 's having power higher than two and taking expectation of both sides, we get the *MSE* of $\hat{\hat{Y}}^{C}_{\mathrm{Re}(dss)}$ to the *fda* as

$$MSE\left(\hat{\hat{Y}}^{C}_{\mathrm{Re}(dss)}\right) = \gamma S_y^2 + \frac{1}{n'}\left\{\theta_y + R^2\theta_x(1-2k)\right\}. \quad (30)$$

From (5) and (30), it is observed that to the *fda*, the *MSE* of *RT* estimator $\hat{\hat{Y}}_{R(dss)}$ and the *MSE* of chain *RT* exponential estimator $\hat{\hat{Y}}^{C}_{\mathrm{Re}(dss)}$ are same *i.e.*, $MSE\left(\hat{\hat{Y}}_{\mathrm{Re}(dss)}\right) = MSE\left(\hat{\hat{Y}}_{R(dss)}\right)$.

## 2.4. Chain PT exponential estimator

On replacing $\bar{y}_{ds}$ by $\hat{\hat{Y}}_{Pe(dss)}$ in (8), we obtain the chain *PT* exponential estimator in *DSS* for population mean $\bar{Y}$ as

$$\hat{\hat{Y}}^{C}_{Pe(dss)} = \hat{\hat{Y}}_{Pe(dss)} \exp\left\{\frac{(\bar{x}_{ds}-\bar{x}')}{(\bar{x}'+\bar{x}_{ds})}\right\} = \bar{y}_{ds} \exp\left\{\frac{2(\bar{x}_{ds}-\bar{x}')}{(\bar{x}'+\bar{x}_{ds})}\right\}. \quad (31)$$

Substituting $\bar{y}_{ds} = \bar{Y}(1+\varepsilon_0), \bar{x}_{ds} = \bar{X}(1+\varepsilon_1)$ and $\bar{x}' = \bar{X}(1+\varepsilon'_1)$ in (31) we have

$$\hat{\bar{Y}}^{C}_{Pe(dss)} = \bar{Y}(1+\varepsilon_0)\exp\left\{(\varepsilon_1 - \varepsilon'_1)\left(1+\frac{(\varepsilon_1+\varepsilon'_1)}{2}\right)^{-1}\right\}. \tag{32}$$

Expanding the *RHS* of (32) neglecting terms of $\varepsilon$'s having power greater than two and subtracting $\bar{Y}$ from both sides, we have

$$\left(\hat{\bar{Y}}^{C}_{Pe(dss)} - \bar{Y}\right) = \bar{Y}\left[\varepsilon_0 + \varepsilon_1 - \varepsilon'_1 + \varepsilon'^2_1 + \varepsilon_0\varepsilon_1 - \varepsilon_0\varepsilon'_1 - \varepsilon_1\varepsilon'_1\right]. \tag{33}$$

Taking expectation of both sides of (33) we get the bias of $\hat{\bar{Y}}^{C}_{Pe(dss)}$ to the *fda* as

$$B\left(\hat{\bar{Y}}^{C}_{Pe(dss)}\right) = \frac{\theta_{yx}}{n'\bar{X}}. \tag{34}$$

Squaring both sides of (33), retaining terms of $\varepsilon$'s up to second degree and then taking expectation of both sides we get the *MSE* of $\hat{\bar{Y}}^{C}_{Pe(dss)}$ to the *fda* as

$$MSE\left(\hat{\bar{Y}}^{C}_{Pe(dss)}\right) = \gamma S_y^2 + \frac{1}{n'}\left[\theta_y + R^2\theta_x(1+2k)\right]. \tag{35}$$

which equals to the *MSE* of $\hat{\bar{Y}}_{P(dss)}$ *i.e.* $MSE\left(\hat{\bar{Y}}^{C}_{Pe(dss)}\right) = MSE\left(\hat{\bar{Y}}_{P(dss)}\right)$.

## 2.5. Chain Ratio-RT exponential estimator

Chain ratio-*RT* exponential estimator in *DSS* for population mean $\bar{Y}$ is obtained on replacing $\bar{y}_{ds}$ by $\hat{\bar{Y}}_{R(dss)}$ in (7) given by

$$\hat{\bar{Y}}^{C}_{R\,\mathrm{Re}(dss)} = \bar{y}_{ds}\left(\frac{\bar{x}'}{\bar{x}_{ds}}\right)\exp\left(\frac{\bar{x}' - \bar{x}_{ds}}{\bar{x}' + \bar{x}_{ds}}\right). \tag{36}$$

Proceeding as earlier the bias and *MSE* of $\hat{\bar{Y}}^{C}_{R\,\mathrm{Re}(dss)}$ to the *fda*, are respectively given by

$$B\left(\hat{\bar{Y}}^{C}_{R\,\mathrm{Re}(dss)}\right) = \frac{R\theta_x}{2n'\bar{X}}\left(\frac{15}{4} - 3k\right), \tag{37}$$

$$MSE\left(\hat{\bar{Y}}^{C}_{R\,\mathrm{Re}(dss)}\right) = \left[\gamma S_y^2 + \frac{1}{n'}\left\{\theta_y + 3R^2\theta_x\left(\frac{3}{4} - k\right)\right\}\right]. \tag{38}$$

It is observed from (2), (5), (9), (16), (30) and (38) it is observed that the estimator $\hat{\bar{Y}}^{C}_{R\,\mathrm{Re}(dss)}$ is more efficient than the estimators $\bar{y}_{ds}, \left(\hat{\bar{Y}}_{R(dss)}, \hat{\bar{Y}}^{C}_{\mathrm{Re}(dss)}\right), \hat{\bar{Y}}_{\mathrm{Re}(dss)}$ and $\hat{\bar{Y}}^{C}_{R(dss)}$ respectively if the conditions $k > \frac{3}{4}, k > \frac{5}{4}, k > 1$ and $k < \frac{7}{4}$ are satisfied. We also conclude that if $k$ is lies between $\left(\frac{5}{4}, \frac{7}{4}\right)$, then chain ratio-*RT* exponential estimator $\hat{\bar{Y}}^{C}_{R\mathrm{Re}(dss)}$ performs better than the estimators $\bar{y}_{ds}$, $\hat{\bar{Y}}_{R(dss)}$, $\hat{\bar{Y}}_{\mathrm{Re}(dss)}$, $\hat{\bar{Y}}^{C}_{R(dss)}$ and $\hat{\bar{Y}}^{C}_{\mathrm{Re}(dss)}$.

## 2.6. Chain product-PT exponential estimator

Inserting $\hat{\bar{Y}}_{P(dss)}$ in place of $\bar{y}_{ds}$ in (8), we define a chain product-*PT* exponential estimator for population mean $\bar{Y}$ in *DSS* as

$$\hat{\bar{Y}}^{C}_{PPe(dss)} = \hat{\bar{Y}}_{P(dss)} \exp\left(\frac{\bar{x}_{ds} - \bar{x}'}{\bar{x}' + \bar{x}_{ds}}\right) = \bar{y}_{ds}\left(\frac{\bar{x}_{ds}}{\bar{x}'}\right)\exp\left(\frac{\bar{x}_{ds} - \bar{x}'}{\bar{x}' + \bar{x}_{ds}}\right) . \tag{39}$$

Using the procedure adopted in preceding sections, we get the bias and *MSE* of the $\hat{\bar{Y}}^{C}_{PPe(dss)}$ to the *fda*, respectively as

$$B\left(\hat{\bar{Y}}^{C}_{PPe(dss)}\right) = \frac{3R\theta_x}{8n'\bar{X}}(1 + 4k), \tag{40}$$

$$MSE\left(\hat{\bar{Y}}^{C}_{PPe(dss)}\right) = \left[\gamma S_y^2 + \frac{1}{n'}\left\{\theta_y + 3R^2\theta_x\left(\frac{3}{4} + k\right)\right\}\right]. \tag{41}$$

It is observed from (2), (6), (10), (25), (35) and (41) that the proposed estimator $\hat{\bar{Y}}^{C}_{PPe(dss)}$ is better than:

(i) the unbiased estimator $\bar{y}_{ds}$ if $k < -\frac{3}{4}$,

(ii) the *PT* estimator $\hat{\bar{Y}}_{P(dss)}$ and the chain *PT* exponential estimator $\hat{\bar{Y}}^{C}_{Pe(dss)}$ if $k < -\frac{5}{4}$,

(iii) the *PT* exponential estimator $\hat{\bar{Y}}_{Pe(dss)}$ if $k < -1$;and

(iv) the chain *PT* estimator $\hat{\bar{Y}}^{C}_{P(dss)}$ if $k > -\frac{7}{4}$.

It is further observed from (i) to (iv) that the proposed estimator $\hat{\bar{Y}}^{C}_{PPe(dss)}$ is always better than the estimators $\bar{y}_{ds}$, $\hat{\bar{Y}}_{P(dss)}$, $\hat{\bar{Y}}_{Pe(dss)}$, $\hat{\bar{Y}}^{C}_{P(dss)}$ and $\hat{\bar{Y}}^{C}_{Pe(dss)}$ if $k \in \left(-\frac{7}{4}, -\frac{5}{4}\right)$.

## 3. A Class of Chain Ratio-RT Exponential Estimators in DSS

We have suggested a class of chain ratio-*RT* exponential estimators for population mean $\bar{Y}$ in *DSS* as

$$\hat{\bar{Y}}_{CR(dss)}^{P} = \bar{y}_{ds}\left(\frac{\xi\,\bar{x}'+\psi}{\xi\,\bar{x}_{ds}+\psi}\right)^{\lambda}\exp\left\{\frac{\omega\xi\left(\bar{x}'-\bar{x}_{ds}\right)}{\xi\left(\bar{x}'+\bar{x}_{ds}\right)+2\psi}\right\}, \tag{42}$$

where $\{\xi(\neq 0),\psi\}$ are real constants or known parameters such as standard deviation $S_x$, coefficient of variation $C_x$, coefficient of skewness $\beta_1(x)$, coefficient of kurtosis $\beta_2(x)$ and $\Delta=(\beta_2(x)-\beta_1(x)-1)$ associated with auxiliary variable $x$ or coefficient of variation $C_y$ of study variable $y$ or $\rho_{yx}$, the coefficient of correlation between $y$ and $x$; $(\lambda,\omega)$ are suitably chosen design parameters. In particular $(\lambda,\omega)$ are to be determined such that *MSE* of the class of chain ratio-*RT* exponential estimators $\hat{\bar{Y}}_{CR(dss)}^{P}$ is minimum.

We note that for different values of scalars $(\xi,\psi,\lambda,\omega)$ a large number of estimators can be generated from the suggested class of estimators $\hat{\bar{Y}}_{CR(dss)}^{P}$.

To the *fda*, the bias and the *MSE* of the estimator $\hat{\bar{Y}}_{CR(dss)}^{P}$ are respectively given by

$$B\left(\hat{\bar{Y}}_{CR(dss)}^{P}\right)=\frac{\eta\tau R\theta_x}{2n'\bar{X}}\left\{\frac{\tau}{4}-k\right\}. \tag{43}$$

$$MSE\left(\hat{\bar{Y}}_{CR(dss)}^{P}\right)=\left[\gamma S_y^2+\frac{1}{n'}\left\{\theta_y+\frac{\tau\eta R^2\theta_x}{4}(\tau\eta-4k)\right\}\right]. \tag{44}$$

where $\tau=\dfrac{\xi\bar{X}}{\xi\bar{X}+\psi},\eta=(\omega+2\lambda)$.

We note from (43) that the bias of $\hat{\bar{Y}}_{CR(dss)}^{P}$ is negligible if the sample size $n'$ is sufficiently large. The proof of the results in (43) and (44) are simple so omitted.

The *MSE* of $\hat{\bar{Y}}_{CR(dss)}^{P}$ is minimum when

$$\eta=\frac{2k}{\tau}=\eta_{(opt)}\text{ say.} \tag{45}$$

Thus, the resulting minimum *MSE* of $\hat{\bar{Y}}_{CR(dss)}^{P}$ is given by

$$MSE_{\min}\left(\hat{\bar{Y}}_{CR(dss)}^{P}\right)=\left[\gamma S_y^2+\frac{\theta_y}{n'}\left(1-\rho^2\right)\right]. \tag{46}$$

where $\rho=\dfrac{\theta_{yx}}{\sqrt{\theta_y\theta_x}}$.

Now we state the following theorem:

**Theorem 1:** Up to first order of approximation,

$$MSE\left(\hat{\bar{Y}}^{P}_{CR(dss)}\right) \geq \left[\gamma S_y^2 + \frac{\theta_y}{n'}\left(1-\rho^2\right)\right]$$

with equality holding if $\eta = \frac{2k}{\tau}$.

### 3.1. Efficiency comparison

From (2) and (44), we have that

$$MSE\left(\hat{\bar{Y}}^{C}_{CR(dss)}\right) < MSE\left(\bar{y}_{ds}\right) \text{if}$$

either $$k > \frac{\tau\eta}{4}, \tau\eta > 0, \tag{47}$$

or $$k < \frac{\tau\eta}{4}, \tau\eta < 0, \tag{48}$$

If we set $(\lambda, \omega) = (\lambda^*, 0)$ then the class of estimators $\hat{\bar{Y}}^{P}_{CR(dss)}$ reduces to

$$\hat{\bar{Y}}^{P1}_{CR(dss)} = \bar{y}_{ds}\left(\frac{\xi\,\bar{x}'+\psi}{\xi\,\bar{x}_{ds}+\psi}\right)^{\lambda^*} \tag{49}$$

where $\lambda^*$ is a constant.

Putting $(\lambda, \omega) = (\lambda^*, 0)$in (43) and (44) we get the bias and *MSE* of $\hat{\bar{Y}}^{P1}_{CR(dss)}$to the *fda*, are respectively given by

$$B\left(\hat{\bar{Y}}^{P1}_{CR(dss)}\right) = \frac{\tau R\lambda^*\theta_x}{n'\,\bar{X}}\left\{\frac{\tau\left(\lambda^*+1\right)}{2} - k\right\}, \tag{50}$$

$$MSE\left(\hat{\bar{Y}}^{P1}_{CR(dss)}\right) = \left[\gamma S_y^2 + \frac{1}{n'}\left\{\theta_y + \tau R^2\lambda^*\theta_x\left(\tau\lambda^* - 2k\right)\right\}\right]. \tag{51}$$

Now from (44) and (51), we have

$$MSE\left(\hat{\bar{Y}}^{P1}_{CR(dss)}\right) - MSE\left(\hat{\bar{Y}}^{P}_{CR(dss)}\right) = \frac{\tau R^2\theta_x}{n'}\left(2\lambda^* - \eta\right)\left[\frac{\tau\left(2\lambda^*+\eta\right)}{4} - k\right] > 0, \text{ if}$$

either $$k < \frac{\tau\left(2\lambda^*+\eta\right)}{4}, \eta < 2\lambda^*, \tag{52}$$

or $$k > \frac{\tau\left(2\lambda^*+\eta\right)}{4}, \eta > 2\lambda^*. \tag{53}$$

If we set $(\lambda, \omega) = (0, \omega^*)$, then the class of estimators $\hat{\hat{Y}}^{P}_{CR(dss)}$ reduces to

$$\hat{\hat{Y}}^{P2}_{CR(dss)} = \bar{y}_{ds} \exp\left\{ \frac{\omega^* \xi(\bar{x}' - \bar{x}_{ds})}{\xi(\bar{x}' + \bar{x}_{ds}) + 2\psi} \right\}, \tag{54}$$

where $\omega^*$ is a constant.

Putting $(\lambda, \omega) = (0, \omega^*)$ in (43) and (44) we get bias and *MSE* of the estimator $\hat{\hat{Y}}^{P2}_{CR(dss)}$ respectively as

$$B\left(\hat{\hat{Y}}^{P2}_{CR(dss)}\right) = \frac{\tau\omega^* R\theta_x}{2n'\bar{X}} \left\{ \frac{\tau(\omega^* + 2)}{4} - k \right\}, \tag{55}$$

$$MSE\left(\hat{\hat{Y}}^{P2}_{CR(dss)}\right) = \left[ \gamma S_y^2 + \frac{1}{n'} \left\{ \theta_y + \frac{\tau R^2 \omega^* \theta_x}{4} (\tau\omega^* - 4k) \right\} \right]. \tag{56}$$

From (44) and (56), we have that

$$MSE\left(\hat{\hat{Y}}^{P}_{CR(dss)}\right) < MSE\left(\hat{\hat{Y}}^{P2}_{CR(dss)}\right), \text{ if}$$

either $$k < \frac{\tau(\omega^* + \eta)}{4}, \eta < \omega^*, \tag{57}$$

or $$k > \frac{\tau(\omega^* + \eta)}{4}, \eta > \omega^*. \tag{58}$$

Thus we conclude that $\hat{\hat{Y}}^{P}_{CR(dss)}$ is better than $\bar{y}_{ds}$, $\hat{\hat{Y}}^{P1}_{CR(dss)}$ and $\hat{\hat{Y}}^{P2}_{CR(dss)}$ if the conditions given in equations (47) or (48), (52) or (53) and (57) or (58) respectively are satisfied.

## 4. Improved Class of Estimators

Motivated by Searls (1964) we consider an improved class of chain *RT* estimators in *DSS* for population mean $\bar{Y}$ as

$$\begin{aligned} P^{I}_{(dss)} &= \alpha_2 \hat{\hat{Y}}^{P}_{CR(dss)} \\ &= \alpha_2 \bar{y}_{ds} \left( \frac{\xi \bar{x}' + \psi}{\xi \bar{x}_{ds} + \psi} \right)^{\lambda} \exp\left\{ \frac{\omega \xi(\bar{x}' - \bar{x}_{ds})}{\xi(\bar{x}' + \bar{x}_{ds}) + 2\psi} \right\} \end{aligned} \tag{59}$$

where $\alpha_2$ is a constant to be determined such that *MSE* of $P^{I}_{(dss)}$ is minimum.

To the *fda*, the bias and *MSE* of $P^{I}_{(dss)}$ are respectively given by

$$B\left(P^{I}_{(dss)}\right) = \bar{Y}(\alpha_2\theta_5 - 1), \tag{60}$$

$$MSE\left(P^{I}_{(dss)}\right) = \bar{Y}^2\left[1 + \alpha_2^2\theta_2 - 2\alpha_2\theta_5\right]. \tag{61}$$

where

$$\theta_2 = \left[1 + \frac{1}{\bar{Y}^2}\left\{\gamma S_y^2 + \frac{1}{n'}\left(\theta_y + \frac{\tau\eta R^2 A_x}{2}\left[\tau(\eta+1) - 4k\right]\right)\right\}\right],$$

$$\theta_5 = \left[1 + \frac{\tau\eta\theta_x}{8n'\bar{X}^2}\{\tau(\eta+2) - 4k\}\right].$$

The $MSE\left(P^I_{(dss)}\right)$ at (61) is minimized for

$$\alpha = \frac{\theta_5}{\theta_2} = \alpha_{2(opt)}\text{, say.} \tag{62}$$

Thus, the resulting minimum $MSE$ of $P^I_{(dss)}$ is given by

$$MSE_{\min}\left(P^I_{(dss)}\right) = \bar{Y}^2\left[1 - \frac{\theta_5^2}{\theta_2}\right]. \tag{63}$$

Now we arrived at the following theorem.

**Theorem 2:** Up to terms of order $O\left(n^{-1}\right)$,

$$MSE\left(P^I_{(dss)}\right) \geq \bar{Y}^2\left[1 - \frac{\theta_5^2}{\theta_2}\right]$$

with equality holding if

$$\alpha_2 = \frac{\theta_5}{\theta_2}.$$

For comparing $\hat{\bar{Y}}^P_{CR(dss)}$ with $P^I_{(dss)}$, we express (44) in terms of $\theta_2$ and $\theta_5$ as

$$MSE\left(\hat{\bar{Y}}^P_{CR(dss)}\right) = \bar{Y}^2\left[1 + \theta_2 - 2\theta_5\right]. \tag{64}$$

Now from (63) and (64), we have

$$MSE\left(\hat{\bar{Y}}^P_{CR(dss)}\right) - MSE_{\min}\left(P^I_{(dss)}\right) = \bar{Y}^2\,\frac{(\theta_2 - \theta_5)^2}{\theta_2} \geq 0,$$

which follows that

$$MSE_{\min}\left(P^I_{(dss)}\right) \leq MSE\left(\hat{\bar{Y}}^P_{CR(dss)}\right). \tag{65}$$

Thus from (65) we conclude that the improved estimator $P^I_{(dss)}$ is more efficient than the suggested estimator $\hat{\bar{Y}}^P_{CR(dss)}$ under the optimum condition (62). Also, the estimators belonging to the improved family of estimators $P_{(dss)}^{I}$ are more efficient as compared to the estimators belonging to suggested class of estimators $\hat{\bar{Y}}_{CR(dss)}^{P}$.

## 5. A General Class of Estimators

Following the procedure adopted by Upadhyaya *et al*. (1985), we define a generalized class of estimators in *DSS* for population mean $\bar{Y}$ as

$$P_{(dss)}^{G} = \alpha_1 \bar{y}_{ds} + \alpha_2 \bar{y}_{ds} \left( \frac{\xi \bar{x}' + \psi}{\xi \bar{x}_{ds} + \psi} \right)^{\lambda} \exp\left\{ \frac{\omega \xi (\bar{x}' - \bar{x}_{ds})}{\xi (\bar{x}' + \bar{x}_{ds}) + 2\psi} \right\}, \tag{66}$$

where $(\alpha_1, \alpha_2)$ are constants to be determined such that the *MSE* of $P_{(dss)}^{G}$ is minimum; and the scalars $(\xi, \psi, \omega, \lambda)$ are same as defined earlier.

To the *fda*, the bias and *MSE* of the generalized class of estimators $P_{(dss)}^{G}$ are respectively given by

$$B\left(P_{(dss)}^{G}\right) = \bar{Y}\left[\alpha_1 \theta_4 + \alpha_2 \theta_5 - 1\right], \tag{67}$$

$$MSE\left(P_{(dss)}^{G}\right) = \bar{Y}^2 \left[1 + \alpha_1^2 \theta_1 + \alpha_2^2 \theta_2 + 2\alpha_1 \alpha_2 \theta_3 - 2\alpha_1 \theta_4 - 2\alpha_2 \theta_5\right], \tag{68}$$

where

$$\theta_1 = \left[1 + \frac{1}{\bar{Y}^2}\left(\gamma S_y^2 + \frac{1}{n'}\theta_y\right)\right],$$

$$\theta_3 = \left[1 + \frac{1}{\bar{Y}^2}\left(\gamma S_y^2 + \frac{1}{n'}\theta_y\right) + \frac{\tau \eta \theta_x}{8n' \bar{X}^2}\left\{(\eta + 2)\tau - 8k\right\}\right],$$

$$\theta_4 = 1.$$

$\theta_2$ and $\theta_5$ are same as defined previously.

Minimization of (68) with respect to $(\alpha_1, \alpha_2)$ gives

$$\begin{bmatrix} \theta_1 & \theta_3 \\ \theta_3 & \theta_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \theta_4 \\ \theta_5 \end{bmatrix}. \tag{69}$$

After simplification of (69), we get the optimum values of $(\alpha_1, \alpha_2)$ respectively as

$$\left.\begin{aligned}\alpha_{1(opt)} &= \frac{\Delta_1}{\Delta_0} \\ \alpha_{2(opt)} &= \frac{\Delta_2}{\Delta_0}\end{aligned}\right\} , \tag{70}$$

where

$$\Delta_0 = \begin{vmatrix}\theta_1 & \theta_3 \\ \theta_3 & \theta_2\end{vmatrix} = \left(\theta_1\theta_2 - \theta_3^2\right),$$

$$\Delta_1 = \begin{vmatrix}\theta_4 & \theta_3 \\ \theta_5 & \theta_2\end{vmatrix} = \left(\theta_2\theta_4 - \theta_3\theta_5\right),$$

$$\Delta_2 = \begin{vmatrix}\theta_1 & \theta_4 \\ \theta_3 & \theta_5\end{vmatrix} = \left(\theta_1\theta_5 - \theta_3\theta_4\right).$$

Thus the resulting minimum *MSE* of $P_{(dss)}^{G}$ is given by

$$MSE_{\min}\left(P_{(dss)}^{G}\right) = \bar{Y}^2\left[1 - \frac{\left\{\theta_2\theta_4^2 - 2\theta_3\theta_4\theta_5 + \theta_1\theta_5^2\right\}}{\left(\theta_1\theta_2 - \theta_3^2\right)}\right]. \tag{71}$$

**Theorem 3:** Up to terms of order $O\left(n^{-1}\right)$,

$$MSE\left(P_{(dss)}^{G}\right) \geq \bar{Y}^2\left[1 - \frac{\left\{\theta_2\theta_4^2 - 2\theta_3\theta_4\theta_5 + \theta_1\theta_5^2\right\}}{\left(\theta_1\theta_2 - \theta_3^2\right)}\right]$$

with equality if

$$\alpha_{1(opt)} = \frac{\Delta_1}{\Delta_0}, \alpha_{2(opt)} = \frac{\Delta_2}{\Delta_0}.$$

Now from (63) and (71), we have

$$MSE_{\min}\left(P_{(dss)}^{I}\right) - MSE_{\min}\left(P_{(dss)}^{G}\right) = \bar{Y}^2 \frac{\left(\theta_2\theta_4 - \theta_3\theta_5\right)^2}{\theta_2\left(\theta_1\theta_2 - \theta_3^2\right)} \geq 0$$

which follows that

$$MSE_{\min}\left(P_{(dss)}^{G}\right) \leq MSE\left(P_{(dss)}^{I}\right). \tag{72}$$

Combining the inequalities (65) and (72) we have

$$MSE_{\min}\left(P_{(dss)}^{G}\right) \leq MSE\left(P_{(dss)}^{I}\right) \leq MSE\left(\hat{\bar{Y}}_{CR(dss)}^{P}\right). \tag{73}$$

From (73), we infer that the generalized estimator $P_{(dss)}^{G}$ is more efficient than the suggested improved estimator $P_{(dss)}^{G}$ and the estimator $\hat{\hat{Y}}_{CR(dss)}^{P}$.

## 6. Numerical Illustration

To demonstrate the performance of the various estimators of the population mean $\bar{Y}$ of $y$, we have taken two data sets. Description of the population data sets are given below.

**Data 1** [Source: Chouhan (2012)]
$y$: Productivity (MT/Hectare), $x$: Production in '000 Tons
$N$=20, $n$=8, $n_1 = 4, n_2 = 4, n'_1 = 7, n'_2 = 7, N_1 = 10, N_2 = 10, \bar{Y}_1 = 1.70, \bar{Y}_2 = 3.67, \bar{Y} = 2.685,$
$\bar{X}_1 = 10.41, \bar{X}_2 = 289.14, \bar{X} = 149.705, S_{x1} = 3.53, S_{x2} = 111.61, S_{y1} = 0.50, S_{y2} = 1.41, S_{yx1} = 1.60,$
$S_{yx2} = 144.87, S_y^2 = 2.20, R = 0.018.$

**Data 2** [Source: Murthy (1967), p228]
$y$: Output, $x$: Fixed capital,
$N$=10, $n$=4, $n_1 = 2, n_2 = 2, n'_1 = 4, n'_2 = 4, N_1 = 5, N_2 = 5, \bar{Y}_1 = 1925.8, \bar{Y}_2 = 3115.6, \bar{Y} = 1260.35$
$\bar{X}_1 = 214.4, \bar{X}_2 = 333.8, \bar{X} = 137.05, S_{x1} = 74.87, S_{x2} = 66.35, S_{y1} = 615.92, S_{y2} = 340.38,$
$S_{yx1} = 39360.68, S_{yx2} = 22356.50, S_y^2 = 668351.00, R = 9.196.$

We have computed the percent relative efficiencies ($PRE$s) of estimators $\hat{\hat{Y}}_{R(dss)}$, $\hat{\hat{Y}}_{\mathrm{Re}(dss)}$, $\hat{\hat{Y}}_{R(dss)}^{C}$, $\hat{\hat{Y}}_{\mathrm{Re}(dss)}^{C}$, $\hat{\hat{Y}}_{R\mathrm{Re}(dss)}^{C}$ with respect to usual unbiased estimator $\bar{y}_{ds}$ by using the following formulae:

$$PRE\left(\hat{\hat{Y}}_{R(dss)}, \bar{y}_{ds}\right) = PRE\left(\hat{\hat{Y}}_{\mathrm{Re}(dss)}^{C}, \bar{y}_{ds}\right) = \frac{\left\{\gamma S_y^2 + \left(\frac{\theta_y}{n'}\right)\right\}}{\left[\gamma S_y^2 + \left(\frac{1}{n'}\right)\left\{\theta_y + R^2\theta_x(1-2k)\right\}\right]} * 100, \tag{74}$$

$$PRE\left(\hat{\hat{Y}}_{\mathrm{Re}(dss)}, \bar{y}_{ds}\right) = \frac{\left\{\gamma S_y^2 + \left(\frac{\theta_y}{n'}\right)\right\}}{\left[\gamma S_y^2 + \left(\frac{1}{n'}\right)\left\{\theta_y + \frac{R^2\theta_x}{4}(1-4k)\right\}\right]} * 100, \tag{75}$$

$$PRE\left(\hat{\hat{Y}}_{R(dss)}^{C}, \bar{y}_{ds}\right) = \frac{\left\{\gamma S_y^2 + \left(\frac{\theta_y}{n'}\right)\right\}}{\left[\gamma S_y^2 + \left(\frac{1}{n'}\right)\left\{\theta_y + 4R^2\theta_x(1-k)\right\}\right]} * 100, \tag{76}$$

$$PRE\left(\hat{\bar{Y}}^{C}_{R\,\mathrm{Re}(dss)}, \bar{y}_{ds}\right) = \frac{\left\{\gamma S_y^2 + \left(\frac{\theta_y}{n'}\right)\right\}}{\left[\gamma S_y^2 + \left(\frac{1}{n'}\right)\left\{\theta_y + 3R^2\theta_x(0.75-k)\right\}\right]} * 100, \tag{77}$$

and findings are displayed in Table 1.

The *PRE*s of proposed class of estimators $\hat{\bar{Y}}^{P}_{CR(dss)}$, improved class of estimators $P^{I}_{(dss)}$ and generalized class of estimators $P^{G}_{(dss)}$ with respect to $\bar{y}_{ds}$ have been computed by using the formulae:

$$PRE\left(\hat{\bar{Y}}^{P}_{CR(dss)}, \bar{y}_{ds}\right) = \frac{\left\{\gamma S_y^2 + \left(\frac{\theta_y}{n'}\right)\right\}}{\bar{Y}^2\left[1+\theta_2 - 2\theta_5\right]} * 100, \tag{78}$$

$$PRE\left(P^{I}_{(dss)}, \bar{y}_{ds}\right) = \frac{\theta_2\left\{\gamma S_y^2 + \left(\frac{\theta_y}{n'}\right)\right\}}{\bar{Y}^2\left(\theta_2 - \theta_5^2\right)} * 100, \tag{79}$$

$$PRE\left(P^{G}_{(dss)}, \bar{y}_{ds}\right) = \frac{\left\{\gamma S_y^2 + \left(\frac{\theta_y}{n'}\right)\right\}}{\bar{Y}^2\left[1 - \frac{\left(\theta_2\theta_4^2 - 2\theta_3\theta_4\theta_5 + \theta_1\theta_5^2\right)}{\left(\theta_1\theta_2 - \theta_3^2\right)}\right]} * 100 \tag{80}$$

for different values of $(\xi, \psi, \lambda, \omega)$. Results are shown in Table 2.

**Table 1: *PRE* of Different estimators with respect to $\bar{y}_{ds}$**

| Estimators | Data 1 | Data 2 |
|---|---|---|
| | ***PRE*** | ***PRE*** |
| $\bar{y}_{ds}$ | 100 | 100 |
| $\hat{\bar{Y}}_{R(dss)}$ or $\hat{\bar{Y}}^{C}_{\mathrm{Re}(dss)}$ | 145.18 | 138.96 |
| $\hat{\bar{Y}}_{\mathrm{Re}(dss)}$ | 168.40 | 152.82 |
| $\hat{\bar{Y}}^{C}_{R(dss)}$ | 41.98 | 48.01 |
| $\hat{\bar{Y}}^{C}_{R\,\mathrm{Re}(dss)}$ | 77.83 | 83.62 |

**Table 2: *PRE*s of $\hat{\hat{Y}}^{P}_{CR(dss)}, P^{I}_{(dss)}$ and $P^{G}_{(dss)}$ with respect to $\bar{y}_{ds}$ for different values of scalars $(\xi, \psi, \lambda, \omega)$**

| Values of scalars | | | | Data 1 | | | Data 2 | | |
|---|---|---|---|---|---|---|---|---|---|
| $\xi$ | $\psi$ | $\lambda$ | $\omega$ | *PRE* $\hat{\hat{Y}}^{P}_{CR(dss)}$ | *PRE* $P^{I}_{(dss)}$ | *PRE* $P^{G}_{(dss)}$ | *PRE* $\hat{\hat{Y}}^{P}_{CR(dss)}$ | *PRE* $P^{I}_{(dss)}$ | *PRE* $P^{G}_{(dss)}$ |
| 1 | 0 | 1 | –1 | 168.4 | 170.13 | 177.37 | 152.82 | 155.05 | 160.94 |
| 1 | 0 | 0.75 | –0.75 | 154.23 | 155.77 | 177.14 | 142.40 | 144.46 | 160.70 |
| 1 | 0 | 0.5 | –0.5 | 136.19 | 137.64 | 176.92 | 128.90 | 130.87 | 160.49 |
| 1 | 0 | 0.25 | –0.25 | 117.44 | 118.89 | 176.73 | 114.28 | 116.25 | 160.28 |
| 1 | 0 | 1 | 0 | 145.18 | 148.54 | 178.45 | 138.96 | 142.81 | 161.99 |
| 1 | 0 | 0.25 | 0.25 | 154.23 | 155.77 | 177.14 | 142.40 | 144.46 | 160.70 |
| 1 | 0 | 0.5 | 0.5 | 172.81 | 175.27 | 177.88 | 157.32 | 160.24 | 161.44 |
| 1 | 0 | 0.75 | 0.75 | 126.45 | 130.17 | 178.76 | 124.98 | 129.27 | 162.29 |
| 1 | 0 | 0.75 | 1 | 121.21 | 112.2 | 179.09 | 110.33 | 115.00 | 162.60 |
| *PRE*s at optimum value | | | | 175.54 | 177.66 | 177.71 | 158.62 | 161.26 | 161.27 |

It is observed from Table 1 that the estimators $\hat{\hat{Y}}_{R(dss)}$ and $\hat{\hat{Y}}_{\mathrm{Re}(dss)}$ are equally efficient. The estimators $\hat{\hat{Y}}^{C}_{R(dss)}, \hat{\hat{Y}}^{C}_{\mathrm{Re}(dss)}$ and $\hat{\hat{Y}}^{C}_{R\mathrm{Re}(dss)}$ are more efficient than the conventional unbiased estimator $\bar{y}_{ds}$ with considerable gain in efficiency. The chain estimators $\hat{\hat{Y}}^{C}_{R(dss)}$ and $\hat{\hat{Y}}^{C}_{R\mathrm{Re}(dss)}$ are even inferior to the unbiased estimator $\bar{y}_{ds}$.

Table 2 shows that the proposed estimators $\hat{\hat{Y}}^{P}_{CR(dss)}, P^{I}_{(dss)}$ and $P^{G}_{(dss)}$ are more efficient than the usual unbiased estimator $\bar{y}_{ds}$ for selected values of $(\xi, \psi, \lambda, \omega)$. The proposed chain estimator $\hat{\hat{Y}}^{P}_{CR(dss)}$ (at optimum value of $\eta$) is more efficient than the estimators $\bar{y}_{ds}$, $\hat{\hat{Y}}_{R(dss)}$, $\hat{\hat{Y}}_{\mathrm{Re}(dss)}, \hat{\hat{Y}}^{C}_{R(dss)}$, $\hat{\hat{Y}}^{C}_{\mathrm{Re}(dss)}$ and $\hat{\hat{Y}}^{C}_{R\mathrm{Re}(dss)}$.

Table 2 also exhibits that the generalized estimator $P^{G}_{(dss)}$ is the best (in the sense of having least *MSE*) among all the estimators $\bar{y}_{ds}, \hat{\hat{Y}}_{R(dss)}, \hat{\hat{Y}}_{\mathrm{Re}(dss)}, \hat{\hat{Y}}^{C}_{R(dss)}, \hat{\hat{Y}}^{C}_{\mathrm{Re}(dss)}, \hat{\hat{Y}}^{C}_{R\mathrm{Re}(dss)}, \hat{\hat{Y}}^{P}_{CR(dss)}$ and $P^{I}_{(dss)}$ discussed here. Thus there is enough scope of selecting the scalars $(\xi, \psi, \lambda, \omega)$ involved in the proposed class of estimators $\hat{\hat{Y}}^{P}_{CR(dss)}, P^{I}_{(dss)}$ and $P^{G}_{(dss)}$ obtaining better than the estimators $\bar{y}_{ds}$, $\hat{\hat{Y}}_{R(dss)}, \hat{\hat{Y}}_{\mathrm{Re}(dss)}, \hat{\hat{Y}}^{C}_{R(dss)}$, $\hat{\hat{Y}}^{C}_{\mathrm{Re}(dss)}$ and $\hat{\hat{Y}}^{C}_{R\mathrm{Re}(dss)}$.

So the proposed estimators $\hat{\hat{Y}}^{P}_{CR(dss)}, P^{I}_{(dss)}$ and $P^{G}_{(dss)}$ are recommended for their use in practice.

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