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Constant Block-Sum Two-Associate Class Group Divisible Designs

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Abstract

It is shown that classes of semi-regular and regular group divisible designs do not lead to constant block-sum designs. Construction of constant block-sum designs using singular group divisible designs is discussed in general. For a given singular group divisible design, the construction method is shown to provide a large number of distinct constant block-sum designs. Construction of constant block-sum designs for equispaced treatment levels is also discussed.

Key words: Balanced incomplete block design; Eigenvalue; Eigenvector; Partially balanced.

1. Introduction

Recently Khattree (2018a,b) discussed the concept of constant block-sum designs for quantitative treatment levels. In these designs, the sum of the treatment levels in each block is constant. Non-existence of constant block-sum balanced incomplete designs was established by Khattree (2018a, 2020). Several methods of construction have been presented by Khattree (2019). A general approach to determine whether or not a design can be transformed into a constant block-sum design and its construction if it exists has been developed in Khattree (2020). Bansal and Garg (2020) derived some conditions for existence of partially balanced constant block-sum designs and gave further combinatorial methods of construction. Khattree (2020) discussed some individual examples, including two-associate class group divisible (GD) designs. The purpose of this note is to present results with respect to the property of constant block-sum that apply to the whole class of GD designs. Non-existence of constant block-sum designs is established for classes of semi-regular and regular GD designs. Construction of constant block-sum singular GD designs is discussed in general. Existence of a large number of distinct constant block-sum solutions for a given singular GD design is illustrated with the help of an example. Singular GD constant block-sum designs for equispaced treatment levels are discussed in Section 3.

2. Group Divisible Designs

In two-associate class GD designs, $v = m_1m_2$ treatments are arranged in m_1 groups of m_2 treatments each. Let the treatments be coded as 1, 2, $\cdots$, m_1m_2. Then it is convenient

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to form the groups as:

Table 1

1	2	. . .	m_2
m_1+1	m_1+2	. . .	$2m_2$
		.	
		.	
		.	
$m_2(m_1-1)+1$	$m_2(m_1-1)+2$	. . .	m_1m_2

The treatments are first associates if they belong to the same group and second associates otherwise. The parameters of a GD design are $v = m_1m_2$, b, r, k, λ_1, λ_2, m_1, m_2, where the symbols have their standard meaning, see Raghavarao (1971) or Dey (1986) for details. Let

$$\boldsymbol{A} = \boldsymbol{N}\boldsymbol{N}' - \frac{rk}{v}\boldsymbol{J}_v$$

where $\boldsymbol{N}$ is the $v \times b$ incidence matrix and $\boldsymbol{J}_t$ denotes a square matrix of one's of size t. Note that $\mathbf{1}_v$, a vector of ones of size $v \times 1$, is an eigenvector of $\boldsymbol{A}$ corresponding to a zero eigenvalue.

For an equireplicate partially balanced design, Khattree (2020) showed that a necessary condition for existence of a constant block-sum design is that

$$\boldsymbol{A}\boldsymbol{w} = \mathbf{0}$$

where $\boldsymbol{w} \neq \mathbf{1}_v$ is an eigenvector of $\boldsymbol{A}$ corresponding to a zero eigenvalue. Note that this is not a sufficient condition, as it is possible that a vector $\boldsymbol{w}$ satisfying the necessary condition does not have all of its elements different from each other. If the v elements of $\boldsymbol{w}$ are all different from each other, they are taken as v treatment levels to yield a constant block-sum design.

As $\boldsymbol{A}$ and $\boldsymbol{N}\boldsymbol{N}'$ are symmetric matrices, they both admit their spectral decompositions. Also, $\boldsymbol{N}\boldsymbol{N}'\mathbf{1}_v = rk\mathbf{1}_v$, so it can be easily seen that if $\boldsymbol{w} \neq \mathbf{1}_v$ is an eigenvector of $\boldsymbol{A}$ corresponding to a zero eigenvalue then it is also an eigenvector of $\boldsymbol{N}\boldsymbol{N}'$ corresponding to a zero eigenvalue and vice versa. Thus, equivalently, we have the following theorem.

Theorem 1: A necessary condition for the existence of a constant block-sum design is that $\boldsymbol{N}\boldsymbol{N}'$ is singular.

Remark 1: Singularity of $\boldsymbol{N}\boldsymbol{N}'$ in turn implies that the rows of $\boldsymbol{N}$ are not linearly independent.

Remark 2: Statement of Remark 1 is automatically satisfied if $v > b$, since $\text{Rank}(\boldsymbol{N}) \leq \min(v, b) < v$.

The structure of $\boldsymbol{N}\boldsymbol{N}'$ for GD designs as given below and its eigenvectors and eigenvalues given in Lemma 1 are well known, see *e.g.* Nigam, Puri and Gupta (1988).

$$\boldsymbol{NN'} = \begin{bmatrix} (r-\lambda_1)\boldsymbol{I}_{m_2}+\lambda_1\boldsymbol{J}_{m_2} & \lambda_2\boldsymbol{J}_{m_2} & \lambda_2\boldsymbol{J}_{m_2} & \cdots & \lambda_2\boldsymbol{J}_{m_2} \\ \lambda_2\boldsymbol{J}_{m_2} & (r-\lambda_1)\boldsymbol{I}_{m_2}+\lambda_1\boldsymbol{J}_{m_2} & \lambda_2\boldsymbol{J}_{m_2} & \cdots & \lambda_2\boldsymbol{J}_{m_2} \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \\ \lambda_2\boldsymbol{J}_{m_2} & \lambda_2\boldsymbol{J}_{m_2} & \lambda_2\boldsymbol{J}_{m_2} & \cdots & (r-\lambda_1)\boldsymbol{I}_{m_2}+\lambda_1\boldsymbol{J}_{m_2} \end{bmatrix}$$

$$= (r-\lambda_1)\boldsymbol{I}_{m_1}\otimes\boldsymbol{I}_{m_2} + (\lambda_1-\lambda_2)\boldsymbol{I}_{m_1}\otimes\boldsymbol{J}_{m_2} + \lambda_2\boldsymbol{J}_{m_1}\otimes\boldsymbol{J}_{m_2}$$

where $\boldsymbol{I}_q$ and $\boldsymbol{J}_q$ denote respectively an identity matrix and a square matrix of one's, both of order q, and $\otimes$ is the (right) kronecker product. Let $\boldsymbol{u}_{1i}$, $i = 1, 2, \cdots, (m_1-1)$ be orthonormal column vectors of size m_1 each, such that $\boldsymbol{u}'_{1i}\boldsymbol{1}_{m_1} = 0$, $\boldsymbol{u}'_{1i}\boldsymbol{u}_{1i} = 1$, and $\boldsymbol{u}'_{1i}\boldsymbol{u}_{1i_1} = 0$, $i \neq i_1 = 1, 2, \cdots, (m_1-1)$. Similarly, let $\boldsymbol{u}_{2j}$, $j = 1, 2, \cdots, (m_2-1)$ be orthonormal column vectors of size m_2 each, such that $\boldsymbol{u}'_{2j}\boldsymbol{1}_{m_2} = 0$, $\boldsymbol{u}'_{2j}\boldsymbol{u}_{2j} = 1$, and $\boldsymbol{u}'_{2j}\boldsymbol{u}_{2j_1} = 0$, $j \neq j_1 = 1, 2, \cdots, (m_2-1)$. Without loss of generality, we take normalized orthogonal polynomial contrasts as $\boldsymbol{u}_{1i}$ and $\boldsymbol{u}_{2j}$, $i = 1, 2, \cdots, (m_1-1)$,$j = 1, 2, \cdots, (m_2-1)$.

Lemma 1:

- (a) $\boldsymbol{w}_{1i} = \boldsymbol{u}_{1i}\otimes\boldsymbol{1}_{m_2}$, $i = 1, 2, \cdots, (m_1-1)$ constitute a set of (m_1-1) eigenvectors of $\boldsymbol{NN'}$ corresponding to the constant eigenvalue of $\theta_1 = (rk - v\lambda_2)$,
- (b) $\boldsymbol{w}_{2j} = \boldsymbol{1}_{m_1}\otimes\boldsymbol{u}_{2j}$, $\boldsymbol{w}_{12ij} = \boldsymbol{u}_{1i}\otimes\boldsymbol{u}_{2j}$, $i = 1, 2, \cdots, (m_1-1)$; $j = 1, 2, \cdots, (m_2-1)$ constitute a set of $m_1(m_2-1)$ eigenvectors of $\boldsymbol{NN'}$ corresponding to the constant eigenvalue of $\theta_2 = (r-\lambda_1)$,
- (c) $\boldsymbol{1}_{m_1}\otimes\boldsymbol{1}_{m_2}$ is an eigenvector of $\boldsymbol{NN'}$ corresponding to the eigenvalue of $\theta_0 = rk$, and
- (d) the m_1m_2 eigenvectors of $\boldsymbol{NN'}$ in (a), (b), and (c) are mutually orthogonal.

GD designs are called singular if $r = \lambda_1$, semi-regular if $r > \lambda_1$ and $rk = v\lambda_2$, and regular if $r > \lambda_1$ and $rk > v\lambda_2$. Let us first consider the class of semi-regular GD (SRGD) designs. It can be seen that $\theta_1 = 0$ and $\theta_2 > 0$ for SRGD designs. From Lemma 1, the following (m_1-1) eigenvectors of $\boldsymbol{NN'}$ correspond to an eigenvalue of zero as required in Theorem 1.

$$\boldsymbol{w}_{1i} = \boldsymbol{u}_{1i}\otimes\boldsymbol{1}_{m_2}, \qquad i = 1, 2, \cdots, m_1-1 \ .$$

However, it is easily seen that none of these eigenvectors on its own satisfies the requirement that all of its v elements be different from each other. Note that a linear combination of these m_1-1 eigenvectors is also an eigenvector of $\boldsymbol{NN'}$ corresponding to zero eigenvalue. So, let us consider the following general linear combination $\boldsymbol{t}_{1w}$, where c_i, $i = 1, 2, \cdots, (m_1-1)$ are some constants.

$$\begin{aligned} \boldsymbol{t}'_{1w} &= \sum_{i=1}^{m_1-1} c_i\left(\boldsymbol{u}'_{1i}\otimes\boldsymbol{1}'_{m_2}\right) \\ &= \left[\left(\sum_{i=1}^{m_1-1} c_iu_{1i1}\right)\boldsymbol{1}'_{m_2} \quad \left(\sum_{i=1}^{m_1-1} c_iu_{1i2}\right)\boldsymbol{1}'_{m_2} \quad \cdots \quad \left(\sum_{i=1}^{m_1-1} c_iu_{1im_1}\right)\boldsymbol{1}'_{m_2}\right] \end{aligned} \qquad (1)$$

where $\boldsymbol{u}'_{1i} = (u_{1i1} \ \ u_{1i2} \ \ \cdots \ \ u_{1im_1})\,,\ \ i = 1,\, 2, \cdots, (m_1 - 1)$. It is clear from equation (1) that there does not exist a linear combination $\boldsymbol{t}_{1w}$ such that all of its $v = m_1 m_2$ elements are different from each other. Thus we can state the following result.

Theorem 2: There does not exist a constant block-sum semi-regular GD design.

Next, turning attention to the class of regular GD designs, note that both of the eigenvalues θ_1 and θ_2 of $\boldsymbol{NN}'$ for these designs are greater than zero. So, an eigenvector $\boldsymbol{w}$ per the necessary condition of Theorem 1 does not exist for the class of regular GD designs. Thus we have the following.

Theorem 3: There does not exist a regular GD constant block-sum design.

Finally, we now consider singular GD (SGD) designs for which the eigenvalue $\theta_2 = r - \lambda_1 = 0$. From Lemma 1, the following $m_1(m_2 - 1)$ eigenvectors satisfy the necessary condition of Theorem 1 for existence of constant block-sum designs.

$$\begin{aligned} \boldsymbol{w}_{2j} &= \boldsymbol{1}_{m_1} \otimes \boldsymbol{u}_{2j}, \quad && j = 1,\, 2, \cdots (m_2 - 1)\,, \\ \boldsymbol{w}_{12ij} &= \boldsymbol{u}_{1i} \otimes \boldsymbol{u}_{2j}, \quad && i = 1,\, 2, \cdots, (m_1 - 1)\,;\ j = 1,\, 2, \cdots, (m_2 - 1) \end{aligned}$$

None of these $m_1(m_2 - 1)$ eigenvectors on its own satisfies the requirement that all of its $m_1 m_2$ elements be different from each other. So, we explore a linear combination $\boldsymbol{t}_{2w}$ of the $m_1(m_2 - 1)$ eigenvectors, that is also an eigenvector of $\boldsymbol{NN}'$ with zero eigenvalue, such that its $m_1 m_2$ elements are different from each other.

$$\boldsymbol{t}_{2w} = \sum_{j=1}^{m_2-1} c_{2j} \boldsymbol{w}_{2j} + \sum_{i=1}^{m_1-1} \sum_{j=1}^{m_2-1} c_{12ij} \boldsymbol{w}_{12ij} \tag{2}$$

where $c_{1j},\, c_{12ij},\ i = 1,\, 2, \cdots, (m_1 - 1)\,;\ j = 1,\, 2, \cdots, (m_2 - 1)$ are some constants. For illustration, we consider the following example.

Example 1: Consider the SGD design S21 in Clatworthy (1973) tables with parameters $v = 9,\ b = 3,\ r = 2,\ k = 6,\ \lambda_1 = 2,\ \lambda_2 = 1,\ m_1 = m_2 = 3:$

Block No.	Block contents					
1	1	2	3	4	5	6
2	1	2	3	7	8	9
3	4	5	6	7	8	9

Here, $m_1(m_2 - 1) = 6$ orthonormal eigenvectors of $\boldsymbol{NN}'$ corresponding to zero eigenvalue are as follows.

$$\begin{bmatrix} \boldsymbol{w}'_{21} \\ \boldsymbol{w}'_{22} \\ \boldsymbol{w}'_{1211} \\ \boldsymbol{w}'_{1212} \\ \boldsymbol{w}'_{1221} \\ \boldsymbol{w}'_{1222} \end{bmatrix} = \begin{bmatrix} (-1 & 0 & +1 & -1 & 0 & +1 & -1 & 0 & +1)\,/\sqrt{6} \\ (+1 & -2 & +1 & +1 & -2 & +1 & +1 & -2 & +1)\,/3\sqrt{2} \\ (+1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & +1)\,/2 \\ (-1 & +2 & -1 & 0 & 0 & 0 & +1 & -2 & +1)\,/2\sqrt{3} \\ (-1 & 0 & +1 & +2 & 0 & -2 & -1 & 0 & +1)\,/2\sqrt{3} \\ (+1 & -2 & +1 & -2 & +4 & -2 & +1 & -2 & +1)\,/6 \end{bmatrix} \tag{3}$$

By taking $c_{21} = -0.03$, $c_{22} = 0.50$, $c_{1211} = -0.42$, $c_{1212} = 0.61$, $c_{1221} = -0.90$, and $c_{1222} = -0.43$, in equation (2) and using (3) we get:

$$\boldsymbol{t}'_{2w} = (-0.0679\ \ 0.2598\ \ -0.1920\ \ -0.2462\ \ -0.5224\ \ 0.7686\ \ 0.7043\ \ -0.4446\ \ -0.2598)$$

Adding a same constant value to all the elements of $\boldsymbol{t}_{2w}$ does not break the constant block-sum property. The elements of $\boldsymbol{t}^*_{2w}$ given below, obtained by adding $c_0 = 0.70$ to the elements of $\boldsymbol{t}_{2w}$, can be taken as treatment levels for constant block-sum property.

$$\boldsymbol{t}^{*\prime}_{2w} = (0.6321\ \ 0.9598\ \ 0.5080\ \ 0.4538\ \ 0.1776\ \ 1.4686\ \ 1.4043\ \ 0.2554\ \ 0.4402).$$

As a matter of fact, a very large number of solutions for $\boldsymbol{t}^*_{2w}$ can be found by varying the values of the six coefficients c_{21}, c_{22}, c_{1211}, c_{1212}, c_{1221}, c_{1222} of the linear combination $\boldsymbol{t}_{2w}$. Any set of six values of these coefficients that results in all the elements of $\boldsymbol{t}_{2w}$ to be different from each other would satisfy the constant block-sum property. Table 2 lists 5 other solutions for the treatment levels vector $\boldsymbol{t}^*_{2w}$ obtained by trial and error. The corresponding values of the six coefficients are listed in Table 3, where c_0 is the constant value added to the elements of $\boldsymbol{t}_{2w}$ to obtain $\boldsymbol{t}^*_{2w}$. Many more solutions can be found simply by taking other values for the coefficients such that all the elements of $\boldsymbol{t}_{2w}$ are different from each other.

Table 2: Further solutions for Example 1

$\boldsymbol{t}^{*\prime}_{2w}$ No.	$\boldsymbol{t}^{*\prime}_{2w}$
1	0.7980 0.3685 1.8336 0.8232 1.1953 0.9815 0.8612 0.0221 2.1168
2	0.7480 0.4685 1.7836 0.9232 0.9953 1.0815 0.8112 0.1221 2.0668
3	0.6980 0.5685 1.7336 1.0232 0.7953 1.1815 0.7612 0.2221 2.0168
4	0.8771 0.6185 1.5044 1.2773 0.6953 1.0274 0.9403 0.2721 1.7876
5	1.5412 1.1447 0.9140 1.5833 1.9138 0.1029 0.7722 0.6829 2.1450

Table 3: Coefficient values for $\boldsymbol{t}^{*\prime}_{2w}$ listed in Table 2

$\boldsymbol{t}^{*\prime}_{2w}$ No.	c_{21}	c_{22}	c_{1211}	c_{1212}	c_{1221}	c_{1222}	c_0
1	1.00	1.00	0.11	0.30	0.57	1.00	1.00
2	1.00	1.00	0.11	0.30	0.57	0.70	1.00
3	1.00	1.00	0.11	0.30	0.57	0.40	1.00
4	0.50	1.00	0.11	0.30	0.57	0.25	1.00
5	-0.30	-0.10	1.00	0.40	1.07	1.00	1.20

Remark 3: For comparing treatments with respect to their effects, it is natural that treatment levels will be determined by subject matter specialists based on the objectives of their study. Example 1 illustrates the conundrum the experimenter is confronted with. What if none of the solutions illustrated in the example is a good choice of treatment levels for the study objectives? Note that for a $\boldsymbol{t}^*_{2w}$ of Table 2, $f_1\boldsymbol{t}^*_{2w} + f_2\mathbf{1}_9$ also satisfies the property of constant block-sum, where $f_1 > 0$ is a constant and f_2 is another constant such that all the treatment levels are greater than zero. Of course, we can also include more solutions

in Table 2 and hope that one of the solutions meets the study objectives. However, a systematic, perhaps algebraic, method of deriving possible solutions for unequally spaced and equispaced treatment levels in general deserves further research. Khattree (2019) has provided a detailed discussion on optimizing constant block-sum and nearly constant block-sum designs.

Sometimes a choice of $\boldsymbol{u}_{1i}$'s and $\boldsymbol{u}_{2i}$'s other than the orthogonal polynomial contrasts may yield a analytical solution directly without the need of forming linear combinations of eigenvectors. For instance, suppose in Example 1 we take $\boldsymbol{u}'_{11} = (1,\ 2,\ -3)/\sqrt{14}$, $\boldsymbol{u}'_{12} = (1,\ -1.25,\ -0.5)/\sqrt{2.8125}$, $\boldsymbol{u}'_{21} = (-5,\ 4,\ 1)/\sqrt{42}$, $\boldsymbol{u}'_{22} = (1,\ 2,\ -3)/\sqrt{14}$. Then, using Lemma 1,

$$\boldsymbol{w}'_{1211} = \boldsymbol{u}_{11} \otimes \boldsymbol{u}_{21} = (-5\ \ 4\ \ 1\ \ -10\ \ 8\ \ 2\ \ 15\ \ -12\ \ -3)/\sqrt{588} \tag{4}$$

is an eigenvector of $\boldsymbol{NN}'$ with zero eigenvalue having all of its elements different from each other. Thus,

$$\boldsymbol{t}^{*\prime}_{2w} = f_1(-5\ \ 4\ \ 1\ \ -10\ \ 8\ \ 2\ \ 15\ \ -12\ \ -3) + c_0\boldsymbol{1}'_9\,,$$

where $f_1 > 0$ and $c_0 > 12$ are some constants, satisfies the property of constant block-sum. The constants f_1 and c_0 can be chosen appropriately to suit experimenter's requirements with respect to the magnitude of treatment levels.

3. Equispaced Treatment Levels

The general approach illustrated in the previous section shows many possibilities for constant block-sum designs with unequally spaced treatment levels. However, if equispaced treatment levels are desired, SGD designs based on BIB designs in particular afford a solution directly without making use of the eigenvectors of $\boldsymbol{NN}'$. Consider a BIB design D with parameters $v_0 = m_1$, b_0, r_0, k_0, λ_0, with treatments coded as 1, 2, $\cdots$, m_1. Let D_{SGD} denote the design obtained by replacing treatment i in the BIB design by m_2 treatments $(i-1)m_2+1$, $(i-1)m_2+1$, $\cdots$, im_2, $i = 1, 2, \cdots, m_1$. Then D_{SGD} is a SGD design (Bose and Connor, (1952)) with parameters $v = m_1m_2$, $b = b_0$, $r = r_0$, $k = m_2k_0$, $\lambda_1 = r$, $\lambda_2 = \lambda_0$, m_1, m_2, with m_1 groups of treatments as given in Table 1. Let $\boldsymbol{T}$ be the vector of treatments given by,

$$\boldsymbol{T} = (1,\ 2,\ \cdots,\ m_2,\ m_2+1,\ m_2+2,\ \cdots,\ 2m_2, \cdots,\ m_1m_2)'\,. \tag{5}$$

Now suppose it is desired to transform SGD design D_{SGD} into a constant block-sum design for m_1m_2 equispaced treatment levels ℓ_i, $i = 1, 2, \cdots, m_1m_2$, where $\ell_i = \ell_1 + (i-1)d$, $d = \ell_i - \ell_{i-1}$, $i = 2, 3, \cdots, m_1m_2$., ℓ_1 being the lowest dose or treatment level. Let the vector of equispaced treatment levels can be written as,

$$\boldsymbol{T}_\ell = \ell_1\boldsymbol{1}_v + d\,\{0,\ 1, 2,\ \cdots,\ (m_1m_2-1)\}'\,. \tag{6}$$

In fact we only need to work with $\boldsymbol{T}_{\ell 0}$ as defined below, since $\boldsymbol{T}_\ell = \ell_1\boldsymbol{1}_v + d\boldsymbol{T}_{\ell 0}$,

$$\boldsymbol{T}_{\ell 0} = \{0,\ 1,\ 2,\ \cdots,\ (m_1m_2-1)\}'\,. \tag{7}$$

The sum of the m_1m_2 elements of $\boldsymbol{T}_{\ell 0}$, say ℓ_{SUM}, is then given by

$$\ell_{SUM} \;=\; \boldsymbol{T}'_{\ell 0}\mathbf{1}_v \;=\; \{m_1m_2\,(m_1m_2-1)\}\,/2\,.$$

Further suppose that it is possible to partition the $v = m_1m_2$ elements of $\boldsymbol{T}_{\ell 0}$ into m_1 groups of size m_2 each such that the sum of the m_2 elements within all the m_1 groups is equal to each other. Clearly, then the sum of m_2 elements in each group is equal to ℓ_{SUM}/m_1. Let the ith group, say G_i be denoted by,

$$\boldsymbol{G}_i \;=\; \left\{\ell^*_{\{(i-1)m_2+1\}},\; \ell^*_{\{(i-1)m_2+2\}},\; \cdots\;,\; \ell^*_{im_2}\right\}\,,$$

$$\sum_{j=1}^{m_2}\ell^*_{\{(i-1)m_2+j\}} \;=\; \frac{\ell_{SUM}}{m_1} \;=\; \frac{m_2\,(m_1m_2-1)}{2}, \qquad\qquad i \;=\; 1,\,2,\,\cdots\,,\,m_1\,,$$

$$\left\{\ell^*_{\{(i-1)m_2+1\}},\; \ell^*_{\{(i-1)m_2+2\}},\; \cdots\;,\; \ell^*_{im_2}\right\} \in \{0,\;1,\;2,\;\cdots\;,\;(m_1m_2-1)\}\;,$$

$$\boldsymbol{G}_1 \cup \boldsymbol{G}_2\;\cdots\;\cup \boldsymbol{G}_{m_1} \;\equiv\; \boldsymbol{T}_{\ell 0} \;=\; \{0,\;1,\;2,\;\cdots\;,\;(m_1m_2-1)\}\;.$$

Then a constant block-sum design equispaced treatment levels vector $\boldsymbol{t}^*_{2w}$ is given by,

$$\boldsymbol{t}^*_{2w} \;=\; \ell_1\mathbf{1}_v + d\left(\ell^*_1,\,\ell^*_2,\,\cdots\,,\,\ell^*_{m_2},\,\ell^*_{m_2+1},\,\ell^*_{m_2+2},\,\cdots\,,\ell^*_{m_1m_2}\right)'\,. \tag{8}$$

An equispaced constant block-sum design D^*_{SGD} is obtained by replacing the ith element of $\boldsymbol{T}$ of (5) in design D_{SGD} by the $i$the element of $\boldsymbol{t}^*_{2w}$ of (8). The block size being m_2k_0, the treatment levels (8) imply that the constant block-sum equals $k_0\ell_{SUM}/m_1$. Alternatively, D^*_{SGD} can be obtained by replacing treatment i in the BIB design D by the m_2 elements of $\ell_1\mathbf{1}_{m_2} + d\boldsymbol{G}_i$, $i = 1,\,2,\,\cdots\,,\,m_1$. For illustration let us consider Example 1 again.

Example 1 continued: Let D be the BIB design with parameters $v_0 = b_0 = 3$, $r_0 = k_0 = 2$, $\lambda_0 = 1$, with blocks given by [1 2], [1 3], [2 3]. Then the SGD design S21 of Clatworthy (1973) is obtain by replacing treatment i in D by $m_2 = 3$ treatments as described above. Thus, replace treatments 1, 2, 3 in D by the treatment groups (1, 2, 3), (4, 5, 6) and (7, 8, 9) respectively to obtain the SGD design S21 or D_{SGD}. From (3.3) we have

$$\boldsymbol{T}_{\ell 0} \;=\; (0,\;1,\;2,\;3,\;4,\;5,\;6,\;7,\;8)'\;,$$

with $\ell_{SUM} = 36$. Taking $\boldsymbol{G}_1 = (0,\,4,\,8)$, $\boldsymbol{G}_2 = (1,\,5,\,6)$ and $\boldsymbol{G}_3 = (2,\,3,\,7)$, gives the sum of elements in each group to be $\ell_{SUM}/m_1 = 12$. Suppose $\ell = 1.5$ and $d = 0.3$. Then the requisite equispaced constant block-sum design D_{SGD} is obtained by replacing treatment i in the BIB design D by $m_2 = 3$ elements of $1.5\mathbf{1}_3 + d\boldsymbol{G}_i$, $i = 1,\,2,\,3$. The designs S22, S23, S24, and S25 in Clatworthy (1973) are obtained by taking replications of design S21. Corresponding constant block-sum designs can then be obtained by taking replications of D_{SGD}

Most of the SGD designs listed in Clatworthy (1973) are constructed using irreducible BIB designs. Let $D^{m_1}_{k_0}$ denote the irreducible BIB design for $v_0 = m_1$ treatments in blocks of size k_0. Then, the groups $\boldsymbol{G}_i$ for $m_2 = 2$ are as below, where the subscript 2 indicates the value of m_2,

$$\boldsymbol{G}_{2i} \;=\; \{(i-1)\,,\;\;(2m_1-i)\}\,, \qquad\qquad i = 1,\,2,\,\cdots\,,\,m_1\,.$$

The D_{SGD} designs corresponding to S1 to S20 can thus be obtained using $\boldsymbol{G}_{2i}, i = 1, 2, \cdots, m_1$. Constant block-sum designs for some other values of m_2 can also be similarly developed. The reader is also referred to Khattree (2019) for constructions of some equispaced SGD constant block-sum designs.

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