Statistics and Applications {ISSN 2454-7395 (online)}
Volume 24, No. 1, 2026 (New Series), pp 221–236
http://www.ssca.org.in/journal

Nonparametric Estimation and Analysis of Conditional Dynamic Failure Extropy in Bivariate Systems

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Received: 05 February 2025; Revised: 28 July 2025; Accepted: 25 September 2025

Abstract

This study introduces Conditional Dynamic Failure Extropy ($CDFE_X$), a novel measure for quantifying uncertainty in bivariate systems. $CDFE_X$ captures interdependence between components via the joint distribution. We develop nonparametric estimators for $CDFE_X$ and establish their asymptotic properties under mild conditions. Through simulations and real-world datasets, we demonstrate the robustness of the proposed estimators. Our results show that $CDFE_X$ outperforms both univariate Dynamic Failure Extropy (DFE_X) and Conditional Dynamic Cumulative Past Entropy, highlighting its potential to enhance reliability analysis in complex systems.

Key words: Conditional dynamic failure extropy; Bivariate reversed hazard rate; Bivariate mean inactivity time.

AMS Subject Classifications: 94A17, 62G07

1. Introduction

Extropy complements Shannon's entropy (Shannon, 1948), serving to quantify uncertainty in probability distributions. Differential extropy (Lad, 2015) extends this concept to continuous variables, enabling analysis of time-to-event data and lifetimes in continuous domains. In fields such as information theory, statistics, and reliability, it offers a valuable alternative for measuring uncertainty.

Extropy has evolved into a dynamic concept with applications in order statistics and record values (Qiu, 2017), and has been extended to residual lifetime analysis (Qiu and Jia, 2018). Estimators and goodness-of-fit tests Qiu and Jia (2017) have been developed, and bounds based on variational distance have also been explored (Yang *et al.*, 2019). Its adaptability has enabled broad use across reliability, risk assessment, and information theory, including in mixed systems (Qiu *et al.*, 2018). Continued research by Raqab and Qiu (2019), Noughabi and Jarrahiferiz (2019), and Krishnan *et al.* (2020) emphasizes the versatility and ongoing relevance of extropy in modern statistical analysis.

To extend extropy to multivariate systems, recent studies have introduced bivari-

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ate survival extropy (Krishnan and Sathar, 2022) and shift-dependent bivariate weighted survival extropy (Sathar and Krishnan, 2023), which capture interdependent behavior in two-component systems. These measures have found applications in reliability engineering, survival analysis, and forensic science, particularly for analyzing time-since-failure scenarios. By quantifying uncertainty in multi-component settings, such tools support decision-making in maintenance and system upgrades.

The remainder of the paper is structured as follows: Section 2 introduces conditional dynamic failure extropy and explores its main properties, including monotonicity and characterization results. Section 3 discusses nonparametric estimation using empirical and kernel methods, investigates their asymptotic behavior, and presents simulation and data-based illustrations.

2. Conditional dynamic failure extropy

In bivariate systems, marginal distributions alone are insufficient to capture joint dependence unless the variables are independent. When conditional distributions are known, both marginal and conditional components are essential for understanding the joint behavior. The following definition provides a conditional extension of bivariate failure extropy, based on the approach of Kayal (2021).

Definition 1: Let $X = (X_1, X_2)$ be a bivariate random vector with joint distribution function F. Define $Y_i = (X_i \mid X_1 < t_1, X_2 < t_2)$, $i = 1, 2$, representing the conditional distribution of X_i given both components fail within intervals $(0, t_1)$ and $(0, t_2)$, respectively. Then the distribution functions of Y_1 and Y_2 are

$$P(Y_1 \leq y_1) = \frac{F(y_1, t_2)}{F(t_1, t_2)}, \quad 0 < y_1 < t_1,$$

$$P(Y_2 \leq y_2) = \frac{F(t_1, y_2)}{F(t_1, t_2)}, \quad 0 < y_2 < t_2.$$

These conditional distributions capture the failure behavior of each component, given that both components have failed within the specified time window. The corresponding conditional dynamic failure extropy measures are

$$J_1^F(X : t_1, t_2) = -\frac{1}{2} \int_0^{t_1} \left[\frac{F(x_1, t_2)}{F(t_1, t_2)} \right]^2 dx_1, \tag{1}$$

$$J_2^F(X : t_1, t_2) = -\frac{1}{2} \int_0^{t_2} \left[\frac{F(t_1, x_2)}{F(t_1, t_2)} \right]^2 dx_2. \tag{2}$$

$CDFE_X$ quantifies the dispersion of uncertainty in a component's conditional lifetime, given joint system failure before thresholds t_1 and t_2. A value closer to zero indicates lower concentration (i.e., a broader spread of failure times), whereas a more negative value reflects greater concentration, meaning failures are more tightly clustered within the conditional window. Thus, $CDFE_X$ captures residual uncertainty under joint failure constraints, supporting applications in reliability modeling, risk assessment, and preventive maintenance planning.

Table 1: Expressions of conditional dynamic failure extropy ($CDFE_X$) for some well-known bivariate distributions

Bivariate Distribution $F(t_1,t_2)$	**Conditional Dynamic Failure Extropy** $J_i^F(X:t_1,t_2)$
$F(t_1,t_2)=t_i^{1+\theta\log t_j}\cdot t_j,\quad 0\le t_i,t_j\le 1$	$J_i^F=\dfrac{-t_i}{2\,(2\theta\log t_j+3)}$
$F(t_1,t_2)=\exp\left(2-\dfrac{1}{t_i}-\dfrac{1}{t_j}\right)$	$J_i^F=\dfrac{-t_i^2}{4},\quad i,j=1,2,\ i\neq j$
$F(t_1,t_2)=\dfrac{t_it_j}{bd},\quad 0\le t_i\le b,\ 0\le t_j\le d$	$J_i^F=\dfrac{-t_i}{6},\quad i,j=1,2,\ i\neq j$
$F(t_1,t_2)=\left(\dfrac{t_i}{b_i}\right)^{c_i}\left(\dfrac{t_j}{b_j}\right)^{c_j+\theta\log\left(\frac{t_i}{b_i}\right)}$ $\theta\le 0,\ 0\le t_i\le b_i,\ 0\le t_j\le b_j$	$J_i^F=\dfrac{-t_i}{2\left[2c_i+2\theta\log\left(\dfrac{t_j}{b_j}\right)+1\right]}$
$F(x_1,x_2)=\left(1+e^{-x_1}+e^{-x_2}\right)^{-1},$ $-\infty<x_1,x_2<\infty$	$J_i^F(X:t_1,t_2)=\dfrac{-1}{2(1+e^{t_j})^2}\,e^{2t_j}\left(1+e^{-t_i}+e^{-t_j}\right)^2 \times\left[\dfrac{-1}{2+e^{-t_j}}+\dfrac{1}{1+e^{t_i}(1+e^{t_j})} -\log(1+2e^{t_j})+\log\left(e^{t_i}e^{t_j}(1+e^{t_i})\right)\right],$ for $i,j=1,2,\ i\neq j$

Example 1: To illustrate the physical meaning of $CDFE_X$, we analyze failure times (in months) for a two-motor parallel system (Table 2). Assuming a bivariate normal distribution fitted to the data, we compute the conditional dynamic failure extropies $J_1^F(t_1,t_2)$ and $J_2^F(t_1,t_2)$. Figure 1 presents a 3D overlay J_1^F for Motor A (black) and J_2^F for Motor B (white). The plot shows that $|J_2^F|$ is consistently greater than $|J_1^F|$, indicating that, conditional on joint failure before (t_1,t_2), Motor B's failure times are more concentrated, while Motor A shows greater variability. This supports the interpretation of $CDFE_X$ as a measure of conditional uncertainty, offering valuable guidance for reliability planning and maintenance prioritization.

Table 2: Failure times (in months) for motors A and B in 11 parallel systems

System	1	2	3	4	5	6	7	8	9	10	11
Motor A	3.4	2.8	2.93	5.2	4.93	4.63	8.16	7.83	7.33	6.9	8.33
Motor B	2.16	4.93	6.73	4.03	4.1	5.0	5.2	5.73	7.33	7.13	7.06

The following theorem establishes an upper bound for $CDFE_X$.

Theorem 1: Let $X=(X_1,X_2)$ be a bivariate random vector with joint distribution function F, and let $t_1,t_2>0$. Consider the event $A=\{X_1<t_1,X_2<t_2\}$, representing the joint failure of both components before times t_1 and t_2. Then, the conditional dynamic failure extropy of each component satisfies

$$J_i^F(X;t_1,t_2)\le m_i^X(t_1,t_2),\quad i=1,2,$$

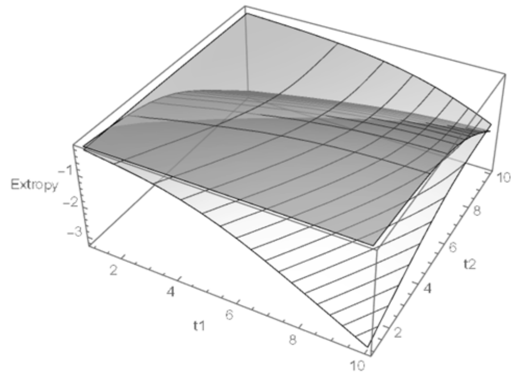

Figure 1: Overlay of conditional dynamic failure extropies J_1^F (Motor A) and J_2^F (Motor B) based on bivariate normal distribution

where the upper bounds $m_i^X(t_1, t_2)$ denote the conditional expected inactivity times

$$m_1^X(t_1, t_2) = \int_0^{t_1} \frac{F(x_1, t_2)}{F(t_1, t_2)}\, dx_1, \quad m_2^X(t_1, t_2) = \int_0^{t_2} \frac{F(t_1, x_2)}{F(t_1, t_2)}\, dx_2.$$

These correspond to the conditional mean waiting times for each component, given joint failure before t_1 and t_2 (Nair and Gopalakrishnan (2008)).

Proof: Since $0 < F(t_1, t_2) \leq 1$, we have

$$\left[\frac{F(x_i, t_j)}{F(t_1, t_2)}\right]^2 \leq \frac{F(x_i, t_j)}{F(t_1, t_2)}, \quad i, j = 1, 2,$$

for $i \neq j$. Integrating both sides over $[0, t_i]$ gives the desired inequality. □

The following example illustrates Theorem 1.

Example 2: Let the bivariate random variable $X = (X_1, X_2)$ have joint distribution function $F(t_1, t_2) = t_1 t_2$, where $0 \leq t_1, t_2 \leq 1$. Then, for $i = 1, 2$, the expected conditional waiting time is $m_i^X(t_1, t_2) = \frac{t_i}{2}$, and the conditional dynamic failure extropy is $J_i^F(X : t_1, t_2) = -\frac{t_i}{6}$. Therefore, $J_i^F(X : t_1, t_2) \leq m_i^X(t_1, t_2)$, verifying the inequality in Theorem 1.

The following theorem establishes an upper bound for conditional dynamic failure extropy ($CDFE_X$), relating it to the expected conditional waiting time and cumulative past entropy. Such bounds are instrumental in reliability modeling, offering insights into the reduction of uncertainty achieved through conditioning.

Theorem 2: Let $X = (X_1, X_2)$ be a non-negative bivariate random vector. Then, for $i = 1, 2$,

$$J_i^F(X : t_1, t_2) \leq \frac{1}{2}\left[\bar{\varepsilon}_i^*(X : t_1, t_2) - m_i^X(t_1, t_2)\right], \tag{3}$$

where $\bar{\varepsilon}_i^*(X : t_1, t_2)$ denotes the conditional dynamic cumulative past entropy (CDCPE), given by

$$\bar{\varepsilon}_1^*(X:t_1,t_2) = -\int_0^{t_1} \frac{F(x_1,t_2)}{F(t_1,t_2)} \log \frac{F(x_1,t_2)}{F(t_1,t_2)}\, dx_1, \tag{4}$$

$$\bar{\varepsilon}_2^*(X:t_1,t_2) = -\int_0^{t_2} \frac{F(t_1,x_2)}{F(t_1,t_2)} \log \frac{F(t_1,x_2)}{F(t_1,t_2)}\, dx_2, \tag{5}$$

and $m_i^X(t_1,t_2)$ is the conditional mean waiting time of component X_i, given joint failure before (t_1,t_2).

Proof: Using the inequality $-\log x \geq 1-x$, we obtain

$$\bar{\varepsilon}_1^*(X:t_1,t_2) \geq \int_0^{t_1} \left[\frac{F(x_1,t_2)}{F(t_1,t_2)} - \left(\frac{F(x_1,t_2)}{F(t_1,t_2)}\right)^2\right] dx_1,$$

which leads directly to the desired bound in (3). The result for $i=2$ follows similarly. □

The following theorem examines the behavior of $CDFE_X$ under strictly monotonic, differentiable transformations, as often encountered in reliability and survival analysis through scaling or shifts.

Theorem 3: Let $V=(V_1,V_2)$ with $V_i=\phi_i(X_i)$, where each ϕ_i is non-negative, strictly monotone, differentiable, and absolutely continuous. Then,

$$J_1^F(V:t_1,t_2) = \begin{cases} -\frac{1}{2}\int_0^{\phi_1^{-1}(t_1)} \left[\frac{F(x_1,\phi_2^{-1}(t_2))}{F(\phi_1^{-1}(t_1),\phi_2^{-1}(t_2))}\right]^2 \phi_1'(x_1)dx_1, & \text{if each } \phi_i \text{ is strictly increasing} \\ -\frac{1}{2}\int_{\phi_1^{-1}(t_1)}^{\infty} \left[\frac{\bar{F}(x_1,\phi_2^{-1}(t_2))}{\bar{F}(\phi_1^{-1}(t_1),\phi_2^{-1}(t_2))}\right]^2 \phi_1'(x_1)dx_1 & \text{if each } \phi_i \text{ is strictly decreasing.} \end{cases} \tag{6}$$

The expression for $J_2^F(V:t_1,t_2)$ follows analogously by symmetry.

In particular, for affine transformations of the form $\phi_i(X_i)=c_iX_i+d_i$ with $c_i>0$, $d_i\geq 0$, we obtain the scaled identity

$$J_1^F(V:t_1,t_2) = c_1\, J_1^F\left(X:\tfrac{t_1-d_1}{c_1},\tfrac{t_2-d_2}{c_2}\right).$$

The connection between $CDFE_X$ and the reversed hazard rate improves interpretability by relating the uncertainty in conditional failure times to the instantaneous recovery rate, as shown in the following theorem.

Theorem 4: Let $J_i^F(X:t_1,t_2)$ denote the conditional dynamic failure extropy, and let the reversed hazard rate be defined by $\bar{h}_i(t_1,t_2)=\frac{\partial}{\partial t_i}\log F(t_1,t_2)$ for $i=1,2$. Then,

$$\bar{h}_i(t_1,t_2) = \frac{2\frac{\partial}{\partial t_i}J_i^F(X:t_1,t_2)+1}{-4J_i^F(X:t_1,t_2)}. \tag{7}$$

Proof: From Equations (1) and (2), consider the case $i=1$

$$\frac{\partial}{\partial t_1}J_1^F(X:t_1,t_2) = \frac{-1}{2}\left[1-2\bar{h}_1(t_1,t_2)\int_0^{t_1}\frac{F(x_1,t_2)}{F(t_1,t_2)}\,dx_1\right]. \tag{8}$$

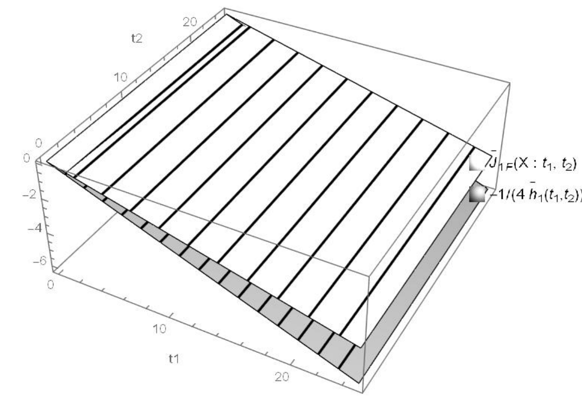

Figure 2: $J_i^F(X : t_1, t_2)$ **with RHS of the inequality** (9)

The result follows directly by solving the above equation for $\bar{h}_1(t_1, t_2)$. The case for $i = 2$ is analogous. □

Nonparametric classification criteria can be developed using the monotonicity properties of conditional dynamic failure extropy ($CDFE_X$). This classification helps characterize the reliability behavior of the system components over time.

Definition 2: A distribution function F is said to be increasing (decreasing) in conditional dynamic failure extropy ($CDFE_X$) if the measure $J_i^F(X : t_1, t_2)$ increases (decreases) monotonically with respect to t_i, for $i = 1, 2$.

Example 3: If X has the distribution function $F(t_1, t_2) = \exp\left(2 - \frac{1}{t_1} - \frac{1}{t_2}\right)$ for $0 \le t_1, t_2 \le 1$, then $J_i^F(X : t_1, t_2) = \frac{-t_i^2}{4}$, $i, j = 1, 2,\ i \neq j$, which is decreasing in t_i. Thus, the distribution is decreasing in $CDFE_X$ ($DCDFE_X$).

A necessary and sufficient condition for a distribution to exhibit increasing ($ICDFE_X$) or decreasing ($DCDFE_X$) conditional dynamic failure extropy can be expressed using the reversed hazard rate, as presented in the following theorem.

Theorem 5: A distribution function F is said to be $ICDFE_X$ ($DCDFE_X$) if and only if, for all $t_i > 0$, $i = 1, 2$,

$$J_i^F(X : t_1, t_2) \le (\ge) \frac{-1}{4\bar{h}_i(t_1, t_2)}, \tag{9}$$

where $\bar{h}_i(t_1, t_2)$ denotes the i-th component of the bivariate reversed hazard rate.

The following example illustrates Theorem 5.

Example 4: If X follows a bivariate uniform distribution with joint distribution function $F(t_1, t_2) = \frac{t_1 t_2}{bd}$, $0 \le t_1 \le b$, $0 \le t_2 \le d$, then $J_i^F(X : t_1, t_2) = \frac{-t_i}{6}$ and $\bar{h}_i(t_1, t_2) = \frac{1}{t_i}$. Thus, the inequality in (9) holds, indicating that F is $DCDFE_X$.

Figure 2 illustrates the relationship between $J_1^F(X : t_1, t_2)$ and the right-hand side

(RHS) of inequality (9) over the range $t_1, t_2 \in [0, 25]$. As shown, $J_1^F(X : t_1, t_2)$ decreases with increasing t_1 and consistently remains above the RHS of the inequality. This confirms the validity of inequality (9) and demonstrates the decreasing trend of $CDFE_X$ with respect to t_1.

Theorem 6: Let $V = (V_1, V_2)$ be a non-negative bivariate random vector such that $V_i = a_i X_i + b_i$, where $a_i > 0$ and $b_i > 0$, for $i = 1, 2$, and $X = (X_1, X_2)$. Then, $J_i^F(V : t_1, t_2)$ is increasing in t_i if and only if $J_i^F(X : t_1, t_2)$ is increasing in t_i.

The following theorem shows that conditional dynamic failure extropy ($CDFE_X$) can uniquely determine the underlying distribution, making it a useful tool for reliability analysis from lifetime data.

Theorem 7: Let X be a non-negative bivariate random vector with absolutely continuous distribution function F and bivariate reversed hazard rate components $\bar{h}_i(t_1, t_2)$, $i = 1, 2$. Then, $J_i^F(X : t_1, t_2)$ uniquely determines the joint distribution function F.

Proof: Let X and V be two non-negative random vectors such that

$$J_i^F(X : t_1, t_2) = J_i^F(V : t_1, t_2), \quad i = 1, 2.$$

Differentiating both sides with respect to t_i yields $\bar{h}_i^X(t_1, t_2) = \bar{h}_i^V(t_1, t_2)$. Since the reversed hazard rate uniquely determines the distribution (see Nair and Gopalakrishnan (2008)), it follows that $F_X = F_V$. □

The following theorem characterizes distributions where $CDFE_X$ is linearly proportional to the conditional mean inactivity time, leading to a specific power-form joint distribution.

Theorem 8: Let $X = (X_1, X_2)$ be a bivariate random vector supported on $[0, a_i]$, $i = 1, 2$, with finite a_i. Then, for all $t_i \in [0, a_i]$, $i = 1, 2$, we have

$$J_i^F(X : t_1, t_2) = c_i\, m_i(t_1, t_2), \quad \text{for } i = 1, 2, \tag{10}$$

if and only if

$$F(t_1, t_2) = \left(\frac{t_1}{a_1}\right)^{\frac{-g_1}{1+g_1}} \left(\frac{t_2}{a_2}\right)^{\frac{-g_2}{1+g_2}}, \tag{11}$$

where $g_i = \frac{2c_i+1}{2c_i}$ and $c_i \in (-1, 0)$, $i = 1, 2$.

Proof: The "if" part follows directly. For the "only if" part, differentiating (10) with respect to t_1 (for $i = 1$), we get

$$\frac{\partial}{\partial t_1} J_1^F(X : t_1, t_2) = c_1 \frac{\partial}{\partial t_1} m_1(t_1, t_2). \tag{12}$$

From this, we obtain

$$\frac{\partial}{\partial t_1} m_1(t_1, t_2) = 1 + g_1, \tag{13}$$

which leads to the form of $F(t_1, t_2)$ in (11). □

Theorem 8 leads to specific distributional forms for suitable choices of c_i. One such case characterizes the bivariate uniform distribution.

Corollary 8.1: A bivariate random vector X follows a uniform distribution on $[0,b]\times[0,d]$ with

$$F(t_1,t_2)=\frac{t_1t_2}{bd},\quad 0\le t_1\le b,\quad 0\le t_2\le d,$$

if and only if $J_i^F(X:t_1,t_2)=-m_i(t_1,t_2)$ for $i=1,2$.

Another class of distributions characterized via J_i^F is given below.

Theorem 9: Let X be a bivariate random vector supported on $(-\infty,a_1)\times(-\infty,a_2)$ with absolutely continuous distribution. Then

$$J_i^F(X:t_1,t_2)=-\frac{1}{4[k_i+k_3(t_j-a_j)]},\quad i\neq j$$

holds if and only if

$$F(t_1,t_2)=\exp\left(k_1(t_1-a_1)+k_2(t_2-a_2)+k_3(t_1-a_1)(t_2-a_2)\right).$$

Remark 1: Many real-world systems involve more than two interdependent components. While the bivariate Conditional Dynamic Failure Extropy ($\mathrm{CDFE_X}$) captures pairwise uncertainty, extending this measure to multivariate settings is essential for analyzing complex systems with higher-dimensional dependencies.

Let $\mathbf{X}=(X_1,X_2,\ldots,X_n)$ be an n-dimensional non-negative random vector with joint distribution function $F(\mathbf{t})=F(t_1,t_2,\ldots,t_n)$. Define the failure threshold vector $\mathbf{t}=(t_1,t_2,\ldots,t_n)$ and the joint failure event $\mathcal{A}=\{X_i<t_i,\ \forall i=1,2,\ldots,n\}$.

Then, the Multivariate Conditional Dynamic Failure Extropy ($CDFE_X$) for component X_i under $\mathcal{A}$ is defined as

$$J_i^F(\mathbf{X}:\mathbf{t})=-\frac{1}{2}\int_0^{t_i}\left[\frac{F(t_1,\ldots,t_{i-1},x_i,t_{i+1},\ldots,t_n)}{F(t_1,\ldots,t_n)}\right]^2dx_i,\quad i=1,2,\ldots,n. \tag{14}$$

Note: All theoretical results, properties, and estimators presented in this paper are derived and validated for the bivariate case ($n=2$). The above multivariate extension is proposed to illustrate its potential applicability, but it is not theoretically or empirically explored in this work.

3. Non parametric estimation

In this section, we introduce a nonparametric framework for estimating Conditional Dynamic Failure Extropy ($CDFE_X$), suitable when the underlying distribution is unknown. We propose both empirical and kernel-based estimators that allow flexible modeling from complete data. The empirical estimator, based on the empirical distribution function, provides a practical and straightforward approach, especially effective for large samples. We also examine the asymptotic behavior of these estimators and validate their performance through simulations and real-world applications.

Definition 3: Let (X_{1i}, X_{2i}), $i = 1, 2, \ldots, n$, be a random sample from a population with joint distribution function F. Based on expressions (1) and (2), the empirical estimators of conditional dynamic failure extropy are defined as

$$\hat{J}_1^F(X : t_1, t_2) = -\frac{1}{2}\int_0^{t_1} \left[\frac{\hat{F}(x_1, t_2)}{\hat{F}(t_1, t_2)}\right]^2 dx_1, \tag{15}$$

$$\hat{J}_2^F(X : t_1, t_2) = -\frac{1}{2}\int_0^{t_2} \left[\frac{\hat{F}(t_1, x_2)}{\hat{F}(t_1, t_2)}\right]^2 dx_2, \tag{16}$$

where $\hat{F}(t_1, t_2) = \frac{1}{n}\sum_{i=1}^{n} I(X_{1i} \le t_1, X_{2i} \le t_2)$ is the bivariate empirical distribution function, and

$$I(X_{1i} \le t_1, X_{2i} \le t_2) = \begin{cases} 1, & \text{if } X_{1i} \le t_1 \text{ and } X_{2i} \le t_2, \\ 0, & \text{otherwise.} \end{cases}$$

To improve estimation accuracy, we propose a kernel-based nonparametric estimator for $CDFE_X$. Kernel methods provide smooth approximations of the underlying distribution, reducing the variance inherent in empirical estimators.

Definition 4: Let (X_{1i}, X_{2i}), $i = 1, 2, \ldots, n$, be a sample from a population with joint distribution function F. The kernel estimator of $CDFE_X$ is defined as

$$\widetilde{J}_1^F(X : t_1, t_2) = -\frac{1}{2}\int_0^{t_1} \left[\frac{\widetilde{F}(x_1, t_2)}{\widetilde{F}(t_1, t_2)}\right]^2 dx_1, \tag{17}$$

$$\widetilde{J}_2^F(X : t_1, t_2) = -\frac{1}{2}\int_0^{t_2} \left[\frac{\widetilde{F}(t_1, x_2)}{\widetilde{F}(t_1, t_2)}\right]^2 dx_2, \tag{18}$$

where

$$\widetilde{F}(t_1, t_2) = \frac{1}{na_n^2}\sum_{j=1}^{n} K_1\left(\frac{t_1 - X_{1j}}{a_n}\right) K_2\left(\frac{t_2 - X_{2j}}{a_n}\right),$$

and $K_i(z) = a_n \int_0^z k_i(v)\,dv$, $i = 1, 2$, with $k_i(v)$ being known kernel density functions. The bandwidth sequence $\{a_n\}$ satisfies $a_n \to 0$ and $na_n \to \infty$ as $n \to \infty$.

To simplify notation, define

$$\hat{A}(t_1, t_2) = \int_0^{t_1} \left[\widetilde{F}(x_1, t_2)\right]^2 dx_1, \quad \hat{B}(t_1, t_2) = \left[\widetilde{F}(t_1, t_2)\right]^2,$$

$$A(t_1, t_2) = \int_0^{t_1} [F(x_1, t_2)]^2 dx_1, \quad B(t_1, t_2) = [F(t_1, t_2)]^2.$$

We employ kernel functions such as the Epanechnikov and Quartic kernels, known for their optimality in minimizing mean squared error.

To ensure consistency of the kernel-based estimator, we impose regularity conditions on the kernel function $k(t)$. These conditions ensure that the estimated density is well-defined and converges to the true distribution as sample size increases. The kernel $k(t)$ must satisfy

$$\int k(t)\,dt = 1, \quad \int t\,k(t)\,dt = 0, \quad \int tt^T\,k(t)\,dt = I,$$

where I is the identity matrix, ensuring proper normalization and moment conditions.

The following lemma, from Jin and Shao (1999), provides the asymptotic properties of the kernel estimator $\widetilde{F}$.

Lemma 1: Let x be a continuity point of F. Then, as $n \to \infty$

1. $\mathbb{E}[\widetilde{F}(x)] \to F(x)$,
2. $n\,\mathrm{Var}[\widetilde{F}(x)] \to F(x)[1 - F(x)]$.

The following theorem establishes the consistency of the kernel-based estimator for Conditional Dynamic Failure Extropy ($CDFE_X$), ensuring convergence in probability to the true value as the sample size increases.

Theorem 10: Let $\widetilde{J}_i^F(X : t_1, t_2)$, $i = 1, 2$, be the kernel-based nonparametric estimator of the conditional dynamic failure extropy $J_i^F(X : t_1, t_2)$. Then $\widetilde{J}_i^F(X : t_1, t_2)$ is a consistent estimator of $J_i^F(X : t_1, t_2)$.

Proof:

$$\int_0^{t_1} \widetilde{F}(x_1, t_2)^2\,dx_1 = \int_0^{t_1} F(x_1, t_2)^2\,dx_1 + 2\int_0^{t_1} F(x_1, t_2)\left[\widetilde{F}(x_1, t_2) - F(x_1, t_2)\right]dx_1 + O\left(\left[\widetilde{F}(x_1, t_2) - F(x_1, t_2)\right]^2\right).$$

Using Taylor expansion, we write

The bias and variance of $\hat{A}(t_1, t_2) = \int_0^{t_1} \widetilde{F}(x_1, t_2)^2 dx_1$ are

$$\mathrm{Bias}[\hat{A}(t_1, t_2)] \simeq 2\int_0^{t_1} F(x_1, t_2)\mathrm{Bias}[\widetilde{F}(x_1, t_2)]dx_1,$$
$$\mathrm{Var}[\hat{A}(t_1, t_2)] \simeq \frac{4}{n}\int_0^{t_1} F^3(x_1, t_2)[1 - F(x_1, t_2)]dx_1.$$

Similarly, for $\hat{B}(t_1, t_2) = \widetilde{F}(t_1, t_2)^2$

$$\mathrm{Bias}[\hat{B}(t_1, t_2)] \simeq 2F(t_1, t_2)\mathrm{Bias}[\widetilde{F}(t_1, t_2)],$$
$$\mathrm{Var}[\hat{B}(t_1, t_2)] \simeq \frac{4}{n}F^3(t_1, t_2)[1 - F(t_1, t_2)].$$

Therefore, we have the convergence in probability

$$\widetilde{J}_1^F(X : t_1, t_2) \xrightarrow{P} J_1^F(X : t_1, t_2) \quad \text{as } n \to \infty.$$

A similar argument holds for $i = 2$. Hence, $\widetilde{J}_i^F(X : t_1, t_2)$ is consistent. □

The following theorem derives the asymptotic bias and variance of the proposed estimator to evaluate its efficiency and understand the bias–variance trade-off.

Theorem 11: The asymptotic bias and variance of $\widetilde{J}_1^F(X : t_1, t_2)$ are given by

$$\text{Bias}(\widetilde{J}_1^F(X : t_1, t_2)) \simeq 0,$$

and

$$\text{Var}(\widetilde{J}_1^F(X : t_1, t_2)) = \frac{1}{nB^2(t_1, t_2)} \int_0^{t_1} F^3(x_1, t_2)[1 - F(x_1, t_2)]dx_1 + \frac{A^2(t_1, t_2)}{nB^4(t_1, t_2)} F^3(t_1, t_2)[1 - F(t_1, t_2)].$$

Proof: Using the approximation

$$\text{Bias}\left(\frac{\hat{A}}{\hat{B}}\right) \simeq \frac{1}{B}\left[\text{Bias}(\hat{A}) - \frac{A}{B}\text{Bias}(\hat{B})\right],$$

and

$$\text{Var}\left(\frac{\hat{A}}{\hat{B}}\right) \simeq \frac{1}{B^2}\left[\text{Var}(\hat{A}) + \frac{A^2}{B^2}\text{Var}(\hat{B})\right],$$

where $A = A(t_1, t_2)$ and $B = B(t_1, t_2)$. Substituting known expressions for bias and variance completes the proof. □

This section develops a flexible nonparametric estimation framework for conditional dynamic failure extropy, applicable to reliability, survival, and risk analysis. The following subsections further explore the implementation and evaluation of these methods using simulated and real-world datasets.

3.1. Simulation study

This subsection evaluates the performance of the empirical and kernel estimators defined in Definitions 3 and 4, using simulation to estimate the bias and mean squared error (MSE) of $CDFE_X$. Simulations were implemented in Mathematica on synthetic datasets of sizes $n = 200$, 500, and 800, generated from the joint distribution

$$F(t_1, t_2) = \left(\frac{t_1}{b_1}\right)^{c_1} \left(\frac{t_2}{b_2}\right)^{c_2} + \theta \log\left(\frac{t_1}{b_1}\right), \quad \theta \leq 0,$$

with parameters $b_1 = b_2 = 0.01$, $c_1 = c_2 = 0.02$, and $\theta = -0.1$. Both Epanechnikov and Quartic kernels were employed to estimate $CDFE_X$. For each n, 1,000 iterations were run.

Table 3: Bias and Mean squared errors (MSEs) of the conditional DFE_X function using empirical estimator, Epanechnikov kernel, and quartic kernel for simulated data

Sample Size	(t_1, t_2)	$CDFE_X$ (Empirical)		$CDFE_X$ (Epanechnikov Kernel)		$CDFE_X$ (Quartic Kernel)	
		Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
200	(0.17, 0.17)	(0.0858, 0.1029)	(0.0073, 0.0105)	(0.0467, 0.0466)	(0.0021, 0.0021)	(0.0536, 0.0573)	(0.0035, 0.0037)
	(0.2, 0.7)	(0.2858, 0.1929)	(0.0173, 0.0305)	(0.0492, 0.0949)	(0.0024, 0.0080)	(0.0592, 0.1049)	(0.0047, 0.0153)
	(0.45, 0.45)	(0.46892, 0.1830)	(0.0891, 0.0483)	(0.1173, 0.0971)	(0.0137, 0.0137)	(0.1571, 0.1071)	(0.0424, 0.0432)
	(0.5, 0.5)	(0.5575, 0.2329)	(0.0941, 0.0778)	(0.1293, 0.1391)	(0.0167, 0.0177)	(0.3215, 0.1991)	(0.0364, 0.0747)
500	(0.17, 0.17)	(0.0849, 0.1018)	(0.0053, 0.0096)	(0.0423, 0.0421)	(0.0009, 0.0009)	(0.0521, 0.0543)	(0.0029, 0.0031)
	(0.2, 0.7)	(0.2851, 0.1922)	(0.0116, 0.0274)	(0.0415, 0.0243)	(0.0011, 0.0052)	(0.0524, 0.0949)	(0.0024, 0.0097)
	(0.45, 0.45)	(0.4681, 0.1797)	(0.0825, 0.0424)	(0.1152, 0.0768)	(0.0122, 0.0124)	(0.1542, 0.1025)	(0.0404, 0.0418)
	(0.5, 0.5)	(0.5524, 0.2285)	(0.0912, 0.0734)	(0.1263, 0.1324)	(0.0126, 0.0145)	(0.2860, 0.1928)	(0.0327, 0.0722)
800	(0.17, 0.17)	(0.0795, 0.0938)	(0.0013, 0.0057)	(0.0315, 0.0247)	(0.0003, 0.0002)	(0.0482, 0.0473)	(0.0023, 0.0016)
	(0.2, 0.7)	(0.2783, 0.1839)	(0.0082, 0.0139)	(0.0361, 0.0172)	(0.0008, 0.0039)	(0.0472, 0.0928)	(0.0015, 0.0081)
	(0.45, 0.45)	(0.4597, 0.1647)	(0.0791, 0.0397)	(0.1145, 0.0752)	(0.0097, 0.0089)	(0.1497, 0.1007)	(0.0375, 0.0373)
	(0.5, 0.5)	(0.5491, 0.2232)	(0.0885, 0.0719)	(0.1246, 0.1312)	(0.0101, 0.0086)	(0.2784, 0.1911)	(0.0287, 0.0699)

In every iteration, empirical and kernel estimates of $CDFE_X$ were computed, and their absolute bias and MSE were evaluated. Table 3 summarizes the simulation results across sample sizes. The simulation results show that kernel estimators consistently yield lower MSE than empirical estimators, especially with larger sample sizes. This demonstrates the reliability and reduced uncertainty of kernel-based methods for estimating $CDFE_X$, validating their applicability in complex reliability analysis. Overall, the study confirms the effectiveness of the proposed nonparametric estimators for practical use.

Table 4: Fit results for both sets of data

	Statistic	P-Value
Anderson-Darling	1.15342	0.28493
Cramér-von Mises	0.0891055	0.640949
Kolmogorov-Smirnov	0.11474	0.656902
Kuiper	0.656902	0.394967
Pearson χ^2	10.3158	0.243557
Watson U^2	0.0840573	0.383813

	Statistic	P-Value
Anderson-Darling	0.486959	0.75891
Cramér-von Mises	0.0558492	0.84009
Kolmogorov-Smirnov	0.102625	0.780647
Pearson χ^2	12.2105	0.142055

Table 5: Conditional DFE_X function bias and MSEs using empirical and Epanechnikov kernel estimators for a real dataset

Measure	**(1.7,1)**	**(2,3)**	**(2,4)**
Empirical Bias	(0.85, 0.5)	(0.12548, 0.63589)	(0.12512, 0.93445)
Empirical MSE	(0.7225, 0.25)	(0.01574, 0.40436)	(0.01565, 0.92869)
Kernel Bias	(0.00068, 0.00037)	(0.00067, 0.00042)	(0.00061, 0.00110)
Kernel MSE	$(4.73354 \times 10^{-7}, 1.42776 \times 10^{-7})$	$(4.5577 \times 10^{-7}, 1.83099 \times 10^{-7})$	$(3.7551 \times 10^{-7}, 1.23225 \times 10^{-6})$
Measure	**(3,3)**	**(3,4)**	
Empirical Bias	(0.12539, 0.63667)	(0.54465, 0.91255)	
Empirical MSE	(0.01572, 0.40536)	(0.29665, 0.9668)	
Kernel Bias	(0.00172, 0.00602)	(0.00260, 0.01090)	
Kernel MSE	$(2.99221 \times 10^{-6}, 0.00003)$	$(6.78643 \times 10^{-6}, 0.00011)$	

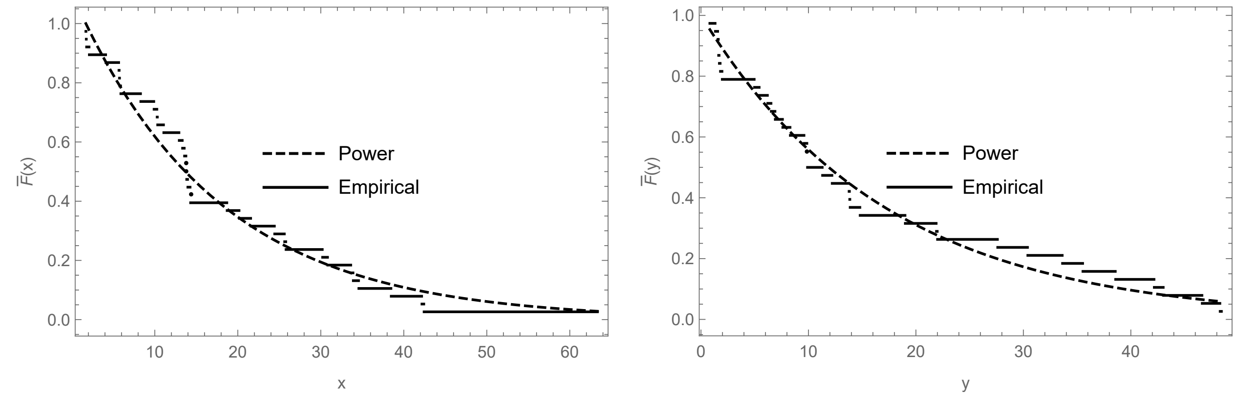

Figure 3: Graphs depicting empirical and fitted distributions for both datasets.

3.2. Application to a real dataset

In the following example, we apply real data to evaluate the empirical and kernel estimators defined in Definitions 3 and 4.

Example 5: We analyze a dataset from the National Eye Institute, involving 38 diabetic patients at risk of blindness, to illustrate the use of $CDFE_X$ estimators. One eye of each patient received laser photocoagulation, and times (in months) until blindness in both treated (X_2) and untreated (X_1) eyes were recorded Kilany and El-Qareb (2023). Each marginal distribution was fitted using a power distribution and assessed via the Kolmogorov–Smirnov test. The fit was further confirmed by comparing empirical and fitted curves in Figure 4, with detailed test statistics in Table 4. We computed $CDFE_X$ using both empirical and kernel methods, and assessed performance using 100,000 bootstrap samples (size 38 each) across (t_1, t_2) values. Bias and MSE of both estimators are summarized in Table 5. Results demonstrate that kernel estimators provide more accurate and stable estimates of $CDFE_X$ with reduced uncertainty. This highlights the practical utility of these methods in clinical reliability studies, particularly in evaluating treatment effects on failure times.

Example 6 demonstrates the superior performance of the proposed kernel estimators for Conditional Dynamic Failure Extropy ($CDFE_X$) over the univariate kernel estimator of Dynamic Failure Extropy by Nair and Sathar (2020).

Example 6: Consider the dataset provided in Kilany and El-Qareb (2023), which records the vision loss times for diabetic retinopathy patients, denoted by X_1 and X_2. Based on the univariate Dynamic Failure Extropy ($\mathrm{DFE_X}$) values reported by Nair and Sathar (2020), estimated as -1.1962 for X_1 and -1.06739 for X_2, we apply equations (17) and (18) to compute the Conditional $\mathrm{DFE_X}$ ($\mathrm{CDFE_X}$) values. For the point $(t_1, t_2) = (1, 2)$, the resulting $\mathrm{CDFE_X}$ values are obtained as $(-0.000828392,\ -0.00198842)$. For $(t_1, t_2) = (2, 3)$, the corresponding univariate $\mathrm{DFE_X}$ values are -1.1999 and -0.728436, with $CDFE_X$ estimates $(-0.000830955, -0.00135683)$. These results show that $CDFE_X$ yields consistently lower uncertainty than the univariate $\mathrm{DFE_X}$, highlighting the advantage of incorporating joint information. While univariate analysis may overstate uncertainty in component failure times, conditional extropy captures interdependence, improving predictability in past lifetime studies. Hence, bivariate failure extropy measures are preferred in such settings.

3.3. Comparison with conditional dynamic cumulative past entropy

This subsection compares the performance of the proposed Bivariate Conditional Dynamic Failure Extropy ($CDFE_X$) estimators with the Conditional Dynamic Cumulative Past Entropy ($CDCPE$), focusing on their ability to capture uncertainty in joint failure distributions. The kernel-based estimators for $CDCPE$ are given by

$$\widetilde{\bar{\varepsilon}}_1^*(X : t_1, t_2) = -\int_0^{t_1} \frac{\widetilde{F}(x_1, t_2)}{\widetilde{F}(t_1, t_2)} \log\left(\frac{\widetilde{F}(x_1, t_2)}{\widetilde{F}(t_1, t_2)}\right) dx_1, \tag{19}$$

$$\widetilde{\bar{\varepsilon}}_2^*(X : t_1, t_2) = -\int_0^{t_2} \frac{\widetilde{F}(t_1, x_2)}{\widetilde{F}(t_1, t_2)} \log\left(\frac{\widetilde{F}(t_1, x_2)}{\widetilde{F}(t_1, t_2)}\right) dx_2. \tag{20}$$

Using the real-world dataset from Example 5, we estimated $CDFE_X$ via both empirical and kernel methods, alongside the $CDCPE$ values. Bias and mean squared error (MSE) were used to evaluate estimator performance, with results shown in Tables 5 and 6. The results show that bivariate $CDFE_X$ consistently achieves lower bias and MSE compared to $CDCPE$, particularly under strong variable dependencies. This highlights $CDFE_X$ as a more robust and informative measure for analyzing joint failure behavior. Bivariate $CDFE_X$ outperforms $CDCPE$ in modeling uncertainty in two-component systems, offering improved accuracy and reliability for real-world reliability analysis.

Table 6: Bias and mean squared errors (MSEs) of the conditional dynamic $CDCPE$ function with Epanechnikov kernel for real data sets

(t_1, t_2)	$CDCPE$ (Kernel)	
	Bias	MSE
(1.7,1)	(0.17196, 0.09877)	(0.02957, 0.00975)
(2,3)	(0.12924, 0.42271)	(0.01671, 0.17869)
(2,4)	(0.12901, 0.54580)	(0.01665, 0.29791)
(3,3)	(0.00651, 0.42287)	(0.00007, 0.17883)
(3,4)	(0.00597, 0.54574)	(0.00007, 0.29784)

4. Conclusions

This paper proposed Conditional Dynamic Failure Extropy ($CDFE_X$) as a novel measure to quantify uncertainty in bivariate systems. We established its theoretical properties and developed nonparametric estimators using empirical and kernel methods. Simulation studies and real data analysis demonstrated the superior performance of $CDFE_X$ over existing measures such as univariate Dynamic Failure Extropy and Conditional Dynamic Cumulative Past Entropy, particularly in capturing joint failure dependencies. Kernel estimators consistently yielded lower bias and MSE than empirical counterparts. Overall, $CDFE_X$ offers a robust and informative framework for reliability analysis in systems with dependent components. Future work may extend this framework to higher dimensions and broader applications.

Acknowledgements

We sincerely thank the reviewers and the editor for their valuable feedback and guidance, which greatly improved the quality of our manuscript.

Conflict of interest

The authors do not have any financial or non-financial conflicts of interest to declare for the research work included in this article.

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