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Parameter Estimation of Generalized Exponential Distribution using Variations in Methods of Ranked Set Sampling

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Abstract

In this study, we estimate the parameters of the generalized exponential distribution using moving extreme ranked set sampling (MERSS) and ranked set sampling (RSS). Under both sampling schemes, we obtain expressions for likelihood functions and derive maximum likelihood equations and the Fisher information matrices. We numerically compute the ML estimates. We compare these estimates with estimates obtained by simple random sampling(SRS) and ranked set sampling using mean square error. Based on simulation studies, we demonstrate that the RSS scheme is more efficient for small set sizes than MERSS and SRS for shape and scale parameters.

Key words: Ranked set sampling; Moving extremes ranked set sampling; Maximum likelihood estimator; Fisher information number; Mean square error.

AMS Subject Classifications: 62F07, 62F10, 62D05

1. Ranked set sampling

When sampling environmental and ecological data, there may be situations in which it is difficult to measure (or quantify) a selected unit with appropriate accuracy, but ranking a few selected units based on the characteristic of interest can be simple. As an example, if one wishes to estimate the mean height of trees, then measuring the height of the sampled trees might be challenging, but there are relatively easy methods to rank small sets of trees based on eye observation of their heights. Rank set sampling (RSS) was developed by McIntyre (1952) as an improvement on random sampling in situations such as these, where some ranking of units may be simple.

A ranked set sample is obtained by randomly selecting m^2 units from an infinite population. These units are then partitioned randomly into m equal samples of m units each. The units in the sample are ranked by judgment or visual inspection or by cheap (low cost) way, or by using auxiliary variables but without actual measurements. The unit with the lowest rank is measured for variable X of interest and the remaining units are discarded; the second sample of m units is ranked without actual measurements. As a result, the second-lowest unit in this set is measured for variable X of interest, while the rest are discarded. Once the largest unit in the last sample of size m has been measured, the entire procedure is repeated h times, yielding $n = mh$ measured units from m^2h selected units. Compared to a simple random sample, this ranked set sample also represents the entire population and is spread throughout the population.

When sampling units are selected without replacement, the estimation of population mean using RSS, and comparison of RSS estimate to SRS estimate, are discussed in Patil et al. (1994).

There are different modifications of the methods of RSS available in the literature. Al-Odat and Al-Saleh (2001) introduced the concept of varied size RSS, which is known as moving extreme ranked set sampling. The described scheme is as given below.

1.1. Moving extreme ranked set sampling

Step 1. Select m simple random samples of size $1, 2, 3, \ldots, m$ respectively.

Step 2. Order the sampling units of each of the samples by eye or by some other relatively inexpensive method, without actual measurements.

Step 3. Measure accurately the maximum order observation from the first set, the maximum order observation from the second set. The process continues in this way until the maximum order observation from the last mth sample is measured.

Step 4. Repeat the Step 1. to Step 3. and then measure the minimum order observation instead maximum order observation.

Step 5. The prcedure described above is one cycle.The entire cycle can be repeated h times to obtain a MERSS of size $n = 2mh$

Hence, this scheme is more simple to implement than RSS. Recently Wangxue et al. (2019), has implemented the MERSS scheme to estimate parameters of Pareto distribution, and He et al. (2021) has implemented the MERSS scheme to estimate parameters of log-logistic distribution and Chen et al. (2021) discussed estimation of location parameter using maximum likelihood under MERSS scheme.

2. Generalized exponential distribution

Consider a continuous random variable X which follows two-parameter generalized exponential distribution having cumulative distribution function and probability diensity function respectively as,

$$H(x; \alpha, \beta) = (1 - e^{-\beta x})^{\alpha}, \quad \alpha, \beta, x > 0 \tag{1}$$

and

$$h(x;\alpha,\beta) = \alpha\beta(1-e^{-\beta x})^{\alpha-1}e^{-\beta x} \quad \alpha,\beta,x>0. \tag{2}$$

Here α is the shape parameter, and β is the scale parameter.

As seen in (1), the distribution is of the type $[F(\cdot)]^{\alpha}$ where $F(\cdot)$ is a cumulative distribution function of exponential distribution with scale parameter β. The distribution (1) is introduced and studied in detail by Gupta and Kundu (1999). According to Gupta and Kundu (2001), while fitting distribution for positive lifetime data, the generalized exponential distribution is used as an alternative to the two-parameter Weibull distribution and two parameter gamma distribution.
The mean and variance of the distribution with density function given in (2) are

$$E(X) = \frac{1}{\beta}(\psi(\alpha+1)-\psi(1)) \tag{3}$$

and

$$V(X) = -\frac{1}{\beta^2}(\psi'(\alpha+1)-\psi'(1)), \tag{4}$$

where $\psi(\cdot)$ is the digamma function and $\psi'(\cdot)$ is the derivative of $\psi(\cdot)$. The skewness and kurtosis both are independent of the scale parameter and they are decreasing function of the shape parameter α.

Consider the transformation $Y=\beta X$ in (2) so that the probability density function of Y is

$$f(y;\alpha) = \alpha(1-e^{-y})^{\alpha-1}e^{-y}, \quad y>0 \tag{5}$$

Let $Y_1,Y_2,\ldots,Y_n$ be a random sample on Y. Then the pdf of r^{th} order statistic $Y_{r:n},(r=1,2,\ldots,n)$ is

$$f_{r:n}(y) = \frac{1}{B(r,n-r+1)}F^{r-1}(y)\,[1-F(y)]^{n-r}\,f(y), \tag{6}$$

where $B(p,q)=\dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p>0,q>0.$

$$\begin{aligned}
f_{r:n}(y) &= \frac{1}{B(r,n-r+1)}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}\alpha\left[1-e^{-y}\right]^{(r+i)\alpha-1}e^{-y},\\
&= \frac{1}{B(r,n-r+1)}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}\alpha(r+i)\frac{\left[1-e^{-y}\right]^{(r+i)\alpha-1}}{r+i}e^{-y},\\
&= \frac{1}{B(r,n-r+1)}\sum_{i=0}^{n-r}\frac{(-1)^i\binom{n-r}{i}}{r+i}f(y;\alpha(r+i)).
\end{aligned}$$

For $r = 1, 2, \ldots, n$, the first order and second order moments are

$$E(Y_{r:n}) = \frac{1}{B(r, n-r+1)} \sum_{i=0}^{n-r} \frac{(-1)^i \binom{n-r}{i}}{r+i} \left[-\psi(1) + \psi(\alpha(r+i)+1)\right]$$
$$= \frac{1}{B(r, n-r+1)} \sum_{i=0}^{n-r} \frac{(-1)^i \binom{n-r}{i}}{r+i} \left[\gamma + \psi(\alpha(r+i)+1)\right] \quad (7)$$

and

$$E(Y_{r:n}^2) = \frac{1}{B(r, n-r+1)} \sum_{i=0}^{n-r} \frac{(-1)^i \binom{n-r}{i}}{r+i} \Big\{ \left[\psi((r+i)\alpha+1) + \gamma\right]^2 - \psi'((r+i)\alpha+1) + \pi^2/6 \Big\}, \quad (8)$$

where $\gamma = -\psi(1) = 0.577215\ldots$ and $\psi'(1) = \pi^2/6$.

Our discussion assumes that the sampling units are ranked without error for the characteristic of interest. In section (3) we discuss the estimation of shape and scale parameters of distribution given in (1) using MERSS scheme. In section (4) we discuss estimation of shape and scale parameters given in (1) using RSS scheme. Section (6) presents our findings.

3. The MERSS sample

Let $X_{j(11)}, X_{j(21)}, X_{j(22)}, X_{j(31)}, X_{j(32)}, X_{j(33)}, \ldots X_{j(m1)}, X_{j(m2)}, \ldots, X_{j(mm)}$ and $X'_{j(11)}, X'_{j(21)}, X'_{j(22)}, X'_{j(31)}, X'_{j(32)}, X'_{j(33)}, \ldots X'_{j(m1)}, X'_{j(m2)}, \ldots, X'_{j(mm)}$ be independent random variables all having the same distribution given in (1) at cycle $j = 1, 2, \ldots, h$ In the case of perfect ranking, for $j = 1$,
$X_{1(1:1)} = \max\{X_{1(11)}\}$,
$X_{1(2:2)} = \max\{X_{1(21)}, X_{1(22)}\}$,
$X_{1(3:3)} = \max\{X_{1(31)}, X_{1(32)}, X_{1(33)}\}, \ldots,$
$X_{1(i:i)} = \max\{X_{1(i1)}, X_{1(i2)}, \ldots, X_{1(ii)}\}$,
denote the $i-$th order statistic from random sample of size $i, i = 1, 2, \ldots, m$ for cycle 1 and
$X'_{1(1:1)} = \min\{X'_{1(11)}\}$,
$X'_{1(1:2)} = \min\{X'_{1(21)}, X'_{1(22)}\}$,
$X'_{1(1:3)} = \min\{X'_{1(31)}, X'_{1(32)}, X'_{1(33)}\}, \ldots,$
$X'_{1(1:i)} = \min\{X'_{1(i1)}, X'_{1(i2)}, \ldots, X'_{1(ii)}\}$
denote the first order statistic from random sample of size i, $i = 1, 2, \ldots, m$ for cycle 1.
Considering $\{X_{1(1:1)}, X_{1(2:2)}, X_{1(3:3)}, \ldots, X_{1(m:m)}\}$ as $MERSS_{\text{Maximum}}$, and
$\{X'_{1(1:1)}, X'_{1(1:2)}, X_{1(1:3)}, \ldots, X'_{1(1:m)}\}$as $MERSS_{\text{Minimum}}$, the MERSS sample (Cycle 1) of size $2m$ is
$$\Big\{X_{1(1:1)}, X_{1(2:2)}, X_{1(3:3)}, \ldots, X_{1(m:m)}; X'_{1(1:1)}, X'_{1(1:2)}, X'_{1(1:3)}, \ldots, X'_{1(1:m)}\Big\}$$

For simplicity let u_{ji} represents the observed values of $X_{j(i:i)}$ and v_{ji} represents the observed values of $X'_{j(1:i)}$, then the probability density functions of

$MERSS_{\text{maximum}}$ and $MERSS_{\text{minimum}}$ respectively are

$$\begin{aligned} f_{X_{j(i:i)}}(u_{ji}) &= i\left[H(u_{ji})\right]^{i-1} h(u_{ji}) \\ &= i\left[(1-e^{-\beta u_{ji}})^{\alpha}\right]^{i-1} \alpha\beta\left(1-e^{-\beta u_{ji}}\right)^{\alpha-1} e^{-\beta u_{ji}} \\ &= \alpha\beta i\left(1-e^{-\beta u_{ji}}\right)^{i\alpha-1} e^{-\beta u_{ji}} \end{aligned} \tag{9}$$

and

$$\begin{aligned} f_{X'_{j(1:i)}}(v_{ji}) &= i\left[1-H(v_{ji})\right]^{i-1} h(v_{ji}) \\ &= i\left[1-(1-e^{-\beta v_{ji}})^{\alpha}\right]^{i-1} \alpha\beta\left(1-e^{-\beta v_{ji}}\right)^{\alpha-1} e^{-\beta v_{ji}} \\ &= \alpha\beta i\left(1-e^{-\beta v_{ji}}\right)^{\alpha-1}\left[1-(1-e^{-\beta v_{ji}})^{\alpha}\right]^{i-1} e^{-\beta v_{ji}} \end{aligned} \tag{10}$$

From (7) and (8), the first order and second order moments are,

$$\mu_{j(i:i)} = E\{X_{j(i:i)}\} = \frac{1}{\beta}\left[\psi(i\alpha+1)+\gamma\right] \tag{11}$$

$$\mu^2_{j(i:i)} = E\{X^2_{j(i:i)}\} = \frac{1}{\beta^2}\left[\{\psi(i\alpha+1)+\gamma\}^2-\psi'(i\alpha+1)+\pi^2/6\right] \tag{12}$$

$$\begin{aligned} \mu^2_{j(1:i)} = E\{X'^2_{j(1:i)}\} = \frac{1}{\beta^2} i\sum_{k=0}^{i-1}\frac{(-1)^k\binom{i-1}{k}}{k+1}&\Big\{\left[\psi((k+1)\alpha+1)+\gamma\right]^2 \\ &-\psi'((k+1)\alpha+1)+\pi^2/6\Big\} \end{aligned} \tag{13}$$

$$\begin{aligned} \mu^2_{j(1:i)} = E\{X'^2_{j(1:i)}\} = \frac{1}{\beta^2} i\sum_{k=0}^{i-1}\frac{(-1)^k\binom{i-1}{k}}{k+1}&\Big\{\left[\psi((k+1)\alpha+1)+\gamma\right]^2 \\ &-\psi'((k+1)\alpha+1)+\pi^2/6\Big\} \end{aligned} \tag{14}$$

3.1. Likelihood function

The likelihood function for $MERSS_{\text{minimum}}$ is,

$$\begin{aligned} L_{MERSS,\text{minimum}} &= \prod_{j=1}^{h}\prod_{i=1}^{m} f_{X'_{j(1:i)}}(v_{ji}) \\ &= \prod_{j=1}^{h}\prod_{i=1}^{m} \alpha\beta i\left(1-e^{-\beta v_{ji}}\right)^{\alpha-1}\left[1-(1-e^{-\beta v_{ji}})^{\alpha}\right]^{i-1} e^{-\beta v_{ji}} \end{aligned}$$

and the likelihood function for $MERSS_{\text{maximum}}$ is,

$$\begin{aligned} L_{MERSS,\text{maximum}} &= \prod_{j=1}^{h}\prod_{i=1}^{m} f_{X_{j(i:i)}}(u_{ji}) \\ &= \prod_{j=1}^{h}\prod_{i=1}^{m} \alpha\beta i\left(1-e^{-\beta u_{ji}}\right)^{i\alpha-1} e^{-\beta u_{ji}} \end{aligned}$$

Thereforethe likelihood function under the scheme of MERSS is,

$$\begin{aligned} L_{MERSS} &= \prod_{j=1}^{h}\prod_{i=1}^{m} \left\{f_{X'_{j(1:i)}}(v_{ji})\right\}\left\{f_{X_{j(i:i)}}(u_{ji})\right\} \\ &= \prod_{j=1}^{h}\prod_{i=1}^{m} \left[\alpha\beta i\left(1-e^{-\beta v_{ji}}\right)^{\alpha-1}\left\{1-(1-e^{-\beta v_{ji}})^{\alpha}\right\}^{i-1} e^{-\beta v_{ji}}\right] \\ &\qquad \left[\alpha\beta i\left(1-e^{-\beta u_{ji}}\right)^{i\alpha-1} e^{-\beta u_{ji}}\right] \end{aligned}$$

and log-likelihood,
$\log L_{MERSS}$

$$\begin{aligned} &= \sum_{j=1}^{h}\sum_{i=1}^{m} \left[\log\left\{f_{X'_{j(1:i)}}(v_{ji})\right\} + \log\left\{f_{X_{j(i:i)}}(u_{ji})\right\}\right] \\ &= \sum_{j=1}^{h}\sum_{i=1}^{m} \left[\log\left\{\alpha\beta i\left(1-e^{-\beta v_{ji}}\right)^{\alpha-1}\left\{1-(1-e^{-\beta v_{ji}})^{\alpha}\right\}^{i-1} e^{-\beta v_{ji}}\right\}\right. \\ &\qquad \left. + \log\left\{\alpha\beta i\left(1-e^{-\beta u_{ji}}\right)^{i\alpha-1} e^{-\beta u_{ji}}\right\}\right] \\ &= \sum_{j=1}^{h}\sum_{i=1}^{m} \left[\left\{\log i + (\alpha-1)\log\left(1-e^{-\beta v_{ji}}\right) + (i-1)\log\left(1-(1-e^{-\beta v_{ji}})^{\alpha}\right)\right.\right. \\ &\qquad \left.\left. -\beta v_{ji}\right\} + \left\{\log i + (i\alpha-1)\log\left(1-e^{-\beta u_{ji}}\right) - \beta u_{ji}\right\}\right] + C \end{aligned} \tag{15}$$

where $C = 2mh\left\{\log\alpha + \log\beta\right\}$

3.2. ML estimates

Differentiating (15) with respect to α we get,

$$\begin{aligned} \frac{\partial \log L_{MERSS}}{\partial\alpha} = \frac{2mh}{\alpha} + \sum_{j=1}^{h}\sum_{i=1}^{m} &\left[\log\left(1-e^{-\beta v_{ji}}\right)\left\{\frac{1-i\left(1-e^{-\beta v_{ji}}\right)^{\alpha}}{1-(1-e^{-\beta v_{ji}})^{\alpha}}\right\}\right. \\ &\left. + \left\{i\log\left(1-e^{-\beta u_{ji}}\right)\right\}\right] \end{aligned} \tag{16}$$

Differentiating (15) with respect to β we get,

$$\frac{\partial \log L_{MERSS}}{\partial\beta} = \frac{2mh}{\beta} + \sum_{j=1}^{h}\sum_{i=1}^{m}\left[v_{ji}\left\{\frac{e^{-\beta v_{ji}}}{(1-e^{-\beta v_{ji}})}\frac{(\alpha-1)-(i\alpha-1)\left(1-e^{-\beta v_{ji}}\right)^{\alpha}}{(1-(1-e^{-\beta v_{ji}})^{\alpha})}-1\right\} - u_{ji}\left\{\frac{(i\alpha-1)e^{-\beta u_{ji}}}{(1-e^{-\beta u_{ji}})}-1\right\}\right] \tag{17}$$

We observe that the equations (16) and (17) can not be solved simultaneously to get closed form solution for α and β. Therefore, we solve numerically these equations simulataneously using R software (Henningsen and Toomet (2011)).

3.3. The observed Fisher information

Differentiating (16) with respect to α we get

$$\frac{\partial^2 \log L_{MERSS}}{\partial\alpha^2} = -\frac{2mh}{\alpha^2} - \sum_{j=1}^{h}\sum_{i=1}^{m}\left[\frac{(i-1)(1-e^{-\beta v_{ji}})^{\alpha}\left\{\log(1-e^{-\beta v_{ji}})\right\}^2}{(1-(1-e^{-\beta v_{ji}})^{\alpha})^2}\right] \tag{18}$$

Differentiating (17) with respect to β we get

$$\frac{\partial^2 \log L_{MERSS}}{\partial\beta^2} = -\frac{2mh}{\beta^2} + \sum_{j=1}^{h}\sum_{i=1}^{m}\left[\left\{\frac{e^{-\beta v_{ji}}v_{ji}}{(1-e^{-\beta v_{ji}})}\right\}^2 \left\{\frac{(\alpha-1)-(\alpha i(i+1)-\alpha-1)\left(1-e^{-\beta v_{ji}}\right)^{\alpha}}{(1-(1-e^{-\beta v_{ji}})^{\alpha})^2}\right\} - \left\{\frac{(i\alpha-1)e^{-\beta u_{ji}}u_{ji}^2}{(1-e^{-\beta u_{ji}})^2}\right\}\right] \tag{19}$$

and differentiating (16) with respect to β we get

$$\frac{\partial^2 \log L_{MERSS}}{\partial\alpha\partial\beta} = \sum_{j=1}^{h}\sum_{i=1}^{m}\left[\left\{\frac{e^{-\beta vji}v_{ji}}{1-e^{-\beta v_{ji}}}\right\} \left\{\frac{1-i\left(1-e^{-\beta v_{ji}}\right)^{\alpha}}{1-(1-e^{-\beta v_{ji}})^{\alpha}} - \frac{\alpha(i-1)\left(1-e^{-\beta v_{ji}}\right)^{\alpha}\log\left(1-e^{-\beta v_{ji}}\right)}{(1-(1-e^{-\beta v_{ji}})^{\alpha})^2}\right\} - \left\{i\frac{e^{-\beta uji}u_{ji}}{1-e^{-\beta u_{ji}}}\right\}\right] \tag{20}$$

We can compute numerically the elements of the observed Fisher information matrix and Variance -Covariance matrix for $\theta = (\alpha, \beta)$

$$I(\hat{\theta}) = \begin{bmatrix} -\frac{\partial^2 \log L_{MERSS}}{\partial \alpha^2} & -\frac{\partial^2 \log L_{MERSS}}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 \log L_{MERSS}}{\partial \beta \partial \alpha} & -\frac{\partial^2 \log L_{MERSS}}{\partial \beta^2} \end{bmatrix}_{(\hat{\alpha}, \hat{\beta})}$$

and $I^{-1}(\hat{\theta})$ respectively.

3.4. Fisher information

In this section, we obtain the Fisher information under the MERSS scheme. Following Azzalini (1996), the sample based on $MERSS_{\text{maximum}}$ and sample based on $MERSS_{\text{minimum}}$ are independent, therefore under certain regularity conditions the Fisher Information of MERSS scheme is given by

$$I_{MERSS}(\alpha, \beta) = I_{MERSS,\text{maximum}}(\alpha, \beta) + I_{MERSS,\text{minimum}}(\alpha, \beta) \tag{21}$$

The components of matrix $I_{MERSS,\text{maximum}}(\alpha, \beta)$ are
$\begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix}$.

Where,

$$I_{11} = -E\left[\frac{\partial^2 \log L_{MERSS,\max}}{\partial \alpha^2}\right] = \frac{hm}{\alpha^2} \tag{22}$$

$$\begin{aligned} I_{22} &= -E\left[\frac{\partial^2 \log L_{MERSS,\max}}{\partial \beta^2}\right] \\ &= \frac{hm}{\beta^2} + \sum_{j=1}^{h}\sum_{i=1}^{m} \frac{(i\alpha)(i\alpha - 1)}{\beta^2}\left[\sum_{k=0}^{\infty}(-1)^k \binom{i\alpha - 3}{k} \frac{2}{(k+2)^3}\right] \end{aligned} \tag{23}$$

$$\begin{aligned} I_{21} &= -E\left[\frac{\partial^2 \log L_{MERSS,\max}}{\partial \beta \partial \alpha}\right] \\ &= -\frac{\alpha}{\beta}\sum_{j=1}^{h}\sum_{i=1}^{m} i^2 \left[\sum_{k=0}^{\infty}(-1)^k \binom{i\alpha - 2}{k} \frac{2}{(k+2)^2}\right] \end{aligned} \tag{24}$$

The components of matrix $I_{MERSS,\text{minimum}}(\alpha, \beta) = \begin{bmatrix} I'_{11} & I'_{12} \\ I'_{21} & I'_{22} \end{bmatrix}$

Case: $i = 2$

$$I'_{11} = -E\left[\frac{\partial^2 \log L_{MERSS,\min}}{\partial \alpha^2}\right]$$
$$= \frac{h}{\alpha^2}\left[1 - 4\sum_{k=0}^{\infty}(-1)^k \frac{1}{(k+2)^3}\right] \tag{25}$$

$$I'_{22} = -E\left[\frac{\partial^2 \log L_{MERSS,\min}}{\partial \beta^2}\right]$$
$$= \frac{h}{\beta^2} + \frac{2h\alpha(\alpha-1)}{\beta^2}$$
$$\frac{1}{(\alpha-2)}\left[\{\psi(\alpha-1)-\psi(1)\}^2 - \{\psi'(\alpha-1)-\psi'(1)\}\right]$$
$$-\frac{1}{2(\alpha-1)}\left[\{\psi(2\alpha-1)-\psi(1)\}^2 - \{\psi'(2\alpha-1)-\psi'(1)\}\right]$$
$$-\frac{1}{(\alpha-1)}\left[\{\psi(\alpha)-\psi(1)\}^2 - \{\psi'(\alpha)-\psi'(1)\}\right]$$
$$-\frac{1}{(2\alpha-1)}\left[\{\psi(2\alpha)-\psi(1)\}^2 - \{\psi'(2\alpha)-\psi'(1)\}\right]$$
$$+ 2h\left[\alpha E'_1 - E'_2\right], \tag{26}$$

where

$$E'_1 = 2\alpha\sum_{k=0}^{\infty}\left[\frac{1}{\alpha(k+2)-2}\left(\{\psi(\alpha(k+2)-1)-\psi(1)\}^2 - \{\psi'(\alpha(k+2)-1)-\right.\right.$$
$$\psi'(1)\}) - \frac{2}{\alpha(k+2)-1}\left(\{\psi(\alpha(k+2))-\psi(1)\}^2 - \{\psi'(\alpha(k+2))-\psi'(1)\}\right)$$
$$\left.+\frac{1}{\alpha(k+2)}\left(\{\psi(\alpha(k+2))-\psi(1)\}^2 - \{\psi'(\alpha(k+2))-\psi'(1)\}\right)\right]$$
$$E'_2 = \frac{2\alpha}{2\alpha-1}\left[\{\psi(2\alpha)-\psi(1)\}^2 - \{\psi'(2\alpha)-\psi'(1)\}\right]$$
$$-\left[\{\psi(2\alpha+1)-\psi(1)\}^2 - \{\psi'(2\alpha+1)-\psi'(1)\}\right]$$

$$I'_{12} = -E\left[\frac{\partial^2 \log L_{MERSS,\min}}{\partial \beta \partial \alpha}\right]$$
$$= -h\left[E'_3 - \{E'_4 + E'_5\}\right]$$

Where

$$E_3' = \frac{2\alpha}{\alpha-1}\left(\{\psi(\alpha)-\psi(1)\}\right) - \frac{2\alpha}{2\alpha-1}\left(\{\psi(2\alpha)-\psi(1)\}\right) - \mu_{j(1:2)}$$

$$E_4' = \frac{2\alpha}{2\alpha-1}\left(\{\psi(2\alpha)-\psi(1)\}\right) - \left(\{\psi(2\alpha+1)-\psi(1)\}\right)$$

$$E_5' = \frac{2\alpha^2}{\beta}\sum_{l=1}^{\infty}\sum_{k=0}^{\infty}\frac{1}{l}\left[\frac{1}{(\alpha(k+2)+l)^2} - \frac{1}{(\alpha(k+2)+l-1)^2}\right]$$

Case: $i = 3$

$$\begin{aligned} I_{11}' &= -E\left[\frac{\partial^2 \log L_{MERSS,\min}}{\partial\alpha^2}\right] \\ &= \frac{5h}{2\alpha^2} \end{aligned} \tag{27}$$

$$\begin{aligned} I_{22}' &= -E\left[\frac{\partial^2 \log L_{MERSS,\min}}{\partial\beta^2}\right] \\ &= \frac{h}{\beta^2} + \frac{3h\alpha(\alpha-1)}{\beta^2}\frac{1}{(\alpha-1)}\left[\{\psi(\alpha)-\psi(1)\}^2 - \{\psi'(\alpha)-\psi'(1)\}\right] \\ &\quad - \frac{2}{(2\alpha-1)}\left[\{\psi(2\alpha)-\psi(1)\}^2 - \{\psi'(2\alpha)-\psi'(1)\}\right] \\ &\quad - \frac{1}{(3\alpha-1)}\left[\{\psi(3\alpha)-\psi(1)\}^2 - \{\psi'(3\alpha)-\psi'(1)\}\right] \\ &\quad - \left[\{\psi(\alpha+1)-\psi(1)\}^2 - \{\psi'(\alpha+1)-\psi'(1)\}\right] \\ &\quad + \frac{1}{\alpha}\left[\{\psi(2\alpha+1)-\psi(1)\}^2 \{\psi'(2\alpha+1)-\psi'(1)\}\right] \\ &\quad - \frac{1}{3\alpha}\left[\{\psi(3\alpha+1)-\psi(1)\}^2 \{\psi'(3\alpha+1)-\psi'(1)\}\right] \\ &\quad + 2h\left[\alpha E_1' - E_2'\right], \end{aligned} \tag{28}$$

where

$$E'_1 = \frac{3\alpha}{\beta^2}\left[\frac{1}{2\alpha-2}\left[\{\psi(2\alpha-1)-\psi(1)\}^2-\{\psi'(2\alpha-1)-\psi'(1)\}\right]\right.$$
$$-\frac{2}{2\alpha-1}\left[\{\psi(2\alpha)-\psi(1)\}^2-\{\psi'(2\alpha)-\psi'(1)\}\right]$$
$$\left.+\frac{1}{2\alpha}\left[\{\psi(2\alpha+1)-\psi(1)\}^2-\{\psi'(2\alpha+1)-\psi'(1)\}\right]\right]$$
$$E'_2 = \frac{3\alpha}{\beta^2}\left[\frac{1}{2\alpha-2}\left[\{\psi(2\alpha-1)-\psi(1)\}^2-\{\psi'(2\alpha-1)-\psi'(1)\}\right]\right.$$
$$-\frac{1}{3\alpha-2}\left[\{\psi(3\alpha-1)-\psi(1)\}^2-\{\psi'(3\alpha-1)-\psi'(1)\}\right]$$
$$-\frac{1}{2\alpha-1}\left[\{\psi(2\alpha)-\psi(1)\}^2-\{\psi'(2\alpha)-\psi'(1)\}\right]$$
$$\left.+\frac{1}{3\alpha-1}\left[\{\psi(3\alpha)-\psi(1)\}^2-\{\psi'(3\alpha)-\psi'(1)\}\right]\right]$$

$$I'_{12} = -E\left[\frac{\partial^2 \log L_{MERSS,\min}}{\partial\beta\partial\alpha}\right]$$
$$= -2h\left[E'_3-\{E'_4+E'_5\}\right],$$

where

$$E'_3 = \frac{3\alpha}{\beta}\left[\frac{1}{\alpha-1}\left[\{\psi(\alpha)-\psi(1)\}\right]-\frac{2}{2\alpha-1}\left[\{\psi(2\alpha)-\psi(1)\}\right]\right.$$
$$\left.+\frac{1}{3\alpha-1}\left[\{\psi(3\alpha)-\psi(1)\}\right]-\mu_{j(1:3)}\right]$$

$$E'_4 = \frac{3\alpha}{\beta}\left[\frac{1}{2\alpha-1}\left[\{\psi(2\alpha)-\psi(1)\}\right]-\frac{1}{3\alpha-1}\left[\{\psi(3\alpha)-\psi(1)\}\right]\right.$$
$$\left.-\frac{1}{2\alpha}\left[\{\psi(2\alpha+1)-\psi(1)\}\right]+\left[\{\psi(3\alpha)-\psi(1)\}\right]\right]$$

$$E'_5 = \frac{3\alpha^2}{\beta}\sum_{l=1}^{\infty}\frac{1}{l}\left[\frac{1}{(2\alpha+l)^2}-\frac{1}{(2\alpha+l-1)^2}\right]$$

In general,

$$\begin{aligned} I'_{11} &= -E\left[\frac{\partial^2 \log L_{MERSS,\min}}{\partial\alpha^2}\right] \\ &= \frac{hm}{\alpha^2} + \sum_{j=1}^{h}\sum_{i=4}^{m}\frac{i(i-1)}{\alpha^2}\sum_{k=0}^{i-3}(-1)^k\binom{i-3}{k}\frac{2}{(k+2)^3} \end{aligned} \tag{29}$$

$$\begin{aligned} I'_{22} &= -E\left[\frac{\partial^2 \log L_{MERSS,\min}}{\partial\beta^2}\right] \\ &= \frac{hm}{\beta^2} + \sum_{j=1}^{h}\sum_{i=4}^{m}\frac{i\alpha(\alpha-1)}{\beta^2} \\ &\qquad \sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k}\frac{1}{(\alpha(k+1)-2)}\Big[\{\psi(\alpha(k+1)-1)-\psi(1)\}^2 \\ &\qquad - \{\psi'(\alpha(k+1)-1)-\psi'(1)\}\Big] \\ &\qquad - \frac{1}{(\alpha(k+1)-1)}\Big[\{\psi(\alpha(k+1))-\psi(1)\}^2 \\ &\qquad - \{\psi'(\alpha(k+1))-\psi'(1)\}\Big] \\ &\quad - (i-1)\alpha\sum_{j=1}^{h}\sum_{i=1}^{m}[\alpha E'_1 - E'_2] \end{aligned} \tag{30}$$

$$\begin{aligned} I'_{12} &= -E\left[\frac{\partial^2 \log L_{MERSS,\min}}{\partial\beta\partial\alpha}\right] \\ &= -\sum_{j=1}^{h}\sum_{i=4}^{m}\left[E'_3 - (i-1)\{E'_4 + E'_5\}\right] \end{aligned} \tag{31}$$

where

$$\begin{aligned} E'_1 = \frac{i\alpha}{\beta^2}\sum_{k=0}^{i-3}(-1)^k\binom{i-3}{k}\Big[&\frac{1}{\alpha(k+2)-1}\{\psi(\alpha(k+2))-\psi(1)\}^2 \\ &- \{\psi'(\alpha(k+2))-\psi'(1)\} + \frac{2}{\alpha(k+2)}\{\psi(\alpha(k+2)+1)-\psi(1)\}^2 \\ &- \{\psi'(\alpha(k+2)+1)-\psi'(1)\} + \frac{1}{\alpha(k+2)+1}\{\psi(\alpha(k+2)+2) \\ &- \psi(1)\}^2 - \{\psi'(\alpha(k+2)+2)-\psi'(1)\}\Big] \end{aligned}$$

$$E_2' = \frac{i\alpha}{\beta^2}\sum_{k=0}^{i-2}(-1)^k\binom{i-2}{k}\left[\frac{1}{\alpha(k+2)-1}\left\{\psi(\alpha(k+2))-\psi(1)\right\}^2 - \left\{\psi'(\alpha(k+2))-\psi'(1)\right\} + \frac{1}{\alpha(k+2)}\left\{\psi(\alpha(k+2)+1)-\psi(1)\right\}^2 - \left\{\psi'(\alpha(k+2)+1)-\psi'(1)\right\}\right]$$

$$E_3' = \frac{i\alpha}{\beta}\sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k}\frac{\psi(\alpha(k+1))-\psi(1)}{\alpha(k+1)-1} - \mu_{j(1:i)}$$

$$E_4' = \frac{i\alpha}{\beta}\sum_{k=0}^{i-2}(-1)^k\binom{i-2}{k}\left[\frac{\psi(\alpha(k+2))-\psi(1)}{\alpha(k+2)-1} - \frac{\psi(\alpha(k+2)+1)-\psi(1)}{\alpha(k+2)}\right]$$

$$E_5' = \frac{i\alpha^2}{\beta}\sum_{l=1}^{\infty}\sum_{k=0}^{i-3}(-1)^k\binom{i-3}{k}\frac{1}{l}\left[\frac{1}{(\alpha(k+2)+l)^2} - \frac{1}{(\alpha(k+2)+l-1)^2}\right]$$

We note that above expectations exist for $\alpha > 2$ as $\psi(\cdot)$ and $\psi'(\cdot)$ exist and finite.

4. Ranked set sample

Let $X_{j1}, X_{j2}, \ldots, X_{jm}, X_{j(m+1)}, X_{j(m+2)}, \ldots, X_{jm^2}$ be independent random variables having the same distribution given in (1) of cycle $j = 1, 2, \ldots, h$. Then from i-th set $\{X_{j((i-1)m+1)}, X_{j((i-1)m+2)j},$
$\ldots, X_{j(im)}\}$, $X_{ji(i:m)}, i = 1, 2, \ldots, m$ denote $i-$th order statistic assuming error free rankings. Let x_{ji} denote observed value of $X_{ji(i:m)}$, then the pdf of i-th order statistic,

$$\begin{aligned} f_{X_{ji(i:m)}}(x_{ji}) &= \frac{1}{B(i, m-i+1)}F^{i-1}(x_{ji})(1-F(x_{ji}))^{m-i}f(x_{ji}) \\ &= \frac{1}{B(i, m-i+1)}\left((1-e^{-\beta x_{ji}})^\alpha\right)^{i-1}\left(1-(1-e^{-\beta x_{ji}})^\alpha\right)^{m-i} \\ &\qquad \alpha\beta(1-e^{-\beta x_{ji}})^{\alpha-1}e^{-\beta x_{ji}}, \alpha > 0, \beta > 0 \end{aligned} \tag{32}$$

From (7) and (8) the first order and second order moments of $X_{ji(i:m)}$ respectively are $\mu_{ji(i:m)} = E\{X_{ji(i:m)}\} = \frac{1}{\beta}E\{Y_{(i:m)}\}$, and $\mu^2_{ji(i:m)} = E\{X^2_{ji(i:m)}\} = \frac{1}{\beta^2}E\{Y^2_{(i:m)}\}$. Then the log-likelihood function is

$$\begin{aligned} \log L_{RSS}(\alpha, \beta) &= \sum_{j=1}^{h}\sum_{i=1}^{m}\log f_{X_{ji(i:m)}}(x_{ji}) \\ &= C_4 + mh(\log\alpha + \log\beta) \\ &\quad + \sum_{j=1}^{h}\sum_{i=1}^{m}\Big\{(i\alpha-1)\log\left(1-e^{-\beta x_{ji}}\right) \\ &\qquad +(m-i)\log\left(1-(1-e^{-\beta x_{ji}})^\alpha\right) - \beta x_{ji}\Big\}, \end{aligned} \tag{33}$$

where $C_4 = mh\log\left(\frac{1}{B(i,m-i+1)}\right)$.

4.1. The likelihood equations

Differentiating (33) with respect to α

$$\frac{\partial \log L_{RSS}(\alpha,\beta)}{\partial\alpha} = \frac{mh}{\alpha} + \sum_{j=1}^{h}\sum_{i=1}^{m}\log(1-e^{-\beta x_{ji}})\left\{\frac{i-m(1-e^{-\beta x_{ji}})}{(1-(1-e^{-\beta x_{ji}})^{\alpha})}\right\} \tag{34}$$

Differentiating (33) with respect to β

$$\frac{\partial \log L_{RSS}(\alpha,\beta)}{\partial\beta} = \frac{mh}{\beta} + \sum_{j=1}^{h}\sum_{i=1}^{m} x_{ji}\left[\frac{e^{-\beta x_{ji}}}{(1-e^{-\beta x_{ji}})}\left\{\frac{i-m(1-e^{-\beta x_{ji}})^{\alpha}}{1-(1-e^{-\beta x_{ji}})^{\alpha}}-1\right\}-1\right] \tag{35}$$

We observe that the equations (34) and (35) can not be solved simultaneously to get a closed-form solution for α and β. Therefore, we solve numerically these equations simulataneously using R software (Henningsen and Toomet (2011)).

Differentiating (34) with respect to α

$$\frac{\partial^2 \log L_{RSS}(\alpha,\beta)}{\partial\alpha^2} = -\frac{mh}{\alpha^2} - \sum_{j=1}^{h}\sum_{i=1}^{m}\left\{\frac{(m-i)(1-e^{-\beta x_{ji}})\left(\log(1-e^{-\beta x_{ji}})\right)^2}{\left(1-(1-e^{-\beta x_{ji}})^{\alpha}\right)^2}\right\} \tag{36}$$

Differentiating (35) with respect to β

$$\begin{aligned}\frac{\partial^2 \log L_{RSS}(\alpha,\beta)}{\partial\beta^2} = -\frac{mh}{\beta^2} - \sum_{j=1}^{h}\sum_{i=1}^{m} x_{ji}^2\frac{e^{-\beta x_{ji}}}{(1-e^{-\beta x_{ji}})^2}&\left[\left\{\frac{i-m(1-e^{-\beta x_{ji}})^{\alpha}}{1-(1-e^{-\beta x_{ji}})^{\alpha}}-1\right\}\right.\\ &\left.+e^{-\beta x_{ji}}\frac{\alpha(m+i)(1-e^{-\beta x_{ji}})^{\alpha}}{\left(1-(1-e^{-\beta x_{ji}})^{\alpha}\right)^2}\right]\end{aligned} \tag{37}$$

and differentiating (34) with respect to β

$$\begin{aligned}\frac{\partial^2 \log L_{RSS}(\alpha,\beta)}{\partial\alpha\partial\beta} = \sum_{j=1}^{h}\sum_{i=1}^{m}\frac{x_{ji}e^{-\beta x_{ji}}}{(1-e^{-\beta x_{ji}})}&\left[\left\{\frac{i-m(1-e^{-\beta x_{ji}})^{\alpha}}{(1-(1-e^{-\beta x_{ji}})^{\alpha})}\right\}\right.\\ &\left.-\frac{\alpha(m-i)(1-e^{-\beta x_{ji}})^{\alpha}\log(1-e^{-\beta x_{ji}})}{\left(1-(1-e^{-\beta x_{ji}})^{\alpha}\right)^2}\right]\end{aligned} \tag{38}$$

We can numerically compute elements of the observed Fisher information matrix,

$$I(\hat{\theta}) = \begin{bmatrix} -\frac{\partial^2 \log L_{RSS}(\alpha,\beta)}{\partial\alpha^2} & -\frac{\partial^2 \log L_{RSS}(\alpha,\beta)}{\partial\alpha\partial\beta} \\ -\frac{\partial^2 \log L_{RSS}(\alpha,\beta)}{\partial\beta\partial\alpha} & -\frac{\partial^2 \log L_{RSS}(\alpha,\beta)}{\partial\beta^2}\end{bmatrix}_{(\hat{\alpha},\hat{\beta})}$$

and variance-covariance matrix $I^{-1}(\hat{\theta})$

5. Comparing information of MERSS and RSS schemes

In this section, we compare the information of MERSS and RSS schemes respectively to estimate shape and scale parameters . Let $\hat{\alpha}_{\text{RSS}}$, $\hat{\beta}_{\text{RSS}}$ and $\hat{\alpha}_{\text{MERSS}}$, $\hat{\beta}_{\text{MERSS}}$ are the estimates of shape and scale parameters under RSS and MERSS schemes respectively. Assuming $\dfrac{\hat{\alpha}_{\text{RSS}}}{\hat{\alpha}_{\text{MERSS}}} \to 1$ and $\dfrac{\hat{\beta}_{\text{RSS}}}{\hat{\beta}_{\text{MERSS}}} \to 1$ then,

RSS:

$$I_\alpha(RSS) = \frac{mh}{\alpha^2} + \sum_{j=1}^{h}\sum_{i=1}^{m} \frac{(m-i)}{B(i,m-i+1)\alpha^2} \sum_{k=0}^{m-i-2} (-1)^k \binom{m-i-2}{k} \frac{2}{(k+i+1)^3} \tag{39}$$

$$\begin{aligned} I_\beta(RSS) = \frac{mh}{\beta^2} + (i\alpha-1)\sum_{j=1}^{h}\sum_{i=1}^{m} \frac{\alpha}{B(i,m-i+1)\beta^2} \sum_{s=0}^{m-i}(-1)^s \binom{m-i}{s} \times \\ \sum_{k=0}^{\infty}(-1)^k \binom{(i+s)\alpha-3}{k} \frac{2}{(k+2)^3} \\ + \alpha(m-i)\Bigg[\frac{\alpha^2}{B(i,m-i+1)\beta^2} \sum_{s=0}^{m-i-2} (-1)^s \binom{m-i-2}{s} \times \\ \sum_{k=0}^{\infty}(-1)^k \binom{(i+s+1)\alpha-3}{k} \frac{2}{(k+3)^3} \\ - \frac{\alpha}{B(i,m-i+1)\beta^2} \sum_{s=0}^{m-i-1} (-1)^s \binom{m-i-1}{s} \times \\ \sum_{k=0}^{\infty}(-1)^k \binom{(i+s+1)\alpha-3}{k} \frac{2}{(k+2)^3}\Bigg] \end{aligned} \tag{40}$$

MERSS:

$$I_\alpha(MERSS) = \frac{2hm}{\alpha^2} + \sum_{j=1}^{h}\sum_{i=1}^{m} \frac{i(i-1)}{\alpha^2} \sum_{k=0}^{i-3}(-1)^k \binom{i-3}{k} \frac{2}{(k+2)^3} \tag{41}$$

and $I_\beta(MERSS)$ is obtained in section (3.4). We note that $I_\beta(MERSS)$ exists and finite for $\alpha > 2$.

The computations are,

When sample size of RSS is $2mh$ and sample size of MERSS is $2mh$, then

For $(m = 3)$

$$I_\alpha(RSS) = h\left[\frac{12.6553}{\alpha^2} + \frac{60}{\alpha^2}\sum_{k=0}^{\infty}\frac{1}{(k+6)^3}\right] = \frac{13.6390h}{\alpha^2}$$

$$I_\beta(RSS) = h\frac{36.47049}{\beta^2}, \quad I_\beta(MERSS) = h\frac{207.6518}{\beta^2} \quad \alpha = 2.5$$

$$I_\beta(RSS) = h\frac{42.74969}{\beta^2}, \quad I_\beta(MERSS) = h\frac{97.40603}{\beta^2} \quad \alpha = 3.0$$

For$(m = 4)$

$$I_\alpha(RSS) = h\left[\frac{158.0665}{\alpha^2} + \frac{112}{\alpha^2}\sum_{k=0}^{\infty}\frac{1}{(k+8)^3}\right] = \frac{159.0577h}{\alpha^2}$$

$$I_\beta(RSS) = h\frac{65.12496}{\beta^2}, \quad I_\beta(MERSS) = h\frac{287.7424}{\beta^2} \quad \alpha = 2.5$$

$$I_\beta(RSS) = h\frac{76.34466}{\beta^2}, \quad I_\beta(MERSS) = h\frac{147.3634}{\beta^2} \quad \alpha = 3.0$$

$$\left[\frac{I_\alpha(RSS)}{I_\alpha(MERSS)}\right]_{m=3} = 1.6416, \quad \left[\frac{I_\alpha(RSS)}{I_\alpha(MERSS)}\right]_{m=4} = 12.8073$$

For $\alpha = 2.5$

$$\left[\frac{I_\beta(RSS)}{I_\beta(MERSS)}\right]_{m=3} = 0.17563, \quad \left[\frac{I_\alpha(RSS)}{I_\alpha(MERSS)}\right]_{m=4} = 0.22633$$

For $\alpha = 3.0$

$$\left[\frac{I_\beta(RSS)}{I_\beta(MERSS)}\right]_{m=3} = 0.43888, \quad \left[\frac{I_\alpha(RSS)}{I_\alpha(MERSS)}\right]_{m=4} = 0.51807$$

We note that information for the shape parameter under the scheme of the RSS is greater than information for the shape parameter under the scheme of MERSS, and information for the scale parameter under the scheme of MERSS is greater than information for the scale parameter under the scheme of RSS.

5.1. Simulated output

In this section, we generate 100000 random numbers from the generalized exponential distribution given in (1) using known values of parameters (α, β) taking $\alpha = 0.8, \alpha = 1.2, \alpha = 1.6, \alpha = 1.8, \alpha = 2.2, \alpha = 3.8$ and $\beta = 0.25$. At each cycle we obtain MERSS sample, by selecting $m^2 + m$ units, SRS sample, by selecting $2m$ units, and RSS sample, by selecting

$4m^2$ units. The ML estimates $(\hat{\alpha}, \hat{\beta})$ are calculated numerically and the comparisons are determined by the mean squared error of estimates, when sample size of SRS, MERSS and RSS are equal to $2mh$ respectively. We define the efficiency of sampling scheme as,

$$E_{\text{Sampling}_1,\text{Sampling}_2}(\hat{\alpha}) = \frac{MSE(\hat{\alpha})_{\text{Sampling}_2}}{MSE(\hat{\alpha})_{\text{Sampling}_1}}$$

and

$$E_{\text{Sampling}_1,\text{Sampling}_2}(\hat{\beta}) = \frac{MSE(\hat{\beta})_{\text{Sampling}_2}}{MSE(\hat{\beta})_{\text{Sampling}_1}}$$

6. Conclusion

1. In this paper, we estimate the parameters of the generalized exponential distribution using moving extreme ranked sampling, ranked set sampling, and simple random sampling. Then we compared the estimates using the mean squared errors of the estimates.

2. From the simulation study, it is found that under the RSS scheme we get smaller MSE as compared to MSE obtained for the MERSS scheme and SRS scheme for both shape and scale parameters. (Annexure Table A2)

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ANNEXURE

Table A1: Biases and MSE's of ML estimates of shape and scale parameters

Sampling	(m,h)	(α,β)	$\hat{\alpha}$	MSE($\hat{\alpha}$)	Bias($\hat{\alpha}$)	$\hat{\beta}$	MSE($\hat{\beta}$)	Bias($\hat{\beta}$)
SRS	(3,3)	(0.8,0.25)	0.9487606	0.1736299	0.1487606	0.2956023	0.0133619	0.0456023
RSS			0.8817586	0.0584896	0.0817586	0.2733177	0.0055097	0.0233177
MERSS			0.7964535	0.0717147	-0.0035465	0.2423628	0.0071007	-0.0076372
SRS	(3,3)	(1.2,0.25)	1.445111	0.4235251	0.2451109	0.2874173	0.0102904	0.0374173
RSS			1.336743	0.1457796	0.1367427	0.2681817	0.0039607	0.0181817
MERSS			1.202511	0.1835160	0.0025113	0.2452994	0.0062704	-0.0047006
SRS	(3,3)	(1.6,0.25)	2.022529	1.1205389	0.4225288	0.2816831	0.0075248	0.0316831
RSS			1.806412	0.3668359	0.2064123	0.2668984	0.0036996	0.0168984
MERSS			1.634325	0.5443902	0.0343248	0.2458775	0.0057707	-0.0041225
SRS	(3,3)	(1.8,0.25)	2.320982	2.1188691	0.5209824	0.2832089	0.0086354	0.0332089
RSS			1.994951	0.4090394	0.1949514	0.2623048	0.0028807	0.0123048
MERSS			1.797398	0.5843733	-0.0026018	0.2401457	0.0046996	-0.0098543
SRS	(3,3)	(2.2,0.25)	2.741057	2.1544205	0.5410568	0.2753242	0.0061387	0.0253242
RSS			2.505455	0.7952925	0.3054547	0.2648720	0.0029338	0.0148720
MERSS			2.167485	0.8985339	-0.0325153	0.2383081	0.0046774	-0.0116919
SRS	(3,3)	(3.8,0.25)	5.034128	10.224889	1.2341283	0.2764764	0.0054923	0.0264764
RSS			4.414143	3.255205	0.6141435	0.2632917	0.0024231	0.0132917
MERSS			3.895133	4.580507	0.0951328	0.2400658	0.0040669	-0.0099342

Table A1(Continued): Biases and MSE's of ML estimates of shape and scale parameters

Sampling	(m,h)	(α,β)	$\hat{\alpha}$	MSE($\hat{\alpha}$)	Bias($\hat{\alpha}$)	$\hat{\beta}$	MSE($\hat{\beta}$)	Bias($\hat{\beta}$)
SRS	(4,2)	(0.8,0.25)	0.9653919	0.1879596	0.1653919	0.2988921	0.0166438	0.0488921
RSS			0.8787682	0.0488045	0.0787682	0.2753540	0.0054171	0.0253540
MERSS			0.7710374	0.0690403	-0.0289626	0.2380351	0.0083744	-0.0119649
SRS	(4,2)	(1.2,0.25)	1.523014	0.5874794	0.3230142	0.2970299	0.0125121	0.0470299
RSS			1.312745	0.1347717	0.1127447	0.2664989	0.0038060	0.0164989
MERSS			1.172552	0.1804458	-0.0274481	0.2371569	0.0060172	-0.0128431
SRS	(4,2)	(1.6,0.25)	1.990566	1.0587583	0.3905659	0.2806856	0.0085753	0.0306856
RSS			1.786667	0.3209858	0.1866674	0.2664005	0.0033632	0.0164005
MERSS			1.556896	0.4012962	-0.0431041	0.2366258	0.0056646	-0.0133742
SRS	(4,2)	(1.8,0.25)	2.300448	1.5793808	0.5004485	0.2871264	0.0087790	0.0371264
RSS			2.033651	0.4139100	0.2336511	0.2685046	0.0033391	0.0185046
MERSS			1.761394	0.6294839	-0.0386061	0.2354936	0.0057293	-0.0145064
SRS	(4,2)	(2.2,0.25)	2.840629	3.684262	0.6406294	0.2808774	0.0070519	0.0308774
RSS			2.427666	0.690291	0.2276657	0.2619086	0.0028063	0.0119086
MERSS			2.190393	1.325949	-0.0096071	0.2392440	0.0054996	-0.0107560
SRS	(4,2)	(3.8,0.25)	5.512445	30.686737	1.7124454	0.2776625	0.0069002	0.0276625
RSS			4.361663	2.803481	0.5616628	0.2609717	0.0022263	0.0109717
MERSS			3.758171	4.772074	-0.0418289	0.2353198	0.0043514	-0.0146802

Table A2: Efficiency of estimators of shape and scale parameters under MERSS and RSS schemes

(α,β)	(m,h)	$E_{\text{MERSS,SRS}}(\hat{\alpha})$	$E_{\text{MERSS,SRS}}(\hat{\beta})$	$E_{\text{RSS,MERSS}}(\hat{\alpha})$	$E_{\text{RSS,MERSS}}(\hat{\beta})$	$E_{\text{RSS,SRS}}(\hat{\alpha})$	$E_{\text{RSS,SRS}}(\hat{\beta})$
(0.8,0.25)	(3,3)	2.421120	1.881772	1.226110	1.288763	2.968560	2.425159
	(4,2)	2.722462	1.987462	1.414630	1.545919	3.851276	3.072456
(1.2,0.25)	(3,3)	2.307837	1.641107	1.258859	1.583154	2.905243	2.598127
	(4,2)	3.255711	2.079389	1.338900	1.580977	4.359071	3.287467
(1.6,0.25)	(3,3)	2.058338	1.303967	1.484016	1.559817	3.054605	2.033950
	(4,2)	2.638346	1.513840	1.250199	1.684289	3.298458	2.549744
(1.8,0.25)	(3,3)	3.625883	1.837476	1.428648	1.631409	5.180110	2.997674
	(4,2)	2.509009	1.532299	1.520823	1.715822	3.815759	2.629152
(2.2,0.25)	(3,3)	2.397706	1.312417	1.129816	1.594315	2.708966	2.092406
	(4,2)	2.778585	1.282257	1.920855	1.959733	5.337259	2.512882
(3.8,0.25)	(3,3)	2.232261	1.350488	1.407133	1.678387	3.141089	2.266642
	(4,2)	6.430482	1.585743	1.702196	1.954543	10.945941	3.099403