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Unifying Constructions of Group Divisible Designs

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Abstract

The purpose of this paper is to unify constructions of group divisible designs by making use of certain balanced incomplete block designs, skew-Hadamard matrices, regular Hadamard matrices, balanced generalized Weighing matrices, Conference matrices and generalized Conference matrices. The constructions unify the results of Dey (1977), Dey and Nigam (1985), Parihar and Shrivastava (1988), De and Roy (1990) and generalize some results of Bhagwandas *et al*. (1985), Sinha (1991*b*) and Kadowaki and Kageyama (2009). In the process of investigations, some group divisible designs in the range of r, $k \leq 10$ are found and catalogued. These designs are obtained from the works of other authors but are not reported in Clatworthy (1973) and Sinha (1991*a*).

Keywords: Balanced incomplete block designs; Group divisible designs; Generalized Hadamard matrices; Generalized Conference matrices; Generalized Weighing matrices.

0. Prologue

Dr. Kishore Sinha had the opportunity of working with Professor Aloke Dey, at IASRI, New Delhi as a Post- doctoral research fellow of CSIR, New Delhi during 1977- 1979. It was during this period that he got fascinated with the research work of Professor Aloke Dey especially in the area of Partially Balanced Incomplete Block (PBIB) Designs. His association with Professor Aloke Dey continued growing in strength even after he left IASRI in 1979. His untimely demise has been a personal loss to Kishore in particular and to statistician's fraternity in general.

Various methods of constructions and trial and error solutions of group divisible designs are available and scattered over the literature. To the best of our knowledge, Dey (1977) for the first-time used matrix approach for the constructions of group divisible designs. His works motivated us to take up unification and generalization of constructions of group divisible designs. It is my proud privilege to pay my most respectful homage by dedicating this research paper to his memory.

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1. Introduction

Some relevant definitions in the context of the paper are as follows:

1.1. Group divisible designs

A *Group divisible* (*GD*) *design* is an arrangement of v (= mn; m, $n \geq 2$) treatments into b blocks such that each block contains k (<v) distinct treatments, each treatment occurs r times and any pair of distinct treatments which are first associates occur together in λ_1 blocks and in λ_2 blocks if they are second associates. Furthermore, if $r–\lambda_1 = 0$ then the GD design is singular; if $r–\lambda_1 > 0$ and $rk–v\lambda_2 = 0$ then it is semi-regular (SR); and if $r–\lambda_1 > 0$ and $rk–v\lambda_2 > 0$, the design is regular (R). Semi- regular and regular GD designs are denoted by SRGD and RGD respectively. Following Cheng (1995), GD designs with parameters satisfying $b = 4(r–\lambda_2)$ are called family (A) GD designs.

1.2. α- Resolvable design

A block design D (v, b, r, k) whose b blocks can be divided into $t = r/\alpha$ classes, each of size $\beta = v\alpha/k$ and such that in each class of β blocks every treatment of D is replicated α times, is called an α- resolvable design. When α=1 the design is said to be resolvable.

1.3. Hadamard matrices

An $n \times n$ matrix $\mathbf{H} = (H_{ij})$ with entries H_{ij} as ± 1 is called a *Hadamard matrix* if $\mathbf{HH'}=\mathbf{H'H}=n\mathbf{I}_n$, where $\mathbf{H'}$ is the transpose of $\mathbf{H}$ and $\mathbf{I}_n$ is the identity matrix of order n. A Hadamard matrix is in normalized form if its first row and first column contain only +1's. A Hadamard matrix $\mathbf{H}$ is said to be of *skew type or skew- Hadamard* if its main diagonal entries are +1 and $\mathbf{H}–\mathbf{I}_n$ is *skew- symmetric*. In other words, a Hadamard matrix is called skew-symmetric if $H_{ij} = –H_{ji}\ \forall i \neq j$ and $H_{ii} = 1\ \forall i$.

Example 1: $\mathbf{H} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$ are Hadamard matrices of order 2 and 4 respectively.

Example 2: $\mathbf{H} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$ are skew- Hadamard matrices of order 2 and 4 respectively.

1.4. Conference matrices

A *Conference matrix* of order n is an $n \times n$ matrix $\mathbf{C}$ with diagonal entries 0 and off-diagonal entries ± 1 such that $\mathbf{CC'} = (n-1)\mathbf{I}_n$. A *Conference matrix* $\mathbf{C}$ is symmetric if $\mathbf{C} = \mathbf{C'}$ and skew- symmetric if $\mathbf{C} = -\mathbf{C'}$. A Conference matrix of order n is denoted as CM (n).

Example 3: $\mathbf{C} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}$ is a symmetric Conference matrix and

$\mathbf{C} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix}$ is a skew – symmetric Conference matrix.

1.5. Regular Hadamard matrices

A Hadamard matrix is *regular* if sum of the elements in any row of the matrix is constant. It is known that the order of a regular Hadamard matrix is a perfect square $4t^2$, t a positive integer. The number of entries +1 in any row is a constant, either $2t^2-t$ or $2t^2+t$. In the first case, any two rows will have t^2-t positions wherein both have entry +1; the second case has t^2+t positions wherein both have entry +1. For methods of construction, see Crnkovic (2006).

Example 4: $\mathbf{H} = \begin{pmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix}$ is a regular Hadamard matarix of order 4.

1.6. Generalized weighing matrix

Let $n \geq w \geq 1$. A *Weighing matrix* $\mathbf{W}(n, w)$ of order n and weight w is an $n \times n$ $(0, \pm 1)$ – matrix such that $\mathbf{WW}' = w\mathbf{I}_n$.

A *generalized Weighing matrix* is a $v \times b$ matrix $\mathbf{M} = (m_{ij})$ with entries 0 and elements of a multiplicative group G of order g such that the inner product of any pair of distinct rows contains every element of G same number of times.

A generalized Weighing matrix $v \times b$ with the additional property that every row contains precisely r nonzero entries, each column contains exactly k nonzero entries and the inner product of any pair of distinct rows contains every group element exactly λ/g times, is known as a *generalized Bhaskar Rao design* GBRD $(v, b, r, k, \lambda; G)$. By replacing its nonzero entries by unity, produces the incidence matrix of a BIB design (v, b, r, k, λ).

A *Bhaskar Rao design* BRD (v, b, r, k, λ) is a $v \times b$ $(0, \pm 1)$ – matrix such that the inner product of any pair of distinct rows is zero and replacing –1 by unity, produces the incidence matrix of a BIB design (v, b, r, k, λ).

A GBRD $(v, b, r, k, \lambda; G)$ with $r = k$ and $v = b$ is also known as a *balanced generalized Weighing matrix* BGWM $(v, k, \lambda; G)$.

If the diagonal entries of BGWM $(v, k, \lambda; G)$ are zero and the inner product of any pair of distinct rows contains each element of G exactly λ times, then it is known as *generalized*

Conference matrix, GCM (*G; λ*). The order of *GCM* (*G; λ*) is λg+2. If $G = \{\pm 1\}$, then *GCM* (*G; λ*) is a Conference matrix of order 2(λ+1). For details, see Colbourn and Dinitz (2007) and Tonchev (2009).

Example 5: A GBRD (4, 12, 9, 3, 6; C_6) over a cyclic group $C_6 = \{1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5\}$ is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & \alpha^4 & \alpha^2 & \alpha^3 & \alpha & \alpha^5 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & \alpha^2 & \alpha^4 & 0 & 0 & 0 & \alpha^3 & \alpha & \alpha^5 & \alpha^3 & \alpha & \alpha^5 \\ 0 & 0 & 0 & 1 & \alpha^2 & \alpha^4 & \alpha^3 & \alpha^5 & \alpha & 1 & \alpha^2 & \alpha^4 \end{pmatrix}.$$

Example 6: BRD (6, 6, 5, 5, 4) = $\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}.$

Example 7: A 5 x 5 BGWM (5, 4, 3; C_3) over a cyclic group $C_3 = \{1, \alpha, \alpha^2\}$ is

$$\begin{pmatrix} 0 & \alpha^2 & \alpha & \alpha^2 & \alpha^2 \\ 1 & 0 & \alpha^2 & \alpha & \alpha^2 \\ 1 & 1 & 0 & \alpha^2 & \alpha \\ \alpha^2 & 1 & 1 & 0 & \alpha^2 \\ 1 & \alpha^2 & 1 & 1 & 0 \end{pmatrix}.$$

Example 8: A GCM (C_3; 2) of order 8 over a cyclic group $C_3 = \{1, \alpha, \alpha^2\}$ is

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \alpha^2 & \alpha & \alpha & \alpha^2 & 1 \\ 1 & 1 & 0 & 1 & \alpha^2 & \alpha & \alpha & \alpha^2 \\ 1 & \alpha^2 & 1 & 0 & 1 & \alpha^2 & \alpha & \alpha \\ 1 & \alpha & \alpha^2 & 1 & 0 & 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha & \alpha^2 & 1 & 0 & 1 & \alpha^2 \\ 1 & \alpha^2 & \alpha & \alpha & \alpha^2 & 1 & 0 & 1 \\ 1 & 1 & \alpha^2 & \alpha & \alpha & \alpha^2 & 1 & 0 \end{pmatrix}.$$

1.7. Generalized Hadamard matrix and Difference matrix

A *generalized Hadamard matrix* GHM (λ, *g*) over a group *G* of order *g* is a balanced generalized weighing matrix with $v = b = k = \lambda$. For GHM we require that the matrix should be square, but if we relax this condition and allow $v \times b$ ($v \leq b$) matrices, along with the conditions imposed on GHM, we obtain *difference matrices*. For details see Lampio (2015).

Example 9: $\text{GHM}\,(6,3) = \begin{pmatrix} 1 & \alpha & \alpha & 1 & \alpha^2 & \alpha^2 \\ \alpha^2 & \alpha^2 & \alpha & 1 & 1 & \alpha \\ \alpha^2 & \alpha & \alpha^2 & 1 & \alpha & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \alpha & \alpha^2 & 1 & 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha^2 & 1 & \alpha^2 & \alpha \end{pmatrix}$ is a generalized Hadamard matrix with elements from the cyclic group $C_3 = \{1, \alpha, \alpha^2\}$.

Example 10: A 3×8 difference matrix over a cyclic group $C_4 = \{1, \alpha, \alpha^2, \alpha^4\}$ is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \alpha & \alpha & \alpha^2 & \alpha^2 & \alpha^3 & \alpha^3 \\ 1 & 1 & \alpha^2 & \alpha^3 & \alpha & \alpha^3 & \alpha & \alpha^2 \end{pmatrix}.$$

1.8. Kronecker sum of two matrices

Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ be two matrices of orders $m\times$n and $p\times q$ respectively over a field. Then the *Kronecker sum* $\mathbf{A} \oplus \mathbf{B}$ is an $mp\times nq$ matrix given by

$$\mathbf{A} \oplus \mathbf{B} = \mathbf{A} \otimes \mathbf{J}_{p,q} + \mathbf{J}_{m,n} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{J}_{p,q} + \mathbf{B} & a_{12}\mathbf{J}_{p,q} + \mathbf{B} & \cdots & a_{1n}\mathbf{J}_{p,q} + \mathbf{B} \\ a_{21}\mathbf{J}_{p,q} + \mathbf{B} & a_{22}\mathbf{J}_{p,q} + \mathbf{B} & \cdots & a_{2n}\mathbf{J}_{p,q} + \mathbf{B} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}\mathbf{J}_{p,q} + \mathbf{B} & a_{m2}\mathbf{J}_{p,q} + \mathbf{B} & \cdots & a_{mn}\mathbf{J}_{p,q} + \mathbf{B} \end{pmatrix}.$$

where $\mathbf{J}_{v\times b}$ is the $v \times b$ matrix all of whose entries are 1, $\mathbf{A}\otimes\mathbf{B}$ is the Kronecker (or tensor) product of two matrices **A** and **B.**

Here, several methods of constructions of series of GD designs from certain BIB designs, skew Hadamard matrices, regular Hadamard matrices, balanced generalized Weighing matrices, Conference matrices and generalized Conference matrices are described. The constructions unify the results of Dey (1977), Dey and Nigam (1985), Parihar and Shrivastava (1988), De and Roy (1990) and generalize several results of Bhagwandas *et al*. (1985), Sinha (1991*b*) and Kadowaki and Kageyama (2009). A comprehensive coverage of constructions of GD designs may also be found in Arasu *et al.* (1991), Dey and Balasubramanian (1991), Dey (1986, 2010), Raghavarao (1971), Raghavarao and Padgett (2005). In the process of investigations, some group divisible designs in the range of r, $k \leq 10$ are found and catalogued. These designs are obtained from the works of other authors but are not reported in Clatworthy (1973) and Sinha (1991*a*).

The following notations are used: $\mathbf{I}_n$ is the identity matrix of order n, $\mathbf{J}_{v\times b}$ is the $v \times b$ matrix all of whose entries are 1 and $\mathbf{J}_{v\times v} = \mathbf{J}_v$, $\mathbf{A}\otimes\mathbf{B}$ is the Kronecker product of two matrices **A** and **B**, $\mathbf{A}'$ is the transpose of matrix **A** and $\mathbf{O}_n$ is null matrix of order n. *SRX* and *RX* numbers are from Clatworthy (1973). The design numbers *SRXa* and *RXa*, *b*, *c*, *d* are not found in Clatworthy (1973); and these designs are supposed to be located between SRX and SR(X+1) and RX and R(X+1) respectively.

2. The Constructions

2.1. From BIB designs

Theorem 1: There exists a GD design with parameters

$$v^* = vs, b^* = stv, r^* = t(k+s-1), k^* = k+s-1, \lambda_1 = (s-2)t, \lambda_2 = \lambda, m = v,$$

$$n = s; m, s \geq 2; t = r/\alpha \quad (1)$$

where v, k, λ are the parameters of an α- resolvable BIB design with $\lambda = t[(k+s-1)(k+s-2)-(s-1)(s-2)]/s(v-1)$.

Proof: Let $\mathbf{N}_i$ $(1 \leq i \leq t)$ represent the incidence matrices corresponding to resolution classes of an α- resolvable balanced incomplete block (BIB) design with parameters v, $b = tv$, r, k and $\lambda = t[(k+s-1)(k+s-2)-(s-1)(s-2)]/s(v-1)$ and also satisfying the condition $\sum_{i=1}^{t}(\mathbf{N}_i + \mathbf{N}_i') = \lambda(\mathbf{J}-\mathbf{I})_v$.

Then the incidence pattern

$$\mathbf{M} = \mathbf{I}_s \otimes \mathbf{N}_{v\times tv} + (\mathbf{J}_s - \mathbf{I}_s) \otimes (\mathbf{I}_v|\mathbf{I}_v|\cdots|\mathbf{I}_v)$$

$$= \begin{bmatrix} (\mathbf{N_1}|\mathbf{N_2}|\cdots|\mathbf{N}_t) & (\mathbf{I}_v|\mathbf{I}_v|\cdots|\mathbf{I}_v) & \cdots & (\mathbf{I}_v|\mathbf{I}_v|\cdots|\mathbf{I}_v) \\ (\mathbf{I}_v|\mathbf{I}_v|\cdots|\mathbf{I}_v) & (\mathbf{N_1}|\mathbf{N_2}|\cdots|\mathbf{N}_t) & \cdots & (\mathbf{I}_v|\mathbf{I}_v|\cdots|\mathbf{I}_v) \\ \vdots & \vdots & \ddots & \vdots \\ (\mathbf{I}_v|\mathbf{I}_v|\cdots|\mathbf{I}_v) & (\mathbf{I}_v|\mathbf{I}_v|\cdots|\mathbf{I}_v) & \cdots & (\mathbf{N_1}|\mathbf{N_2}|\cdots|\mathbf{N}_t) \end{bmatrix}$$

represents a GD design with parameters (1).

For $t = s = 2$ in Theorem 1 we obtain:

Corollary 1: There exists a GD design with parameters

$$v^* = 2v, b^* = 4v, r^* = 2(k+1), k^* = k+1, \lambda_1 = 0, \lambda_2 = \lambda = k(k+1)/(v-1),$$

$$m = v, n = 2.$$

For $t = 2$, $s = 3$ in Theorem 1 we obtain:

Corollary 2: There exists a GD design with parameters

$$v^* = 3v, b^* = 6v, r^* = 2(k+2), k^* = k+2, \lambda_1 = 2, \lambda_2 = \lambda = 2k(k+3)/3(v-1),$$

$$m = v, n = 3.$$

Table 1 lists GD designs constructed using Corollaries 1 and 2:

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Table 1: RGD from BIB designs

No.	GD: $(v, r, k, b, \lambda_1, \lambda_2, m, n)$	Source: 3- resolvable BIB design (v, r, k, b, λ)
1	$R106$: (10, 8, 4, 20, 0, 3, 5, 2)	(5, 6, 3, 10, 3), Corollary 1
2	$R128$: (26, 8, 4, 52, 0, 1, 13, 2)	(13, 6, 3, 26, 1), Corollary 1
3	$R150$: (15, 10, 5, 30, 2, 3, 5, 3)	(5, 6, 3, 10, 3), Corollary 2
4	$R160$: (39, 10, 5, 78, 2, 1, 13, 3)	(13, 6, 3, 26, 1), Corollary 2

The incidence matrix of a 3- resolvable BIB design with parameters (5, 6, 3, 10, 3) used for constructions of $R106$ and $R150$ in Table 1 can be partitioned as:

$$\mathbf{N}_{5\times 10} = (\mathbf{N}_1|\mathbf{N}_2) = \left(\begin{array}{ccccc|ccccc} 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right)$$

i. e. $\mathbf{N}_1$= *circ* (0 1 0 1 1) and $\mathbf{N}_2$= *circ* (0 1 1 1 0).

Example 11: The blocks of $R106$ using Corollary 1 are given as:
[(2, 3, 5, 6), (1, 3, 4, 7), (2, 4, 5, 8), (1, 3, 5, 9), (1, 2, 4, 10), (3, 4, 5, 6), (1, 4, 5, 7), (1, 2, 5, 8), (1, 2, 3, 9), (2, 3, 4, 10), (1, 7, 8, 10), (2, 6, 8, 9), (3, 7, 9, 10), (4, 6, 8, 10), (5, 6, 7, 9), (1, 8, 9, 10), (2, 6, 9, 10), (3, 6, 7, 10), (4, 6, 7, 8), (5, 7, 8, 9)].

The GD scheme is given as the 5 x 2 array: $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \end{bmatrix}'$.

Example 12: The blocks of $R150$ using Corollary 2 are given as:
[(2, 3, 5, 6, 11), (1, 3, 4, 7, 12), (2, 4, 5, 8, 13), (1, 3, 5, 9, 14), (1, 2, 4, 10, 15), (3, 4, 5, 6, 11), (1, 4, 5, 7, 12), (1, 2, 5, 8, 13), (1, 2, 3, 9, 14), (2, 3, 4, 10, 15), (1, 7, 8, 10, 11), (2, 6, 8, 9, 12), (3, 7, 9, 10, 13), (4, 6, 8, 10, 14), (5, 6, 7, 9, 15), (1, 8, 9, 10, 11), (2, 6, 9, 10, 12), (3, 6, 7, 10, 13), (4, 6, 7, 8, 14), (5, 7, 8, 9, 15), (1, 6, 12, 13, 15), (2, 7, 11, 13, 14), (3, 8, 12, 14, 15), (4, 9, 11, 13, 15), (5, 10, 11, 12, 14), (1, 6, 13, 14, 15), (2, 7, 11, 14, 15), (3, 8, 11, 12, 15), (4, 9, 11, 12, 13), (5, 10, 12, 13, 14)].

The GD scheme is given as the 5 x 3 array: $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \end{bmatrix}'$.

A 3- resolvable solution of BIB design with parameters (13, 6, 3, 26, 1) may be found in Kageyama and Mohan (1983). This solution for the construction of $R128$ and $R160$ in Table 1 can be partitioned as:

$\mathbf{N}_{13\times 26} = (\mathbf{N}_1|\mathbf{N}_2)$ where $\mathbf{N}_1$= *circ* (0 0 0 0 1 0 0 0 0 0 1 0 1) and

$\mathbf{N}_2$ = *circ* (0 0 0 0 0 0 0 1 1 0 0 1 0).

Remark 1: The Corollary 1 gives patterned construction for $R106$ whereas an individual solution is given in Dey (1977).

2.2. From skew- Hadamard matrices

Skew- Hadamard matrices are known to exist for the order 2^n, where $n>0$ is an integer; order $4t$, where $4t$-1 is a prime or prime power. For details on existence of skew- Hadamard matrices see Koukouvinos and Stylianou (2008).

Lemma 1: Let **N** be the incidence matrix of a BIB design obtained from the core of a normalized skew- Hadamard matrix of order $4t$. Then

(i) $\mathbf{N}+\mathbf{N}'=(\mathbf{J}-\mathbf{I})_{\mathbf{4}t-\mathbf{1}}$ (ii) $\mathbf{N}^{\mathbf{2}}+\mathbf{N}=t(\mathbf{J}-\mathbf{I})_{\mathbf{4}t-\mathbf{1}}$.

Proof: Let **C** be the core of a normalized skew- Hadamard matrix of order $4t$ obtained by deleting first row and first column. Then the diagonal entries of **C** are -1 and

(a) $\mathbf{C}+\mathbf{I}_{4t-1}$ is a skew- symmetric matrix *i.e.* $\mathbf{C}+\mathbf{I}_{\mathbf{4}t-\mathbf{1}}=-(\mathbf{C}+\mathbf{I}_{\mathbf{4}t-\mathbf{1}})'$

$$\Rightarrow \mathbf{C}+\mathbf{C}'=-2\mathbf{I}_{\mathbf{4}t-\mathbf{1}}.$$

(b) $\mathbf{CC}'=4t\mathbf{I}_{\mathbf{4}t-\mathbf{1}}-\mathbf{J}_{\mathbf{4}t-\mathbf{1}}$.

Clearly $\mathbf{N}=(\mathbf{C}+\mathbf{J}_{\mathbf{4}t-\mathbf{1}})/2$ represents a symmetric ($4t$–1, $2t$–1, t–1) – design and **N**, $\mathbf{N}'$ have zeros in diagonals. Then

$$\mathbf{N}+\mathbf{N}'=(\mathbf{C}+\mathbf{C}'+2\mathbf{J}_{\mathbf{4}t-\mathbf{1}})/2=(\mathbf{J}-\mathbf{I})_{\mathbf{4}t-\mathbf{1}}$$

$$\begin{aligned}\mathbf{N}^{\mathbf{2}}+\mathbf{N}&=(\mathbf{C}^{\mathbf{2}}+2\mathbf{CJ}_{\mathbf{4}t-\mathbf{1}}+\mathbf{J}^{\mathbf{2}}_{\mathbf{4}t-\mathbf{1}}+2\mathbf{C}+2\mathbf{J}_{\mathbf{4}t-\mathbf{1}})/4\\
&=[\mathbf{C}(\mathbf{C}+2\mathbf{I}_{\mathbf{4}t-\mathbf{1}})-2\mathbf{J}_{\mathbf{4}t-\mathbf{1}}+(4t-1)\mathbf{J}_{\mathbf{4}t-\mathbf{1}}+2\mathbf{J}_{\mathbf{4}t-\mathbf{1}}]/4\\
&=(-\mathbf{CC}'+(4t-1)\mathbf{J}_{\mathbf{4}t-\mathbf{1}})/4=(-(4t\mathbf{I}_{\mathbf{4}t-\mathbf{1}}-\mathbf{J}_{\mathbf{4}t-\mathbf{1}})+(4t-1)\mathbf{J}_{\mathbf{4}t-\mathbf{1}})/4\\
&=t(\mathbf{J}-\mathbf{I})_{\mathbf{4}t-\mathbf{1}}.\end{aligned}$$

Theorem 2: The existence of a skew- Hadamard matrix of order $4t$ implies the existence of a GD design with parameters

$$v=b=6(4t-1),\ r=k=2(5t-2),\ \lambda_1=5(t-1),\ \lambda_2=2(2t-1),\ m=6,\ n=4t-1. \quad (2)$$

Proof: Let **N** be the incidence matrix of a BIB design obtained from the core of a normalized skew-Hadamard matrix of order $4t$ and **C** be a conference matrix of order 6. Then replacing 0 by $\mathbf{I}_{\mathbf{4}t-\mathbf{1}}$, 1 by **N** and –1 by $\mathbf{N}'$ in **C** we obtain a (0, 1) - matrix

$$\mathbf{M}=\begin{pmatrix}\mathbf{I}_{\mathbf{4}t-\mathbf{1}} & \mathbf{N} & \mathbf{N} & \mathbf{N} & \mathbf{N} & \mathbf{N}\\ \mathbf{N}' & \mathbf{I}_{\mathbf{4}t-\mathbf{1}} & \mathbf{N}' & \mathbf{N}' & \mathbf{N} & \mathbf{N}\\ \mathbf{N}' & \mathbf{N}' & \mathbf{I}_{\mathbf{4}t-\mathbf{1}} & \mathbf{N} & \mathbf{N} & \mathbf{N}'\\ \mathbf{N}' & \mathbf{N}' & \mathbf{N} & \mathbf{I}_{\mathbf{4}t-\mathbf{1}} & \mathbf{N}' & \mathbf{N}\\ \mathbf{N}' & \mathbf{N} & \mathbf{N} & \mathbf{N}' & \mathbf{I}_{\mathbf{4}t-\mathbf{1}} & \mathbf{N}'\\ \mathbf{N}' & \mathbf{N} & \mathbf{N}' & \mathbf{N} & \mathbf{N}' & \mathbf{I}_{\mathbf{4}t-\mathbf{1}}\end{pmatrix}.$$

Using the relations $\mathbf{NN}' = \mathbf{N}'\mathbf{N} = \mathrm{t}\mathbf{I}_{4t-1} + (t-1)\mathbf{J}_{4t-1}$and $\mathbf{N}^2 + \mathbf{N} = (\mathbf{N}')^2 + \mathbf{N}' = t(\mathbf{J} - \mathbf{I})_{4t-1}$ one can see that **M** represents the incidence matrix of a GD design with the parameters (2).

Remark 2: For *t*=3 in Theorem 2 we obtain a BIB design with parameters *v*=*b*=66, *r*=*k*=26, λ=10, reported in Hall (1998) as design number 214.

Remark 3: For $\mathbf{N} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and *t*=1 in the incidence matrix **M** of the Theorem 2 we obtain *SR*72.

Following Dey and Balasubramanian (1991), series 25 (rephrased), p. no. 288:

If there exists a symmetric BIB design with parameters $v' = 4t - 1, k' = 2t - 1, \lambda' = t - 1 (t \geq 1)$ such that the incidence matrix **N** of the BIB design satisfies $\mathbf{N} + \mathbf{N}' = (\mathbf{J} - \mathbf{I})_{v'}$, then there exists a GD design with parameters: $v = b = pv', r = k = pk' + 1, \lambda_1 = pk', \lambda_2 = p\lambda' + 1, m = v', n = p(\geq 2)$.

It is known that the incidence matrix **N** of a BIB design obtained from the core of a normalized skew- Hadamard matrix of order 4t satisfies $\mathbf{N} + \mathbf{N}' = (\mathbf{J} - \mathbf{I})_{4t-1}$, see Lemma 4 above.

Theorem 3: The existence of a skew- Hadamard matrix of order 4*t* implies the existence of a 2- parameter GD design with parameters

$$v = b = p(4t - 1), r = k = p(2t - 1) + 1, \lambda_1 = p(2t - 1), \lambda_2 = p(t - 1) + 1,$$

$$m = 4t - 1, n = p, t \geq 1. \quad (3)$$

Proof: Let **N** be the incidence matrix of a BIB design obtained from the core of a normalized skew-Hadamard matrix of order 4*t*. Then using the relations $\mathbf{NN}' = \mathbf{N}'\mathbf{N} = t\mathbf{I}_{4t-1} + (t - 1)\mathbf{J}_{4t-1}$and $\mathbf{N} + \mathbf{N}' = (\mathbf{J} - \mathbf{I})_{4t-1}$ it can be verified that $\mathbf{M} = \mathbf{I}_p \otimes (\mathbf{I}_{4t-1} + \mathbf{N}) + (\mathbf{J} - \mathbf{I})_p \otimes \mathbf{N}$ is the incidence matrix of a GD design with parameters (3).

The following Table lists regular GD designs constructed using Theorem 3:

Table 2: RGD from skew- Hadamard matrices

No.	GD: ($v, r, k, b, \lambda_1, \lambda_2, m, n$)	p, t	Reference
1	*R*177: (14, 7, 7, 14, 6, 3,7, 2)	$p = t = 2$	Clatworthy (1973)
2	R206*a:* (21, 10, 10, 21, 9, 4, 7, 3)	$p = 3, t = 2$	Freeman (1976)

Remark 4: Following Theorem 7 of Bush (1979) and Corollary 4.1.1 of Kageyama and Tanaka (1981) we get:

A GD design with parameters

$$v=b=3(4t-1), r=k=2t+1, \lambda_1=t-1, \lambda_2=1, m=3, n=4t-1. \quad (4)$$

is obtained from the core of a normalized skew- Hadamard matrix.

2.3. From Conference matrices

Symmetric conference matrices are known to exist for orders 2, 4, 6, 10, … and skew-symmetric conference matrices are known to exist for 2, 4, 8, 12,….

Theorem 4: The existence of a conference matrix of order t (≥4) implies the existence of family (A) regular GD designs with parameters

(i) $v = b = 2t$, $r = k = t-1$, $\lambda_1 = 0$, $\lambda_2 = (t-2)/2$, $m = t$, $n = 2$. (5)

(ii) $v = b = 2t$, $r = k = t+1$, $\lambda_1 = 2$, $\lambda_2 = (t+2)/2$, $m = t$, $n = 2$. (6)

Proof: Let **C** be a conference matric of order t (≥4) and $\mathbf{N_1} = (\mathbf{J}_t - \mathbf{I}_t + \mathbf{C})/2\,, \mathbf{N_2} = (\mathbf{J}_t - \mathbf{I}_t - \mathbf{C})/2$ then we claim that $\mathbf{N} = \begin{pmatrix} \mathbf{N_1} & \mathbf{N_2} \\ \mathbf{N_2} & \mathbf{N_1} \end{pmatrix}$ is the incidence matrix of the GD design with parameters (5). We have

$$\mathbf{N_1}\mathbf{N}_1' + \mathbf{N_2}\mathbf{N}_2' = [(\mathbf{J}_t - \mathbf{I}_t + \mathbf{C})/2]\,[(\mathbf{J}_t - \mathbf{I}_t + \mathbf{C})/2]' + [(\mathbf{J}_t - \mathbf{I}_t - \mathbf{C})/2]\,[(\mathbf{J}_t - \mathbf{I}_t - \mathbf{C})/2]'$$

$$= (t-1)\mathbf{I}_t + [(t-2)/2](\mathbf{J}_t - \mathbf{I}_t).$$

$$\mathbf{N_1} + \mathbf{N_2} = \mathbf{J}_t - \mathbf{I}_t \Rightarrow (\mathbf{N_1} + \mathbf{N_2})(\mathbf{N_1} + \mathbf{N_2})' = (\mathbf{J}_t - \mathbf{I}_t)^2 = (\mathrm{t}-1)\,\mathbf{I}_t + (\mathrm{t}-2)\,(\mathbf{J}_t - \mathbf{I}_t)$$

$$\Rightarrow \mathbf{N_1}\mathbf{N}_2' + \mathbf{N_2}\mathbf{N}_1' = [(\mathrm{t}-1)\mathbf{I}_t + (\mathrm{t}-2)(\mathbf{J}_t - \mathbf{I}_t)] - [\mathbf{N_1}\mathbf{N}_1' + \mathbf{N_2}\mathbf{N}_2']$$

$$\Rightarrow \mathbf{N_1}\mathbf{N}_2' + \mathbf{N_2}\mathbf{N}_1' = [(t-2)/2](\mathbf{J}_t - \mathbf{I}_t)\,.$$

Thus $\mathbf{N_1}$ and $\mathbf{N_2}$ satisfy the conditions given in Dey (1977). Hence N is the incidence matrix of the GD design with parameters (5). The GD design with parameters (6) is complementary of the design with parameters (5).

The following Table lists GD designs obtained using Theorem 4:

Table 3: RGD from Conference Matrices

No.	GD: (v, r, k, b, λ_1, λ_2, m, n)	Source	Reference
1	*R*54: (8, 3, 3, 8, 0, 1, 4, 2)	CM (4)	Clatworthy (1973)
2	*R*144: (12, 5, 5, 12, 0, 2, 6, 2)	CM (6)	Dey (1977)
3	*R*117*a*: (16, 7, 7, 16, 0, 3, 8, 2)	CM (8)	Dey (1977)
4	*R*197*a*: (20, 9, 9, 20, 0, 4, 10, 2)	CM (10)	Dey (1977)

Theorem 5: The existence of a Conference matrix of order t (≥4) and a BIB design with

$v = 2k$, b, r, k, λ implies the existence of a GD design with parameters

$v^* = tv, b^* = tb, r^* = r(t-1), k^* = k(t-1), \lambda_1^* = (t-1)\lambda, \lambda_2^* = r(t-2)/2,$

$$m = t, n = v. \tag{7}$$

Proof: Let **N** be the incidence matrix of a BIB design with $v = 2k, b, r, k, \lambda$. Then replacing 0 by $\mathbf{O}_t$, 1 by **N** and –1 by $\bar{\mathbf{N}} = \mathbf{J}_{v\times b} - \mathbf{N}$ in a Conference matrix of order t we obtain a GD design with parameters (7).

For $(t-1)\lambda = r(t-2)/2$ in Theorem 5, we obtain:

Corollary 3: The existence of a conference matrix of order $t = 2(r-\lambda)/(r-2\lambda)$; ($t \geq 4$) and a BIB design with $v=2k, b, r, k, \lambda$ implies the existence of a BIB design with parameters

$$v^* = tv, b^* = tb, r^* = r(t-1), k^* = k(t-1), \lambda^* = (t-1)\lambda.$$

Using BIB design (4, 6, 3, 2, 1) and $t = 4$ in Corollary 3 produces *MR*35; and a BIB design (6, 10, 5, 3, 2) and $t = 6$ produces *MR*427. *MRX* denotes design number *X* in Mathon and Rosa (2007). It is not known if these solutions are isomorphic to theirs.

Remark 5: For $\mathbf{N} = \mathbf{I}_2$ in Theorem 5 we obtain Theorem 4 (i).

2.4. From balanced generalized Weighing matrices and generalized Conference matrices

Let $C_n = \{1, \alpha, \alpha^2, \cdots, \alpha^{n-1}\}$ denote a cyclic group of order n and $\boldsymbol{\beta}$ = *circ* (0 1 0…0) denote a circulant matrix of order n.

Replacing 1 by $\mathbf{I}_n$ and α^i by $\boldsymbol{\beta}^i$ ($1 \leq i \leq n-1$) in *BGWM* ($v, k, \lambda; C_n$) we obtain:

Theorem 6: The existence of a BGWM ($v, k, \lambda; C_n$) implies the existence of a GD with parameters

$$v^* = b^* = vn, r^* = k^* = k, \lambda_1 = 0, \lambda_2 = \lambda/n, m = v, n. \tag{8}$$

Further replacing 0 by $\mathbf{O}_n$, 1 by $\mathbf{I}_n$ and α^i by $\boldsymbol{\beta}^i$ ($1 \leq i \leq n-1$) in *GCM* ($C_n; \lambda$) of order v we obtain:

Theorem 7: The existence of a GCM ($C_n; \lambda$) of order $v = n\lambda+2$ implies the existence of a GD design with parameters

$$v^* = b^* = vn, r^* = k^* = k, \lambda_1 = 0, \lambda_2 = \lambda, m = v, n \tag{9}$$

where k is the number of nonzero entries in each column of *GCM* ($C_n; \lambda$).

The following Table lists GD designs obtained using Theorems 6 and 7:

Table 4: RGD from balanced generalized Weighing matrices and generalized Conference matrices

No.	GD: ($v, r, k, b, \lambda_1, \lambda_2, m, n$)	Source	Reference
1	$R112$: (14, 4, 4, 14, 0, 1, 7, 2)	BGWM (7, 4, 2; C_2)	Clatworthy (1973)
2	$R114$: (15, 4, 4, 15, 0, 1, 5, 3)	BGWM (5, 4, 3; C_3)	Clatworthy (1973)
3	$R180b$: (24, 7, 7, 24, 0, 2, 8, 3)	GCM (C_3; 2), Order = 8	F (1976)
4	$R182b$: (45, 7, 7, 45, 0, 1, 15, 3)	BGWM (15, 7, 3; C_3)	DR (1990)
5	$R191$: (63, 8, 8, 63, 0, 1, 9, 7)	GCM (C_7; 1), Order = 9	Clatworthy (1973)
6	$R200a$: (38, 9, 9, 38, 0, 2, 19, 2)	BGWM (19, 9, 4; C_2)	DR (1990)
7	$R200c$: (40, 9, 9, 40, 0, 2, 10, 4)	BGWM (10, 9, 8; C_4)	DN (1985)

F (1976), DN (1985) and DR (1990) stand for Freeman (1976), Dey and Nigam (1985) and De and Roy (1990) respectively. The balanced generalized Weighing matrices and generalized Conference matrices used in Table 3 may be found in Colbourn and Dinitz (2007).

2.5. From Kronecker Sum of Hadamard matrices and incidence matrices of BIB designs

Theorem 8 given below gives an algebraic representation of Theorem 1.6 of Parihar and Shrivastava (1988).

Theorem 8: The existence of a Hadamard matrix of order 4t and a BIB design with $v = 2k, b, r, k, \lambda$ implies the existence of a SRGD design with parameters

$$v^* = (4t-1)v, b^* = 4tb, r^* = 4tr, k^* = (4t-1)k, \lambda_1^* = 4t\lambda, \lambda_2^* = 2tr, m = 4t-1,$$

$$n = v. \tag{10}$$

Proof: Let $\mathbf{H}^*$ be a $(4t–1)$ x $4t$ matrix obtained by deleting the first row of a normalized Hadamard matrix and $\mathbf{N}$ be the incidence matrix of a BIB design with $v = 2k, b, r, k, \lambda$. Considering Kronecker sum $\mathbf{M} = \mathbf{H}^* \oplus \mathbf{N}$ of $\mathbf{H}^*$ and $\mathbf{N}$. Then under the transformation: $-1 \rightarrow 1$ in $-\bar{\mathbf{N}} = -(\mathbf{J}_{v\times b} - \mathbf{N})$ and $1 \rightarrow 0$, $2 \rightarrow 1$ in $\mathbf{J}_{v\times b} + \mathbf{N}$, it is easy to see that $\mathbf{M}$ represents incidence matrix of a SRGD with parameters (10).

Removing α ($1 \leq \alpha \leq 4t–3$) rows of blocks of the incidence matrix of the design with parameters (10) we obtain:

Corollary 4: There exists a SRGD design with parameters

$$v^* = (4t-\alpha-1)v, b^* = 4tb, r^* = 4tr, k^* = (4t-\alpha-1)k, \lambda_1^* = 4t\lambda, \lambda_2^* = 2tr,$$

$$m = 4t-\alpha-1, n = v. \tag{11}$$

Remark 6: The Corollary 4 unifies the Theorems 1.2, 1.3, 1.4 and 1.5 of Parihar and Shrivastava (1988).

Theorem 9: The existence of a regular Hadamard matrix of order $4t^2$ and a BIB design with $v=2k, b, r, k, \lambda$ implies the existence of a SRGD design with parameters

$$v^* = 4t^2v, b^* = 4t^2b, r^* = 4t^2r, k^* = 4t^2k, \lambda_1^* = 4t^2\lambda, \lambda_2^* = 2t^2r, m = 4t^2, n = v. \tag{12}$$

Proof: Let **H** be a regular Hadamard matrix of order $4t^2$ and **N** be the incidence matrix of a BIB design with $v = 2k, b, r, k, \lambda$. Considering Kronecker sum $\mathbf{M} = \mathbf{H} \oplus \mathbf{N}$ of **H** and **N**. Then under the transformation: $-1 \rightarrow 1$ in $-\bar{\mathbf{N}} = (\mathbf{J}_{v\times b} - \mathbf{N})$ and $1 \rightarrow 0$, $2 \rightarrow 1$ in $\mathbf{J}_{v\times b} + \mathbf{N}$, it is easy to see that **M** represents incidence matrix of a SRGD with parameters (12).

For $\mathbf{N} = \mathbf{I}_2$ in Theorem 9 we obtain:

Corollary 5: There exists a resolvable SRGD design with parameters

$$v^* = b^* = 8t^2, r^* = k^* = 4t^2, \lambda_1^* = 0, \lambda_2^* = 2t^2, m = 4t^2, n = 2. \tag{13}$$

Removing α ($1 \le \alpha \le 4t^2$-2) rows of blocks of the incidence matrix of design with parameters (12) we obtain:

Corollary 6: There exists a SRGD design with parameters

$$v^* = (4t^2 - \alpha)v, b^* = 4t^2b, r^* = 4t^2r, k^* = (4t^2 - \alpha)k, \lambda_1^* = 4t^2\lambda, \lambda_2^* = 2t^2r,$$

$$m = 4t^2 - \alpha, n = v. \tag{14}$$

Remark 7: This theorem is generalization and algebraic representation of the Theorem 2.2 of Bhagwandas *et al.* (1985). For $t = 1$ in Theorem 9 we obtain Theorem 2.2 of Bhagwandas *et al.* (1985).

Theorem 10: The existence of a Hadamard matrix of order $2t$ implies the existence of a resolvable SRGD design with parameters

$$D_i: v^* = b^* = 2^{i+2}t, r = k = 2^{i+1}t, \lambda_1 = 0, \lambda_2 = 2^it, m = 2^{i+1}t, n = 2\ (i \ge 0). \tag{15}$$

Proof: Kadowaki and Kageyama (2009, Theorem 3.3.4) constructed a resolvable SRGD design with parameters

$$D_0: v = b = 4t, r = k = 2t, \lambda_1 = 0, \lambda_2 = t, m = 2t, n = 2. \tag{16}$$

Let $\mathbf{N_0}$ be incidence matrix of a SRGD design D_0 with parameters (16). Considering Kronecker sum $\mathbf{N}_i = \mathbf{H_2} \oplus \mathbf{N}_{i-1} (i \ge 1)$ of $\mathbf{H_2}$ and $\mathbf{N}_{i-1}$, where $\mathbf{H_2}$ is a Hadamard matrix of order 2 and $\mathbf{N}_{i-1}$ represents the incidence matrix of a SRGD design with parameters

$$v' = b' = 2^{i+1}t, r' = k' = 2^it, \lambda_1 = 0, \lambda_2 = 2^{i-1}t, m = 2^it, n = 2\ (i \ge 1).$$

Then under the transformation: $-1 \rightarrow 1$ in $-(\mathbf{J}_{v'\times b'} - \mathbf{N}_{i-1})$ and $1 \rightarrow 0$, $2 \rightarrow 1$ in $\mathbf{J}_{v'\times b'} + \mathbf{N}_{i-1}$, it is easy to see that $\mathbf{N}_i$ represents incidence matrix of a SRGD with parameters (15).

Remark 8: This Theorem generalizes the Theorem 3.3.4 of Kadowaki and Kageyama (2009) and Theorem 2.1 of Sinha (1991*b*). For $i = 0$ we obtain Theorem 3.3.4 of Kadowaki and Kageyama (2009) and for $i = 1$ and 2 we obtain series 2.1 and 2.2 respectively of Sinha (1991*b*).

3. A Catalogue of Group Divisible Designs

In the process of present investigation, some GD designs scattered in literature are found; and those not found in Clatworthy (1973) and Sinha (1991a) are catalogued below, to make them available at one place for the convenience of researchers, looking for GD designs in the practical range of $r, k \leq 10$.

Table 5: A Catalogue of GD designs

No.	GD: ($v, r, k, b, \lambda_1, \lambda_2, m, n$)	Reference
1	$SR109a$: (50, 10, 10, 50, 0, 2, 10, 5)	GD (1995)
2	$R208b$: (49, 10, 10, 49, 1, 2, 7, 7)	S (1991)
3	$R206b$: (21, 10, 10, 21, 8, 3, 3, 7)	MD (1995)
4	$R200b$: (39, 9, 9, 39, 0, 2, 13, 3)	SS (2021)
5	$R198a$: (24, 9, 9, 24, 6, 3, 12, 2)	DK (1993)
6	$R200d$: (45, 9, 9, 45, 3, 1, 3, 15)	t = 4 in (4), Bush (1979)
7*	$SR103a$: (45, 10, 9, 50, 0, 2, 9, 5)	GD (1995)
8*	$SR95a$: (40, 10, 8, 50, 0, 2, 8, 5)	GD (1995)
9*	$SR86a$: (35, 10, 7, 50, 0, 2, 7, 5)	GD (1995)

S (1991), DK (1993), MD (1995), GD (1995) and SS (2021) stand for Sastry (1991), Duan and Kageyama (1993), Midha and Dey (1995), Ghosh and Divecha (1995) and Saurabh and Sinha (2020) respectively. The design numbers 1, 7, 8 and 9 were later on also reported by Kadowaki and Kageyama (2009).

*Design No. 7 of Table 5 is obtained by deleting the set of treatments 46, 47, 48, 49, 50 from design No. 1; design No. 8 is obtained by deleting the set of treatments 41, 42, 43, 44, 45 from design No. 7; and design No. 9 is obtained by deleting the set of treatments 36, 37, 38, 39, 40 from design No. 8.

As a special case having t=4, in Remark 4, we get a regular group divisible design with parameters: $v = b = 45, r = k = 9, \lambda_1 = 3, \lambda_2 = 1, m = 3, n = 15$ and the average efficiency E = 0.90. The solution given below is not found elsewhere:

(4 6 7 9 12 14 15 16 31), (1 5 7 9 10 13 15 17 32), (1 2 6 9 10 11 14 18 33),
(2 3 7 10 11 12 15 19 34), (1 3 4 9 11 12 13 20 35), (2 4 5 10 12 13 14 21 36),
(3 5 6 11 13 14 15 22 37), (4 6 7 8 10 11 13 24 39), (1 5 7 8 11 12 14 25 40),
(1 2 6 8 12 13 15 26 41), (1 2 3 4 5 6 7 23 38), (2 3 7 8 9 13 14 27 42),
(1 3 4 8 10 14 15 28 43), (2 4 5 8 9 11 15 29 44), (3 5 6 8 9 10 12 30 45),
(1 19 21 22 24 27 29 30 31), (2 16 20 22 24 25 28 30 32), (3 16 17 21 24 25 26 29 33),
(4 17 18 22 25 26 27 30 34), (5 16 18 19 24 26 27 28 35), (6 17 19 20 25 27 28 29 36),
(7 18 20 21 26 28 29 30 37), (8 16 17 18 19 20 21 22 38), (9 19 21 22 23 25 26 28 39),
(10 16 20 22 23 26 27 29 40), (11 16 17 21 23 27 28 30 41), (12 17 18 22 23 24 28 29 42),
(13 16 18 19 23 25 29 30 43), (14 17 19 20 23 24 26 30 44), (15 18 20 21 23 24 25 27 45),
(1 16 34 36 37 39 42 44 45), (2 17 31 35 37 39 40 43 45), (3 18 31 32 36 39 40 41 44),
(4 19 32 33 37 40 41 42 45), (5 20 31 33 34 39 41 42 43), (6 21 32 34 35 40 42 43 44),

(7 22 33 35 36 41 43 44 45), (8 23 31 32 33 34 35 36 37), (9 24 34 36 37 38 40 41 43), (10 25 31 35 37 38 41 42 44), (11 26 31 32 36 38 42 43 45), (12 27 32 33 37 38 39 43 44), (13 28 31 33 34 38 40 44 45), (14 29 32 34 35 38 39 41 45), (15 30 33 35 36 38 39 40 42).

The GD scheme is defined by the array: 1 2 3 4 … 15
16 17 18 19… 30
31 32 33 34 …45.

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