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# A Bimodal Extension of Suja Distribution with Applications

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**Abstract**

This paper introduces a new lifetime distribution called the Bimodal Extension of Suja (BES) distribution using the Quadratic Rank Transmutation Map. The proposed distribution has Suja distribution as a special case. Some statistical and reliability properties of the new distribution were derived and the method of maximum likelihood was employed for estimating the model parameters. The usefulness and flexibility of the BES distribution were illustrated with two real lifetime data sets. Results based on the log-likelihood and goodness of fit statistics values showed that the BES provides a better fir to the data than the other competing (lifetime) distributions considered in this study. Also, the consistency of the parameters of the new distribution was demonstrated through a simulation study. The BES distribution is therefore recommended foe effective modelling of the unimodal or bimodal continuous lifetime data with a non decreasing or bathtub shaped hazard rate function ...

*Key words:* Bimodal data; Hazard rate function; Maximum likelihood method; Quadratic rank transformation map; Suja distribution; BES distribution.

**AMS Subject Classifications:** 62B15, 60E05

## 1. Introduction

One of the activities of statisticians is to make informed decisions about a population on the basis of a sample drawn from that population. Obviously, several phenomena upon which decisions are taken often occur by chance and the best way to account for uncertainties surrounding them is to adopt probabilistic models. Probability models serve as mathematical structures for describing physical phenomena. A necessary step in the use of probabilistic models for modelling real-life problems is to ensure that the observed sample data follow certain probability distribution(s). Standard probability distributions commonly used for modelling several real-life problems include exponential, Weibull, gamma, two-parameter Odoma (Enogwe *et al.*, 2020) and so on. Unfortunately, so many datasets do not come from the existing probability distributions and this has engendered a demand for alternative

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distributions, especially for the extension of the existing distributions which can be moore appropriate for fitting real-life data.

Recently, Shanker (2017) introduced and studied a new distribution, called the Suja distribution with probability density function (PDF) and cumulative distribution function (CDF) given, respectively, by

$$g(x;\eta) = \frac{\eta^5}{\eta^4+24}(1+x^4)e^{-\eta x}; \quad x>0, \eta>0 \tag{1}$$

and

$$G(x;\eta) = 1 - \left[1 + \frac{\eta x(\eta^3 x^3 + 4n^2x^2 + 12\eta x + 24)}{\eta^4+24}\right]e^{-\eta x}; \quad x>0, \eta>0 \tag{2}$$

An application of the Suja distribution to lifetime analysis of engineering data was presented by Shanker (2017) which showed that the Suja distribution outperforms the Akash (Shanker, 2015a), Shanker (Shanker, 2015b), Amarendra (Shanker, 2016a), Aradhana (Shanker, 2016b), Devya (Shanker, 2016c), Sujatha (Shanker, 2016d), Lindley (Ghitany, *et al.*, 2008) and exponential distributions in modelling lifetime data.

In spite of the utility of the Suja distribution, it cannot be used for statistical modelling of datasets with varieties of tails due its dependency on only one parameter. This limitation of Suja distribution can be overcome by obtaining some of its generalization so as to provide greater flexibility in modelling observed data. The work of Al-Omari and Alsmairan (2019) introduced a length-biased Suja distributionn. Also, a power lenght-biased Suja distribtion was developed by Al-Omari *et al.* (2019). Further, Alsmairan and Al-Omari (2020) used the weighted method to extend the Suja distribution, which was applied to ball bearing data to show that the weighted Suja distribution is better than the Suja distribution. It is evident that these extensions of Suja distribution cannot be used to model data with bimodal shape. To obtain an extension of Suja distribution that can model bimodal data, the quadratic rank transformatio map (QRTM) proposed by Shaw and Buckey (2007) is utilized.

According to Shaw and Buckley (2007), the QRTM provides distributions that are more flexible than baseline distributions in modelling real-life datasets with complex structure. The cumulative distribution function (CDF) and probability density function (PDF) of the quadratic transmuted family of distributions may be written as

$$F(x) = (1+\lambda)G(x) - \lambda G^2(x) \tag{3}$$

and

$$f(x) = g(x)((1+\lambda) - 2\lambda G(x)) \tag{4}$$

respectively, where $|\lambda| \leq 1, G(x)$ is the baseline CDF of $X$ and $g(x) = dG(x)/dx$, the baseline PDF of $X$. Observe from (3) and (4) that if $\lambda = 0$, the quadratic transmuted family of distributions reduces to the baseline distribution.

Apart from the work of Shaw and Buckley (2007), other researchers have explored some members of the quadratic transmuted family of distributions. The members of the family of distributions include transmuted extreme value distribution (Aryal and Tsokos, 2009), transmuted

Weibull distribution (Aryal and Tsokos, 2011) transmuted log-logistic distribution (Aryal, 2013), transmuted Lindley distribution (Merovci, 2013a) and transmuted Rayleigh distribution by Merovci (2013b), transmuted Lomax distribution (Ashour and Eltehiwy, 2013), transmuted Pareto distribution (Merovci and Puka, 2014), transmuted two-parameter Lindley distribution due to Al-khazaleh *et al.* (2016), transmuted Dagum distribution (Shahzad and Asghar, 2016), transmuted Janardan distribution by Al-Omari *et al.* (2016), transmuted Burr XII distribution (Maurya *et al.*, 2017), transmuted Mukherjee-Islam (Rather and Subramanian, 2018), transmuted ArcSine distribution (Bleed and Abdelali, 2018), transmuted Ishita distribution (Gharaibeh and Al-Omari, 2019), transmuted Pranav distribution (Odom *et al.*, 2019), transmuted Garima distribution (Mohiuddin *et al.*, 2020), transmuted Aradhana (Gharaibeh, 2020), among others.

The aim of this article is to propose a new distribution, called a BES distribution, which is more flexible than the Suja distribution and some other competing lifetime distributions for modelling complex lifetime datasets. Specifically, this study reveals that the QRTM can be used to generalize a one-parameter continuous distribution to obtain a bimodal two-parameter distribution that has a monotone or non-monotone hazard rate function, especially the bathtub shape. As expected in the proposed distribution, the QRTM has been adopted in previous researches to generate new distributions that are more flexible than the baseline distributions. In Section 2, we define the expressions for the PDF and CDF of the BES distribution. The statistical and reliability properties of the BES distribution are discussed in Section 3. The quantile function and entropies of the BES distribution are given in Section 4. Section 5 provides the distribution of order statistics. In Section 6, the parameters of the BES distribution are estimated through the method of maximum likelihood estimation. Section 7 discusses the asymptotic confidence intervals of the parameters of the BES distribution. A simulation study is conducted in Section 8. In Section 9, two real datasets, methods of model selection, applications of the BES distribution to the data sets and the results are presented. In Section 10, we give the concluding remarks.

## 2. Definition of BES distribution

Inserting (2) into (3), we get the CDF of the new distribution. Also, inserting (1) and (2) into (4), we obtain the PDF of the new distribution. Consequently, a random variable $X$ is said to have the BES distribution if its CDF and PDF are defined as

$$F_{BES}(x;\eta.\lambda) = (1+\lambda)\left[1-\left(1+\frac{\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)}{\eta^4+24}\right)e^{\eta x}\right] - \lambda\left[1-\left(1+\frac{\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)}{\eta^4+24}\right)e^{\eta x}\right]^2 \quad (5)$$

and

$$f_{BES}(x;\eta,\lambda) = \frac{\eta^5}{\eta^4+24}(1+x^4)e^{-\eta x}\left[1-\lambda+2\lambda\left(1+\frac{\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)}{\eta^4+24}\right)e^{\eta x}\right] \quad (6)$$

respectively, for $x > 0$, $\eta > 0$ and $\mid \lambda \mid \leq 1$. The BES distribution reduces to the Suja distribution when $\lambda = 0$ . Figure 1 shows the plots of the PDF of the BES variable based on several sets of values of the parameters of the distribution. Figure 1 indicates that the PDF of the BES distribution has unimodal shape if $\lambda = 0.1, \eta = 0.6, \lambda = 0.3, \eta = 0.7$. The biomodal shape of the BES distribution is observed when $\lambda = -0.9, \eta = 2.0, \lambda = 0.4, \eta = 1.6$, among others. Again, the shape of the BES is nondecreasing if $\lambda = 0.9, \eta = 0.1$.

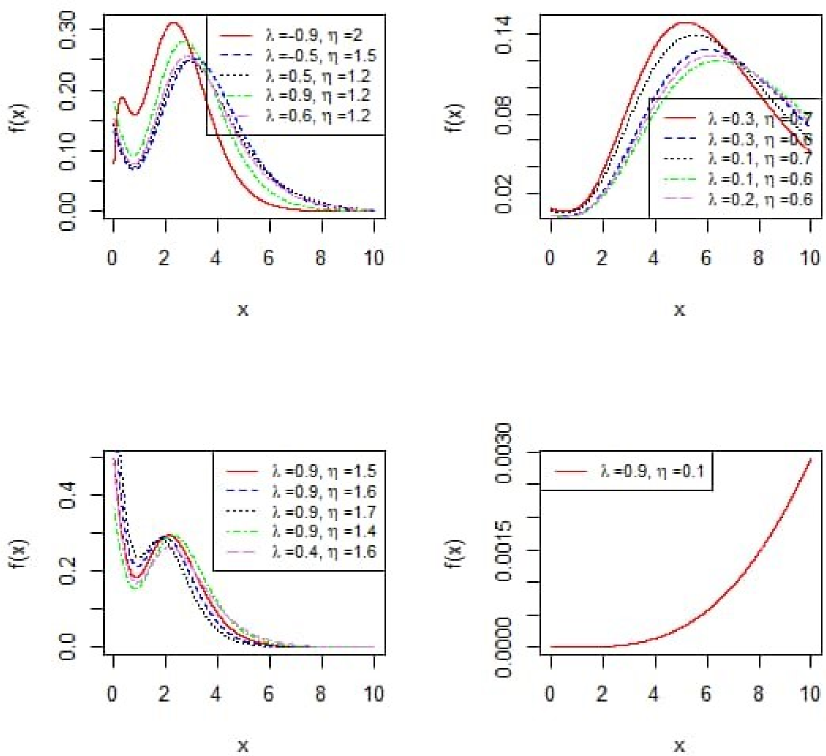


**Figure 1: Various shapes of the PDF of BES**

The graphs depicted as Figure 2 show that the Cumulative Distribution of BES is nondecreasing.

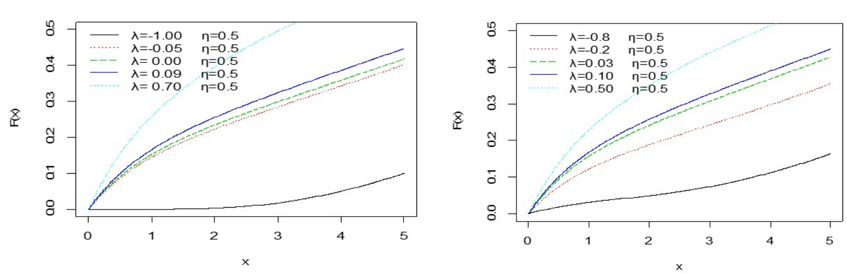


**Figure 2: Various shapes of CDF of BES**

## 3. Statistical and reliability properties of BES distribution

### 3.1. Statistical properties

The moment generating function of $X \sim BES(\eta,\lambda)$ is given by

$$
\begin{aligned}
M_X(t) &= \int_0^\infty e^{tx} f_{BES}(x;\eta,\lambda)dx \\
&= \frac{\eta^5}{\eta^4+24}\int_0^\infty e^{tx}(1+x^4)e^{-\eta x}\left[1-\lambda+2\lambda\left(1+\frac{\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)}{\eta^4+24}\right)e^{-\eta x}\right]dx \\
&= \frac{\eta^5}{\eta^4+24}\int_0^\infty \left[2\lambda(1+x^4)\left(1+\frac{24}{\eta^4+24}\sum_{r=1}^{4}\frac{(\eta x)^r}{r!}\right)e^{-(2\eta-t)x}+(1-\lambda)(1+x^4)e^{-(\eta-t)x}\right]dx \\
&= \frac{2\lambda\eta^5}{\eta^4+24}\left[\int_0^\infty e^{-(2\eta-t)x}+\frac{24}{\eta^4+24}\sum_{r=1}^{4}\frac{\eta^r}{r!}\left(\int_0^\infty x^r e^{-(2\eta-t)x}+\int_0^\infty x^{r+4}e^{-(2\eta-t)x}\right)\right. \\
&\quad \left.+\int_0^\infty x^4 e^{-(2\eta-t)x}\right]dx+\frac{(1-\lambda)\eta^5}{\eta^4+24}\left[\int_0^\infty e^{-(2\eta-t)x}+\int_0^\infty x^4e^{-(2\eta-t)x}\right]dx \\
&= \frac{2\lambda\eta^5}{\eta^4+24}\left[\frac{1}{(2\eta-t)}+\frac{24}{\eta^4+24}\sum_{r=1}^{4}\frac{\eta^r}{r!}\left(\frac{\Gamma(r+1)}{(2\eta-t)^{r+1}}+\frac{\Gamma(r+5)}{(2\eta-t)^{r+5}}\right)+\frac{24}{(2\eta-t)^5}\right] \\
&\quad +\frac{(1-\lambda)\eta^5}{n^4+24}\left[\frac{1}{(\eta-t)}+\frac{24}{(\eta-t)^5}\right] \qquad (7)
\end{aligned}
$$

The $r^{th}$ non-central moment of $X \sim BES(\eta,\lambda)$ is given by

$$\mu_r' = E(X^r) = \int_0^\infty x^r f_{BES}(x;\eta,\lambda)dx$$

$$= \frac{\eta^5}{(\eta^4+24)^2}\int_0^\infty x^r(1+x^4)e^{-\eta x}\left[1-\lambda+2\lambda\left(1+\frac{\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)}{\eta^4+24}\right)e^{-\eta x}\right]dx$$

$$= \frac{\eta^5}{(\eta^4+24)^2}\int_0^\infty \begin{bmatrix}(1-\lambda)(\eta^4+24)x^re^{-\eta x}+(1-\lambda)(\eta^4+24)x^{r+4}e^{-\eta x}\\ +2\lambda(\eta^4+24)x^re^{-2\eta x}+2\lambda(\eta^4+24)x^{r+4}e^{-2\eta x}+8\lambda\eta^3x^{r+3}e^{-2\eta x}\\ +24\lambda\eta^2x^{r+2}e^{-2\eta x}+48\lambda\eta x^{r+1}e^{-2\eta x}+2\lambda\eta^4x^{r+8}e^{-2\eta x}+8\lambda\eta^3x^{r+7}e^{-2\eta x}\\ +24\lambda\eta^2x^{r+6}e^{-2\eta x}+48\lambda\eta x^{r+5}e^{-2\eta x}\end{bmatrix}dx$$

$$= \frac{\eta^5}{(\eta^4+24)^2}\begin{bmatrix}(1-\lambda)(\eta^4+24)\left(\frac{\Gamma(r+1)}{\eta^{r+1}}\right)+(1-\lambda)(\eta^4+24)\left(\frac{\Gamma(r+5)}{\eta^{r+5}}\right)+2\lambda(\eta^4+24)\left(\frac{\Gamma(r+1)}{(2\eta)^{r+1}}\right)\\ +2\lambda(\eta^4+24)\left(\frac{\Gamma(r+5)}{(2\eta)^{r+5}}\right)+8\lambda\eta^3\left(\frac{\Gamma(r+4)}{(2\eta)^{r+4}}\right)+24\lambda\eta^2\left(\frac{\Gamma(r+3)}{(2\eta)^{r+3}}\right)+48\lambda\eta\left(\frac{\Gamma(r+2)}{(2\eta)^{r+2}}\right)\\ 2\lambda\eta^4+\left(\frac{\Gamma(r+9)}{(2\eta)^{r+9}}\right)+8\lambda\eta^3\left(\frac{\Gamma(r+8)}{(2\eta)^{r+8}}\right)+24\lambda\eta^2\left(\frac{\Gamma(r+7)}{(2\eta)^{r+7}}\right)+48\lambda\eta\left(\frac{\Gamma(r+6)}{(2\eta)^{r+6}}\right)\end{bmatrix}$$

$$\therefore \mu_r' = \frac{\eta^5}{(\eta^4+24)^2}\left[\begin{array}{l}(1-\lambda)(\eta^4+24)\left(\frac{\Gamma(r+1)}{\eta^{r+1}}+\frac{\Gamma(r+5)}{\eta^{r+5}}\right)+\frac{\lambda(\eta^4+24)\Gamma(r+1)}{2^r\eta^{r+1}}\\ +\frac{\lambda(\eta^4+24)\Gamma(r+5)}{2^{r+4}\eta^{r+5}}+\frac{\lambda\Gamma(r+4)}{2^{r+1}\eta^{r+1}}+\frac{3\lambda\Gamma(r+3)}{2^r\eta^{r+1}}+\frac{12\lambda\Gamma(r+2)}{2^r\eta^{r+1}}\\ +\frac{\lambda\Gamma(r+9)}{2^{r+8}\eta^{r+4}}+\frac{\lambda\Gamma(r+8)}{2^{r+5}\eta^{r+5}}+\frac{3\lambda\Gamma(r+7)}{2^{r+4}\eta^{r+5}}+\frac{\lambda\Gamma(r+6)}{2^{r+2}\eta^{r+5}}\end{array}\right] \tag{8}$$

Substituting $r = 1, 2, 3, 4$ in (8), yields the first four crude moments of the BES distribution as

$$\mu_1' = \frac{(\theta^4+24)[(\theta^4+120)-\lambda(\theta^4+103)]+\lambda(4725\theta+3600)+108\theta^2(\theta^4+24)^2}{4\theta(\theta^4+24)^2} \tag{9}$$

$$\mu_2' = \frac{(\theta^4+24)(8\theta^4-6\lambda\theta^4-2835\lambda+2880)-408\theta^4\lambda+263655\lambda}{4\theta^2(\theta^4+24)^2} \tag{10}$$

$$\mu_3' = \frac{2(\theta^4+24)(54\theta^4+48\lambda\theta^4-40005\lambda+40320)-2106\theta^4+2835\theta\lambda+423360\lambda}{160^3(\theta^4+24)} \tag{11}$$

$$\mu_4' = \frac{16(\theta^4+24)(48\theta^4-45\lambda\theta^4-80325\lambda+80640)+12240\theta^4\lambda+3742200\Theta\lambda+4399920\lambda}{32\Theta^4(\theta^4+24)^2} \tag{12}$$

The rth central moment of $X \sim BES(\eta, \lambda)$ can be obtained from the relation

$$\mu_r = \sum_{j=0}^{r}(-1)^j\binom{r}{j}\mu_j'(\mu)^{r-1} \tag{13}$$

where $\mu_j'$ is deduced from (8) by replacing $r$ with $j$ and $\mu$ is defined in (9). The following central moments are obtained by letting $r = 2, 3, 4$ in (13):

$$\mu_2 = \sum_{j=0}^{2}(-1)^j\binom{2}{j}\mu_j'(\mu)^{2-1} \tag{14}$$

$$\mu_3 = \sum_{j=0}^{3}(-1)^j\binom{3}{j}\mu_j'(\mu)^{3-1} \tag{15}$$

$$\mu_4 = \sum_{j=0}^{4}(-1)^j\binom{4}{j}\mu_j(\mu)^{4-1} \tag{16}$$

The coefficient of variation ($\gamma_0$), skewness ($\gamma_1$) and kurtosis ($\gamma_2$) of the BES distribution could be obtained by evaluating

$$\gamma_0 = \frac{(\mu_2)^{\frac{1}{2}}}{\mu} \tag{17}$$

$$\gamma_1 = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} \tag{18}$$

$$\gamma_2 = \frac{\mu_4}{(\mu_2)^2} \tag{19}$$

### 3.2. Reliability properties

Suppose $X \sim BES(x;\eta,\lambda)$, then the reliability function may be written as

$$\begin{aligned} R_{BES}(x;\eta,\lambda) &= 1-F_{BES}(x;\eta,\lambda) \\ &= \frac{(\lambda-1)}{\eta^4+24}(\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)+\eta^4+24)e^{-\eta x} \\ &+ \frac{\lambda}{(\eta^4+24)^2}(\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)+\eta^4+24)^2e^{-2\eta x} \end{aligned} \tag{20}$$

Taking the ratio of (6) to (20), we obtain the hazard rate function for $X \sim BES(\eta,\lambda)$ as

$$h_{BES}(x;\eta,\lambda) = \frac{f_{BES}(x;\eta,\lambda)}{R_{BES}(x;\eta,\lambda)}$$

$$h_{BES}(x;\eta,\lambda) = \frac{\eta^5\begin{bmatrix}(1-\lambda)(\eta^4+24)e^{-\eta x}+(1-\lambda)(\eta^4+24)x^4e^{-\eta x} \\ +2\lambda(\eta^4+24)e^{-2\eta x}+2\lambda(\eta^4+24)x^4e^{-2\eta x}+8\lambda\eta^3x^3e^{-2\eta x} \\ +24\lambda\eta^2x^2e^{-2\eta x}+48\lambda\eta xe^{-2\eta x}+2\lambda\eta^4x^8e^{-2\eta x}+8\lambda\eta^3x^7e^{-2\eta x} \\ +24\lambda\eta^2x^6e^{-2\eta x}+48\lambda\eta x^5e^{-2\eta x}\end{bmatrix}}{\begin{bmatrix}(\lambda-1)(\eta^4+24)(\eta^4x^4+4\eta^3x^3+12\eta^2x^2+24\eta x+\eta^4+24)e^{-\eta x} \\ +\lambda(\eta^4x^4+4\eta^3x^3+12\eta^2x^2+24\eta x+\eta^4+24)^2e^{-2\eta x}\end{bmatrix}} \tag{21}$$

The graphical representation of the hazard rate function of the BES distribution is presented as Figure 3. In accordance with Figure 3, the distribution is quite flexible as its hazard rate function is capable of possessing different shapes depending on the values of the associated parameters. Specifically, the figure reveals that the hazard rate function can be nondecreasing or bathtub shaped. It can also have an s-shaped curve or be a bimodal function.

The cumulative hazard function of $X \sim BES(\eta,\lambda)$ can be written as

$$\begin{aligned} Ch_{BES}(x;\eta,\lambda) &= -\ln(1-F_{BES}(x;\eta,\lambda)) = -\ln(R_{BES}(x;\eta,\lambda)) \\ &= -\ln[(\lambda-1)(\eta^4+24)+\lambda(\eta^4x^4+4\eta^3x^3+12\eta^2x^2+24\eta x+\eta^4+24)e^{-\eta x}] \\ &+2\ln(\eta^4+24)-\ln(\eta^4x^4+4\eta^3x^3+12\eta^2x^2+24\eta x+\eta^4+24)+\eta x \end{aligned} \tag{22}$$

The reverse hazard function of $X \sim BES(\eta,\lambda)$ is given by

$$Rh_{BES}(x;\eta,\lambda) = \frac{f_{BES}(x;\eta,\lambda)}{F_{BES}(x;\eta,\lambda)}$$

$$h_{BES}(x;\eta,\lambda) = \frac{\eta^5\begin{bmatrix}(1-\lambda)(\eta^4+24)e^{-\eta x}+(1-\lambda)(\eta^4+24)x^4e^{-\eta x} \\ +2\lambda(\eta^4+24)x^4e^{-2\eta x}+2\lambda(\eta^4+24)x^4e^{-2\eta x}+8\lambda\eta^3x^3e^{-2\eta x} \\ +24\lambda\eta^2x^2e^{-2\eta x}+48\lambda\eta xe^{-2\eta x}+2\lambda\eta^4x^8e^{-2\eta x}+8\lambda\eta^3x^7e^{-2\eta x} \\ +24\lambda\eta^2x^6e^{-2\eta x}+48\lambda\eta x^5e^{-2\eta x}\end{bmatrix}}{\begin{bmatrix}(\eta^4+24)-\lambda(\eta^4x^4+4\eta^3x^3+12\eta^2x^2+24\eta x+\eta^4+24)^2e^{-2\eta x} \\ -(1-\lambda)(\eta^4+24)(\eta^4x^4+4\eta^3x^3+12\eta^2x^2+24\eta x+\eta^4+24)e^{-\eta x}\end{bmatrix}} \tag{23}$$

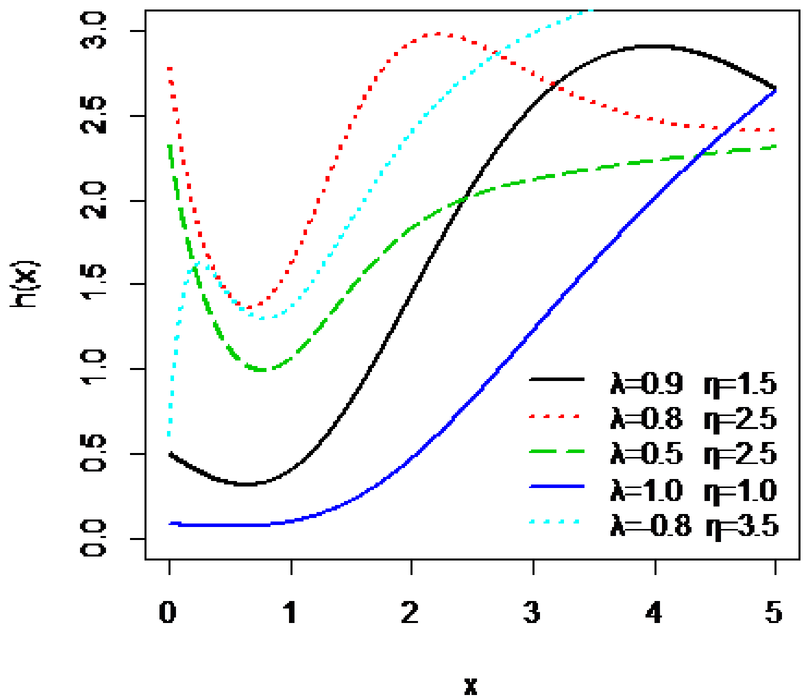


**Figure 3: Various shapes of the hazard function of the BES distribution**

The odds function of $X \sim BES(\eta, \lambda)$ is given by

$$O_{BES}(x; \eta, \lambda) = \frac{F_{BES}(x; \eta, \lambda)}{1 - F_{BES}(x; \eta, \lambda)}$$

$$O_{BES}(x; \eta, \lambda) = \begin{bmatrix} (1-\lambda)(\eta^4+24)^{-1}(\eta^4x^4+4\eta^3x^3+12\eta^2x^2+24\eta x+\eta^4+24)e^{-\eta x} \\ +\lambda(\eta^4+24)^{-2}(\eta^4x^4+4\eta^3x^3+12\eta^2x^2+24\eta x+\eta^4+24)^2e^{-2\eta x} \end{bmatrix}^{-1} - 1 \tag{24}$$

## 4. Quantile function and entropy measures of BES distribution

### 4.1. Quantile function of BES distribution

The $x_\omega^{th}$ quantile function of BES distribution satisfies the equation

$$F_{BES}(x; \eta, \lambda) = \omega, \quad 0 < \omega < 1 \tag{25}$$

Plugging (5) into (25), we have

$$(1+\lambda)\left[1-\left(1+\frac{\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)}{\eta^4+24}\right)e^{-\eta x}\right]^2 - \lambda\left[1-\left(1+\frac{\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)}{\eta^4+24}\right)e^{-\eta x}\right]^2 = \omega \tag{26}$$

Let

$$z = 1-\left(1+\frac{\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)}{\eta^4+24}\right)e^{-\eta x} \tag{27}$$

Then

$$\begin{aligned}(1+\lambda)z-\lambda z^2 &= \omega\\ \lambda z^2-(1+\lambda)z+\omega &= 0\end{aligned} \tag{28}$$

Applying the quadratic formula on (28), we obtain

$$z = \frac{1+\lambda\pm\sqrt{(1+\lambda)^2-4\lambda\omega}}{2\lambda} \tag{29}$$

Substituting (29) into (27), one obtains

$$\frac{1+\lambda\pm\sqrt{(1+\lambda)^2-4\lambda\omega}}{2\lambda} = 1-\left(1+\frac{\eta x_\omega(\eta^3x_\omega^3+4\eta^2x_\omega^2+12\eta x_\omega+24)}{\eta^4+24}\right)e^{-\eta x_\omega}$$

Thus, the quantile is obtained by solving the equations:

$$\frac{1+\lambda\pm\sqrt{(1-\lambda)^2}-4\lambda\omega}{2\lambda} = \left(1+\frac{24}{n^4+24}\sum_{r=1}^{4}\frac{(\eta x_\omega)^r}{r!}\right)e^{-\eta x_\omega} \tag{30}$$

Therefore, the $\omega th$ quantile, denoted by $x_\omega$, for BES distribution, is a positive solution of (30), which can be found by numerical method.

### 4.2. Entropy measures of the BES distribution

The Renyi entropy may be defined for the BES as

$$E_R = \frac{1}{1-\beta}\log\left(\int_0^\infty f_{BES}^{\beta}(x;\eta,\lambda)dx\right), \quad \beta\neq 1, \quad \beta>0$$

$$= \frac{1}{1-\beta}log\left[\begin{array}{l}\left(\frac{\eta^5}{\eta^4+24}\right)^{\beta}\int_0^\infty(1+x^4)^{\beta}e^{-\beta\eta x}\\ \times\left((1-\lambda)+2\lambda\left(1+\frac{\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)}{\eta^4+24}\right)e^{\eta x}\right)^{\beta}dx\end{array}\right] \tag{31}$$

Applying binomial expansion to the terms in (31) and simplifying, one gets

$$E_R = \frac{1}{1-\beta} log \left[ \begin{array}{l} \left(\frac{\eta^5}{\eta^4+24}\right)^{\beta} \sum_{i=0}^{\infty} \sum_{j=0}^{\beta} \sum_{k=0}^{j} \sum_{l=0}^{k} \sum_{m=0}^{l} \sum_{n=0}^{m} \binom{\beta}{i}\binom{\beta}{j}\binom{j}{k}\binom{k}{l}\binom{l}{m}\binom{m}{n} \eta^{k+l+m+n} \\ \frac{(24)^{k-l}(12)^{l-m}(4)^{m-n}(1-\lambda)^j(2\lambda)^{\beta-j}}{(\eta^4+24)^k} \int_0^{\infty} x^{k+l+m+n} e^{-\eta(\beta+j)x} dx \end{array} \right]$$

$$= \frac{1}{1-\beta} log \left[ \begin{array}{l} \left(\frac{\eta^5}{\eta^4+24}\right)^{\beta} \sum_{i=0}^{\infty} \sum_{j=0}^{\beta} \sum_{k=0}^{j} \sum_{l=0}^{k} \sum_{m=0}^{l} \sum_{n=0}^{m} \binom{\beta}{i}\binom{\beta}{j}\binom{j}{k}\binom{k}{l}\binom{l}{m}\binom{m}{n} \\ \frac{(24)^{k-l}(12)^{l-m}(4)^{m-n}(1-\lambda)^j(2\lambda)^{\beta-j}\eta^{k+l+m+n-1}\Gamma(k+l+m++1)}{(\eta^4+24)^k(\beta+j)^{k+l+m+n+1}} \end{array} \right] \quad (32)$$

The Tsallis entropy for the BES distribution may be defined as

$$E_S = \frac{1}{\beta-1}\left(1 - \int_0^{\infty} f_{BES}^{\beta}(x;\eta,\lambda)dx\right), \quad \beta \neq 1, \quad \beta > 0$$

$$= \frac{1}{\beta-1}\left(1 - \left[ \begin{array}{l} \left(\frac{\eta^5}{\eta^4+24}\right)^{\beta} \sum_{i=0}^{\infty} \sum_{j=0}^{\beta} \sum_{k=0}^{j} \sum_{l=0}^{k} \sum_{m=0}^{l} \sum_{n=0}^{m} \binom{\beta}{i}\binom{\beta}{j}\binom{j}{k}\binom{k}{l}\binom{l}{m}\binom{m}{n} \\ \frac{(24)^{k-l}(12)^{l-m}(4)^{m-n}(1-\lambda)^j(2\lambda)^{\beta-j}\eta^{k+l+m+n-1}\Gamma(k+l+m++1)}{(\eta^4+24)^k(\beta+j)^{k+l+m+n+1}} \end{array} \right]\right) \quad (33)$$

## 5. Distributions of order statistics of BES distribution

The PDF of the $rth$ order statistic for $X \sim BES(\eta, \lambda)$ is given by

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)$$

$$= \frac{r\binom{n}{r}\eta^5(1+x^4)e^{-(n-r+1)x}}{\eta^4+24}\left(1+\frac{\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)}{\eta^4+24}\right)^{\eta-r}$$
$$\times\left[(1-\lambda)+2\lambda\left(1+\frac{\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)}{\eta^4+24}\right)e^{-\eta x}\right]$$

$$\times\left[ \begin{array}{l} 1-(1-\lambda)\left(1+\frac{\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)}{\eta^4+24}\right)e^{-\eta x} \\ -\lambda\left(1+\frac{\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)}{\eta^4+24}\right)^2 e^{-2\eta x} \end{array} \right]^{r-1}$$

$$\times\left[(1-\lambda)+\lambda\left(1+\frac{\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)}{\eta^4+24}\right)e^{-\eta x}\right]^{n-r} \quad (34)$$

Putting $r = 1$ in (34), we get the PDF of the first order statistic $X_{(1)}$ as

$$f_{X_{(1)}}(x) = \frac{\eta^5(1+x^4)ne^{-\eta nx}}{\eta^4+24}\left(1+\frac{\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)}{\eta^4+24}\right)^{n-1} \times\left[(1-\lambda)+\lambda\left(1+\frac{\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)}{\eta^4+24}\right)e^{-\eta x}\right]^{n-1} \times\left[(1-\lambda)+2\lambda\left(1+\frac{\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)}{\eta^4+24}\right)e^{-\eta x}\right] \quad (35)$$

Putting $n = r$ in (34), we get the PDF of the largest order statistic $X_{(n)}$ as

$$f_{X_{(n)}}(x) = \frac{\eta^5(1+x^4)ne^{(-\eta-r+1)x}}{\eta^4+24}\left((1-\lambda)+2\lambda\left(1+\frac{\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)}{\eta^4+24}\right)\right)^{n-1} \times\left[1-(1-\lambda)\left(1+\frac{\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)}{\eta^4+24}\right)e^{-\eta x} -\lambda\left(1+\frac{\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)}{\eta^4+24}\right)^2e^{-2\eta x}\right]^{n-1} \quad (36)$$

## 6. Maximum likelihood estimates of parameters of BES distribution

Consider a random sample of a sample size, $n$, $X_1, X_2, ..., X_n$ drawn from the BES distribution. Obviously, the likelihood function of the random sample is

$$L(\eta,\lambda) = \prod_{i=1}^{n} f_{BES}(x_i;\eta,\lambda)$$

$$= \left(\frac{\eta^5}{\eta^4+24}\right)^n e^{-n\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}(1+x_i^4)\left[(1-\lambda)+2\lambda\left(1+\frac{\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)}{\eta^4+24}\right)e^{-\eta x_i}\right] \quad (37)$$

The log-likelihood function is

$$\ln L(\eta,\lambda) = \sum_{i=1}^{n}\ln\left[(1-\lambda)+2\lambda\left(1+\frac{\eta x(\eta^3x^3+4\eta^2x^2+12\eta x+24)}{\eta^4+24}\right)e^{-\eta x}\right] + \sum_{i=1}^{n}\ln(1+x_i^4)+n[5\ln(\eta)-\ln(\eta^4+24)]-\eta\sum_{i=1}^{n}x_i \quad (38)$$

Taking the partial derivatives of (38) with respect to $\eta$ and $\lambda$, and equating the results to zero, yields

$$\frac{\partial \ln L(\eta,\lambda)}{\partial\eta} = \sum_{i=1}^{n}\frac{2\lambda((\eta^3x_i^3+4\eta^2x_i^2+12\eta x_i+24)+\eta(3\eta^2x_i^2+8\eta x_i^2+12x_i))x_ie^{-\eta x_i}}{2\lambda(\eta x(\eta^3x_i^3+4\eta^2x_i^2+12\eta x_i+24)+\eta^4+24)e^{-\eta x_i}+(1-\lambda)(\eta^4+24)}$$

$$-\sum_{i=1}^{n}\frac{8\lambda\eta^4x_i((\eta^3x_i^3+4\eta^2x_i^2+12\eta x_i+24)e^{-\eta x_i}}{(\eta^4+24)[2\lambda(\eta x(\eta^3x_i^3+4\eta^2x_i^2+12\eta x_i+24)\eta^4+24)e^{-\eta x_i}+(1-\lambda)(\eta^4+24)]} \tag{39}$$
$$-\sum_{i=1}^{n}\frac{2\lambda x_i(4\eta^4x_i(\eta^3x_i^3+4\eta^2x_i^2+12\eta x_i+24)+(\eta^4+24))e^{-\eta x_i}}{2\lambda(\eta x(\eta^3x_i^3+4\eta^2x_i^2+12\eta x_i+24)+\eta^4+24)e^{-\eta x_i}+(1-\lambda)(\eta^4+24)]}+\frac{2(\eta^4+120)}{\eta(\eta^4+24)}$$
$$-\sum_{i=1}^{n}x_i=0$$

$$\frac{\partial\ln L(\eta,\lambda)}{\partial\lambda}=\sum_{i=1}^{n}\frac{2(\eta x(\eta^3x_i^3+4\eta^2x_i^2+12\eta x_i+24)+\eta^4+24)e^{-\eta x_i}-(\eta^4+24)}{(1-\lambda)(\eta^4+24)+2\lambda(\eta x(\eta^3x_i^3+4\eta^2x_i^2+12\eta x_i+24)+\eta^4+24)e^{-\eta x_i}}=0 \tag{40}$$

Due to the complex nature of (39) and (40), an iterative method such as Newton-Raphson method is adopted for finding its solution.

## 7. Asymptotic confidence intervals of the parameters of BES distribution

Let $\hat{\Theta}=(\hat{\eta},\hat{\lambda})^T$ be the MLE of $\Theta=(\eta,\lambda)^T$ for the BES distribution. To construct the confidence intervals, the Fisher information, denoted by $I(\Theta)$ is required. Consequently

$$I(\Theta)=\begin{pmatrix}I_{\hat{\eta}\hat{\eta}} & I_{\hat{\eta}\hat{\lambda}}\\ I_{\hat{\lambda}\hat{\eta}} & I_{\hat{\lambda}\hat{\lambda}}\end{pmatrix} \tag{41}$$

The elements of (41) are the second derivatives of (38) with respect to the parameters of the BES distribution. Notice that the asymptotic distribution of $\sqrt{n}(N_2(1,I^{-1}(\Theta))$, under certain regularity conditions. Consequently, the approximate $100(1-\omega)\%$ two sided confidence intervals for $\eta$ and $\lambda$ are given, respectively, by

$$\hat{\eta}\pm Z_{\omega/2}\sqrt{I_{\eta\eta}^{-1}(\hat{\Theta})}\quad\text{and}\quad\hat{\lambda}\pm Z_{\omega/2}\sqrt{I_{\lambda\lambda}^{-1}(\hat{\Theta})} \tag{42}$$

where $I_{\eta\eta}^{-1}(\hat{\Theta})$ and $I_{\lambda\lambda}^{-1}(\hat{\Theta})$ are diagonal elements of the matrix $I_n^{-1}(\hat{\Theta})$ and $Z_{r/2}$ is the upper $(\omega/2)th$ percentile of a standard normal distribution.

## 8. Monte-Carlo simulation study of the BES distribution

To investigate the effect of sample size on the maximum likelihood estimates of parameters of the BES distribution and assess the stability of the parameter estimates, it is essential to conduct a Monte-Carlo simulation on the BES distribution.
The simulation procedure as outlined below was performed using R package:

**Step 1:** Simulate a random sample of size $n$ from the BES distribution with parameters $\lambda=0.8$ and $\eta=1.4$ using the inversion of the CDF method with Equation (30)

**Step 2:** Set intial values for the parameters of the BES distribution.

**Step 3:** Compute the MLE of the parameters of the BES distribution.

**Step 4:** Repeat steps 1-3 $N = 10,000$ times.

**Step 5:** Compute the mean, standard error, average bias and average mean square error (MSE) of the 10,000 maximum likelihood estimates of each parameter $\lambda$ and $\eta$. The mean estimate of the maximum likelihood estimator $\hat{\tau}$ of the parameter $\tau = (\lambda, \eta)$ is given by

$$\bar{\hat{\tau}} = \frac{1}{N}\sum_{i=1}^{N}\hat{\tau}_i \tag{43}$$

The standard error of $\bar{\hat{\tau}}$ is given by

$$SE_{\bar{\hat{\tau}}} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\tau}_i - \bar{\hat{\tau}})^2} \tag{44}$$

The Bias of $\bar{\hat{\tau}}$ is given by

$$Bias(\bar{\hat{\tau}}) = \bar{\hat{\tau}}_i - \tau, \quad i = 1, 2, ..., n \tag{45}$$

The average bias of the MLE $\hat{\tau}$ of the parameter $\tau = (\lambda, \eta)$ is given by

$$Ave.Bias(\hat{\tau}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\tau}_i - \tau) \tag{46}$$

The average mean square error (MSE) of the MLE $\hat{\tau}$ of the parameter $\tau = (\lambda, \eta)$ is given by

$$Ave.MSE(\hat{\tau}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\tau}_i - \tau)^2 \tag{47}$$

**Step 6:** Repeat Steps 1-5 with different sample sizes ($n = 20, 30, 50, 100, 500, 1000$).

**Table 1: Simulation results of the estimates, bias and mean square error of the BES distribution parameters for different sample sizes**

| $n$ | $\hat{\eta}$ | $\hat{\lambda}$ | $Bias(\hat{\eta})$ | $MSE(\hat{\eta})$ | $Bias(\hat{\lambda})$ | $MSE(\hat{\lambda})$ |
|---|---|---|---|---|---|---|
| 20 | 1.5334 | 0.4270 | 0.1338 | 0.0458 | -0.3730 | 0.1951 |
| 30 | 1.5977 | 0.2602 | 0.1977 | 0.0645 | -0.1718 | 0.1879 |
| 50 | 1.4559 | 0.6282 | 0.0559 | 0.0237 | -01718 | 0.1879 |
| 100 | 1.4559 | 0.7406 | 0.0220 | 0.0192 | -0.0594 | 0.1514 |
| 500 | 1.4091 | 0.7761 | 0.0091 | 0.0042 | -0.0239 | 0.0306 |
| 1000 | 1.4030 | 0.7917 | 0.0003 | 0.0014 | -0.0083 | 0.0091 |

As shown in Table 1, the parameter estimates tend toward the actual parameter values as the sample size increases. Also, average bias and mean squared error tend to zero with increasing sample size.

## 9. Applications

In this section, we illustrate the flexibility of the BES distribution with two real datasets. The first dataset comprises the failure times of mechanical components as reported in Javed *et al.* (2018).

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663

The second dataset depicts the fatigue life of some aluminium coupons cut in specific manner reported in Birnbaum and Saunders (1969). The dataset (after subtracting 65) is:

5, 25, 31, 32 ,34 ,35 ,38, 39, 39, 40, 42, 43, 43, 43, 44, 44, 47, 47, 48, 49, 49, 49, 51, 54, 55, 55, 55, 56, 56, 56, 58, 59, 59, 59, 59, 59, 63, 63, 64, 64, 65, 65, 65, 66, 66, 66, 66, 66, 67, 67, 67, 68, 69, 69, 69, 69, 71, 71, 72, 73, 73, 73, 74, 74, 76, 76, 77, 77, 77, 77, 77, 77, 79, 79, 80, 81, 83, 83, 84, 86, 86, 87, 90, 91, 92, 92, 92, 92, 93, 94, 97, 98, 98, 99, 101, 103, 105, 109, 136, 147

Consequently, we fit the BES distribution (BESD) as well as the competing distributions, such as gamma distribution (GD) and each of the following distributions (in each case $g(x)$ is the PDF while $G(x)$ is the CDF of the concerned distribution) to each of the two data sets listed above. The reason for choosing these distributions is because they all belong to the same family of the proposed distribution; so we chose them for comparison to illustrate the flexibility achieved as a result of the generalization. (1) Transmuted Lindley distribution (TLD) (Merovci, 2013a)

$$g(x) = \frac{\eta^2}{\eta+1}(1+x)e^{-\eta x}\left(1-\lambda+2\lambda\left(\frac{\eta+1+\eta x}{\eta+1}\right)e^{-\eta x}\right) \tag{48}$$

and

$$G(x) = \left(1-\frac{\eta+1+\eta x}{\eta+1}e^{-\eta x}\right)\left(1+\lambda\left(\frac{\eta+1+\eta x}{\eta+1}\right)e^{-\eta x}\right) \tag{49}$$

(2) Transmuted Exponential distribution (TED) (Owoloko, *et al.*, 2015)

$$g(x) = \frac{1}{\eta}e^{-\eta x}(1-\lambda+2\lambda e^{-nx}) \tag{50}$$

and

$$G(x) = (1-e^{-\eta x})(1+\lambda e^{-\eta x}), \quad x>0, \eta>0, |\lambda|\leq 1 \tag{51}$$

(3) Transmuted Aradhana distribution (TAD) (Gharaibeh, 2020)

$$g(x) = \frac{\eta^3}{\eta^2+2\eta+2}(1+x)^2e^{-\eta x}\left(1-\lambda+2\lambda\left(\frac{\eta x(\eta x+2\eta+2)}{\eta^2+2\eta+2}\right)e^{-\eta x}\right) \tag{52}$$

and

$$G(x) = (1+\lambda)\left(1-\left(1+\frac{\eta x(\eta x+2\eta+2)}{\eta^2+2\eta+2}\right)e^{-\eta x}\right)-\lambda\left(1-\left(1+\frac{\eta x(\eta x+2\eta+2)}{\eta^2+2\eta+2}\right)e^{-\eta x}\right)^2 \quad (53)$$

(4) Transmuted Ishita distribution (TID) (Sharaibeh and Al-Omari, 2019)

$$g(x) = \frac{\eta^3}{\eta^3+2}(\eta+x^2)e^{-\eta x}\left(1-\lambda+2\lambda\left(1+\frac{\eta x(\eta x+2)}{\eta^3+2}\right)e^{-\eta x}\right) \quad (54)$$

and

$$G(x) = (1+\lambda)\left(1-\left(1+\frac{\eta x(\eta x+2)}{\eta^2+2\eta+2}\right)e^{-\eta x}\right)-\lambda\left(1-\left(1+\frac{\eta x(\eta x+\lambda)}{\eta^3+2}\right)e^{-\eta x}\right)^2 \quad (55)$$

(5) Transmuted Pranav distribution (TPD) (Odom *et al.*, 2019)

$$g(x) = \frac{\eta^4}{\eta^4+6}(\eta+x^3)e^{-\eta x}\left(1-\lambda+2\lambda\left(1+\frac{\eta x(\eta^2x^2+3\eta x+6)}{\eta^4+6}\right)e^{-\eta x}\right) \quad (56)$$

and

$$G(x) = (1+\lambda)\left(1-\left(1+\frac{\eta x(\eta^2x^2+3\eta x+6)}{\eta^4+6}\right)e^{-\eta x}\right)-\lambda\left(1-\left(1+\frac{\eta x(\eta^2x^2+3\eta x+6)}{\eta^4+6}\right)e^{-\eta x}\right)^2 \quad (57)$$

Comparison of the fitted models was basd on the following goodness-of-fit measures: the Akaike Information Criterion (AIC) due to Akaike (1992), given by

$$AIC = -2l + 2k, \quad (58)$$

the Bayesian Information Criterion (BIC) due to Schwarz (1978), given by

$$BIC = k\ln(n) - 2l, \quad (59)$$

and the generalized Carmer-von Mises W* statistics; due to Chen and BAlakrishnan (1995), given by

$$CVM = \frac{1}{12n} + \sum\left[\frac{2i-1}{2n} - \hat{F}(x_i)\right] \quad (60)$$

where $k$ is the number of parameters in the BES distribution, $l$ is the maximized value of the log-likelihood function of the BES distribution,$\hat{F}(x_i)$ is the value of the CDF of the BES distribution and $n$ is the sample size. The smaller the criterion statistics the better the model.

Maximum likelihood estimates of the parameters of the BES distribution and the other seven distributions fitted to both data and the associated results are given in Table 2 and Table 3 for the first and second data respectively.

A comparison of AIC and BIC values of the eight lifetime distributions in Tables 2 and 3 shows that the BES distribution gives a better fit for the lifetime datasets as it has smaller AIC and BIC values than the others. The estimated parameters also satisfy the theoretical range of the parameters as expected.

**Table 2: Maximum likelihood fit of the failure times of mechanical components data**

| Models | Estimates | SE | $\ell$ | AIC | BIC | KS | CVM | AD |
|---|---|---|---|---|---|---|---|---|
| BES | | | | | | | | |
| $\hat{\eta}$ | 1.9390 | 0.0657 | -131.4167 | 266.9234 | 271.8087 | 0.0811 | 0.0910 | 0.8088 |
| $\hat{\lambda}$ | -0.9844 | 0.0762 | | | | | | |
| SD | | | | | | | | |
| $\hat{\eta}$ | 1.6098 | 0.0667 | -138.8172 | 279.6344 | 282.0770 | 0.1488 | 0.3886 | 2.8274 |
| GD | | | | | | | | |
| $a$ | 3.5284 | 0.5177 | -138.3853 | 280.7907 | 285.6760 | 0.1037 | 0.1779 | 1.4110 |
| $b$ | 1.3769 | 0.2171 | | | | | | |
| TLD | | | | | | | | |
| $\hat{\eta}$ | 0.8495 | 0.0538 | -140.1180 | 284.2359 | 289.1212 | 0.1440 | 0.4764 | 3.0160 |
| $\hat{\lambda}$ | -0.9686 | 0.0622 | | | | | | |
| TED | | | | | | | | |
| $\hat{\eta}$ | 0.5718 | 0.0470 | -146.8122 | 297.6244 | 302.5097 | 0.1855 | 0.8876 | 5.0645 |
| $\hat{\lambda}$ | -0.9970 | 0.5029 | | | | | | |
| TID | | | | | | | | |
| $\hat{\eta}$ | 1.1561 | 0.0503 | -136.8579 | 277.7159 | 282.6012 | 0.9735 | 26.4719 | 241.0491 |
| $\hat{\lambda}$ | -0.9720 | 0.0689 | | | | | | |
| TAD | | | | | | | | |
| $\hat{\eta}$ | 1.1838 | 0.0646 | -155.8115 | 315.6229 | 320.5082 | 0.9548 | 23.1526 | 195.3768 |
| $\hat{\lambda}$ | -0.9528 | 0.0678 | | | | | | |
| TPD | | | | | | | | |
| $\hat{\eta}$ | 1.4740 | 0.0518 | -135.5607 | 275.1213 | 280.0066 | 0.9877 | 27.4735 | 306.8742 |
| $\hat{\lambda}$ | -0.9871 | 0.0672 | | | | | | |

## 10. Conclusion

This paper introduces a new lifetime distribution, named the BES distribution. The new distribution generalizes the Suja distribution. We have provided explicit mathematical expressions for some of its basic statistical properties such as the probability density function, cumulative distribution function, rth crude and central moments, variance, coefficient of variation, skewness, kurtosis, and quantile function and some reliability characteristics like the survival, hazard rate, cumulative hazard and reverse hazard functions. Rényi and Tsallis entropies were discussed. Also, the distributions of rth, first and largest order statistics were introduced. Estimation of the model parameters was approached through the method of maximum likelihood estimates. A Monte-Carlo simulation was performed to verify the stability of the maximum likelihood estimates of the model parameters. The flexibility and applicability of the new lifetime distribution were illustrated with two real data sets and the results obtained revealed that the BES distribution provides the best fit among all the compared related distributions. We recommend the transmuted distribution for modelling unimodal or bimodal continuous lifetime data with a nondecreasing or bathtub shaped hazard rate function and hope that it would receive significant applications in the future.

**Table 3: Maximum likelihood fit of the fatigue life of some aluminium coupons data**

| Models | Estimates | SE | $\ell$ | AIC | BIC | KS | CVM | AD |
|---|---|---|---|---|---|---|---|---|
| BES $\hat{\eta}$ $\hat{\lambda}$ | 0.0888 -0.8960 | 0.0035 0.0967 | - 454.9401 | 913.8802 | 919.0906 | 0.0950 7 | 0.1431 | 0.8679 |
| SD $\hat{\eta}$ | 0.0732 | 0.0327 | - 462.1056 | 926.2112 | 928.8163 | 0.1360 | 0.5334 | 3.2141 |
| GD $a$ $b$ | 1.0000 0.9677 | 0.9677 1.0000 | - 457,8804 | 919.7608 | 924.9712 | 0.0998 | 0.1629 | 0.9871 |
| TLD $\hat{\eta}$ $\hat{\lambda}$ | 0.0390 -0.1010 | 0.0021 0.0389 | - 471.1273 | 946.2546 | 951.465 | 0.1695 | 1.0042 | 5.9380 |
| TED $\hat{\eta}$ $\hat{\lambda}$ | 0.0231 -1.2477 | 0.0016 0.0444 | -488.351 | 980.702 | 985.9123 | 0.2304 | 2.0347 | |
| TID | 0.5627 -0.9565 | 0.0026 0.0026 | - 558.2248 | 112045 | 1125.66 | 0.9036 | 26.5551 | |
| TAD $\hat{\eta}$ $\hat{\lambda}$ | 1.1838 -0.9528 | 0.0646 0.0678 | - 155.8115 | 315.6229 | 320.5082 | 0.9548 | 25.1526 | |
| TPD $\hat{\eta}$ $\hat{\lambda}$ | 0.7284 -0.9348 | 0.0029 0.0662 | -456.6 | 9.7200 | 922.4104 | 0.9342 | 28.2863 | 200.1088 |

## References


Abdul Moniem, I. B. and Seham, M. (2015). Transmuted Gompertz distribution. *Computational and Applied Mathematics Journal*, **1**, 36-38.

Akaike, H. (1992). Information theory and an extension of the maximum likelihood principle. in: Breakthroughs in Statistics, *Springer, New York*. 610-624

Al-khazaleh, M., Al-Omari, A. I., and Al-khazaleh, A. H. (2016). Transmuted two-parameter Lindley distribution. *Journal of Statistics Applications and Probability*, **5**, 1-11.

Al-Omari, A. I. and Alsmairan, I. K. (2019). Length-biased Suja distribution and its application. *Journal of Applied Probability and Statistics*, **14**, 95-116.

Al-Omari, A. I., Alhyasat, K. M., Ibrahim, K., and Bakar, A. A. M. (2019). Power length-biased Suja distribution: properties and application. *Electronic Journal of Applied Statistical Analysis,* **12**, 429-452.

Alsmairan, I. K. and Al-Omari, A. I. (2020). Weighted Suja distribution with applications to ball bearings data. *Life Cycle Reliability and safety Engineering,* **9**, 195-211.

Aryal, G. R. (2013). Transmuted log-Logistic distribution. *Journal of Statistics Applications and Probability,* **2**, 11–20.

Aryal, G. R. and Tsokos C. P. (2011). Transmuted Weibull distribution: A generalization of Weibull probability distributions. *European Journal of Pure and Applied Mathematics*,**4**, 89–102.

Aryal, G. R. and Tsokos, C. P. (2009). On the transmuted extreme value distribution with applications. *Nonlinear Analysis: Theory, Methods and Applications*, **71**, 1401-1407.

Ashour, S. K. and Eltehiwy, M. A. (2013). Transmuted Lomax distribution. *American Journal of Applied Mathematics and Statistics,* **1**, 121-127.

Birnabaum, Z. W. and Saunders, S. C. (1969). Estimation for a family of life distributions with applications to fatigue. Journal of Applied Probability, **6**, 328-347.

Bleed, S. O. and Abdelali, A. E. A. (2018). Transmuted Arcsine distribution properties and application. *International Journal of Research-Granthaalayah*, **6**, 38-47.

Bose, R. C. and Nair, K. R. (1939). Partially balanced incomplete block designs. *SankhyA: The Indian Journal of Statistics*, **4**, 337-372.

Chen, G. and Balakrishnan, N. (1995). A general purpose approximate goodness-of-fit test. *Journal of Quality Technology*, **27**, 154-161.

Enogwe, S. U., Nwosu, D. F., Ngome, E. C., Onyekwere, C. K., and Omeje, I. L. (2020). Two-Parameter Odoma distribution with applications. *Journal of Xidian University,* **14**, 740-764.

Gharaibeh, M. M. (2020). Transmuted Aradhana distribution: Properties and Application. *Jordan Journal of Mathematics and Statistics*, **13**, 287-304.

Gharaibeh, M. M. and Al-Omari, A. (2019). Transmuted Ishita distribution and its applications. *Journal of Statistics Applications and Probability*, **8**, 67-81.

Ghitany, M. E., Atieh, B., and Nadarajah, S. (2008). Lindley distribution and its applications. *Mathematics and Computers in Simulation,* **78**, 493-506.

Javed, M., Nawaz, T., and Irfan, M. (2018). The Marshall-Olkin Kappa distribution: properties and applications. *Journal of King Saudi University-Science*, **31**, 684-691.

Mahmoud, M. R. and Mandouh, R. M. (2013). On the transmuted Frechet distribution. *Journal of Applied Sciences Research*, **9**, 5553-5561.

Maurya, R. K., Tripathi, Y. M., and Rastogi, M. K. (2017). Transmuted Burr XII distribution. *Journal of the Indian Society for Probability and Statistics*, **18**, 177-193.

Merovci, F. (2013a). Transmuted Lindley distribution. *International Journal of Open Problems in Computer Science and Mathematics,* **6**, 63-72.

Merovci, F. (2013b). Transmuted Rayleigh distribution. *Austrian Journal of Statistics,* **22**, 21-30.

Merovci, F. and Puka, I. (2014). Transmuted Pareto distribution. *ProbStat Forum*, **7**, 1-11.

Mohiuddin, M., Rather, A. A., Subramanian, C., and Dar, S. A. (2020). Transmuted Garima distribution: Properties and Applications. *Journal of Xidian University*, **14**, 112-123.

Odom, C. C., Nduka, E. C., and Ijomah, M. A. (2019). A modification of Pranav distribution using quadratic rank transmutation map approach. *International Journal of Scientific Research in Mathematical and Statistical Sciences*, **6**, 193-202.

Owoloko, E. A., Oguntunde, P. E., and Adejumo, A. O. (2015). Performance rating of the transmuted exponential distribution: an analytical approach. *Springer Plus*, **4**, 1-15.

Rather, A. A. and Subramanian, C. (2018). Transmuted Mukherjee-Islam failure model, *Journal of Statistics Applications and Probability*, **7**, 343-347.

Schwarz, G. (1978). Estimating the dimension of a model. *Annals of Statistics,* **6**, 461–464.

Shahzad, M. N. and Asghar, Z. (2016). Transmuted Dagum distribution: A more flexible and broad shaped hazard function model. *Hacettepe Journal of Mathematics and Statistics*, **45**, 227-244.

Shanker, R. (2015a). Akash distribution and its applications. *International Journal of Probability and Statistics,* **4**, 65-75.

Shanker, R. (2015b). Shanker distribution and its applications. *International Journal of Statistics and Applications,* **5**, 338-348.

Shanker, R. (2016a). Amarenha distribution and its applications. *American Journal of Mathematics and Statistics*, **6**, 44-56.

Shanker, R. (2016b). Aradhana distribution and its applications. *International Journal of Statistics and Applications,* **6**, 23-34.

Shanker, R. (2016c). Devya distribution and its applications. *International Journal of Statistics,* **6**, 189-202.

Shanker, R. (2016d) Sujatha distribution and its applications. *Statistics in Transition New Series,* **17**, 391-410.

Shanker, R. (2017). Suja distribution and its application. *International Journal of Probability and Statistics*, **6**, 11-19.

Shaw, W. and Buckley, I. (2007). The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic normal distribution from a rank transmutation map. *Research Report.*