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# Robustness of Bayes Estimation of Coefficient of Variation for Normal Distribution for a Class of Moderately Non-Gamma Prior Distributions

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**Abstract**

In this paper, we propose to examine sensitivity of the Bayes estimate of normal coefficient of variation to a moderately non-gamma prior distribution of the unknown precision. Non-negativity and unimodality region of the considered K-prior distributions are computed for illustration purpose. Kullback-Leibler Divergence measure is employed to study the effect as the K-prior becomes much different from the conjugate gamma prior.

*Key words*: Positive and unimodal region; Kullback-Leibler divergence; Bayes estimate; Coefficient of variation; K-prior; MELO approach.

**AMS Subject Classification Code:** 62F15, 62F10

## 1. Introduction

The concept of coefficient of variation (CV) has been intriguing researchers for many years because of its use in assessing the variability of a series since it is independent of the unit of measurement. It has applications in various areas ranging from medical sciences to finance. Here, we study Bayesian estimation of CV for Normal distribution, with mean and precision both unknown, using Zellner's Minimum Posterior Expected Loss (MELO) approach. Zellner (1978) addressed the problem of estimating the reciprocals and the ratios of the population mean and the regression coefficients. He pointed out the situations in which maximum likelihood and other estimators of these problems do not possess finite moments and have infinite risk relative to quadratic and other loss functions, whereas MELO estimators using relative squared error loss function (RSELF) have finite moments and risk, and are hence, admissible.

In Bayes estimation for normal distribution with unknown precision, a conjugate gamma prior is used to obtain the posterior distribution. However, subjectivity involved in choosing a single prior distribution, as observed by Berger (1984), has drawn severe criticism of Bayesian methodology. A reasonable approach is to consider a family of plausible priors that are in the neighbourhood of a specific assessed approximation to the 'true' prior. Not much attention has been paid by the investigators to study the problem of sensitivity to a possible misspecification of the gamma distribution as the conjugate prior distribution in Bayesian analysis. In this paper, we follow Bansal and Singh (1999) and Aggarwal and

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Bansal (2017) to use Khamis' class (K-class) of moderately non-gamma prior distributions for the unknown precision of the normal distribution and study the robustness of the Bayes estimate with respect to the prior.

Many researchers including Barton and Dennis (1952), and Draper and Tierney (1972) exhibited the importance of deriving the conditions under which Gram-Charlier and Edgeworth curves are positive definite and unimodal. Spiring (2011) determined the regions where Edgeworth expansion and Gram-Charlier series upto the $4^{th}$ moment is positive and unimodal. Till now, no attempt has been made in this direction for K-class of moderately non-gamma densities. In this paper, the boundaries of positive and unimodal regions are obtained for K-class of moderately non-gamma densities. The corresponding plot of the region is also displayed.

In Section 2, Bayes estimate of the CV of the normal distribution using MELO approach is derived. In Section 3, we discuss the positive definite and unimodal region for K-class of non-gamma densities. In Section 4, the distance between gamma density and some non-gamma densities are computed using KLD for arbitrarily chosen values of parameters. The derived results are further illustrated using hypothetical data in Section 5.

## 2. Bayes Estimate of Coefficient of Variation of the Normal Distribution

In this Section, the Bayes estimate of Coefficient of Variation (CV) using MELO approach is obtained for Normal distribution with mean and precision both unknown. The conditional normal prior for unknown mean and K-prior for the unknown precision of the normal distribution are used. The posterior distribution is derived below which shall be further used to obtain Bayes estimate of CV.

### 2.1. Likelihood function

Let us suppose that $\mathbf{X} = (X_1, X_2, \dots, X_n)$ is a random sample from $N(\theta, r)$ with mean $\theta$ and precision $r$, both unknown. The likelihood function of $\theta$ and $r$, given observed sample $\mathbf{X} = \mathbf{x}$, is

$$\ell(\theta, r|\mathbf{x}) = \left(\frac{r}{2\pi}\right)^{\frac{n}{2}} exp\left(-\frac{r}{2}\sum_{i=1}^{n}(x_i - \theta)^2\right)$$

$$= \left(\frac{r}{2\pi}\right)^{\frac{n}{2}} exp\left(-\frac{r}{2}\sum_{i=1}^{n}(x_i - \bar{x})^2 - \frac{nr}{2}(\bar{x} - \theta)^2\right); \theta \in (-\infty, \infty), r > 0. \quad (1)$$

### 2.2. Prior distributions

#### 2.2.1. Conditional normal prior for unknown mean

The prior distribution of unknown mean $\theta$, given $r$, is $N(\mu, \tau r)$, both $\mu$ and $\tau$ known, given by

$$g(\theta|r) = \sqrt{\frac{\tau r}{2\pi}} exp\left(-\frac{\tau r}{2}(\theta - \mu)^2\right); -\infty < (\theta, \mu) < \infty, (r, \tau) > 0.$$

### 2.2.2. Khamis' class of moderately non-gamma distributions as a prior for unknown precision (K-prior)

To study the sensitivity of Bayes estimator with respect to the prior when the 'true' prior is not the conventional natural conjugate gamma prior, we consider a class of K-prior for the unknown precision of the normal distribution. Khamis (1960), in his pioneering work, obtained a class of non-gamma densities using Laguerre expansion with Gamma function as the weight function. The application of such series expansion was discussed in Tiku and Tan (1999). Recently, Aggarwal and Bansal (2017) used K-class of moderately non-gamma distributions as a prior (K-prior) for the unknown mean of the Poisson regression super population model.

Consider density $h(r)$ (may be unknown) with first k moments about origin known for $r \in (0,\infty)$ and the Laguerre expansion

$$h_m(r) = \sum_{j=0}^{m} C_j L_j(r) p(r|\alpha,\beta) \text{ with } m \le k$$

where $p(r|\alpha,\beta) = Gamma(\alpha,\beta)$, and

$$L_j(r) = \sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{\Gamma(\alpha+j)}{\Gamma(\alpha+j-i)} (\beta r)^{j-i} \quad, L_0(r) = 1, j = 1,2,\dots,m$$

is the Laguerre polynomial of degree $j$ and $C_j$ are arbitrary constants. Using Khamis (1960)'s expression for $C_j = \frac{\int_0^\infty L_j(r)h(r)dr}{\int_0^\infty \left(L_j(r)\right)^2 h(r)dr}, j = 0,1,\dots,m$, the expansion $h_m(r)$ can be used to approximate $h(r)$ for appropriate values of $\alpha$. Bansal and Singh (1999) considered a particular case of Khamis' class of non-gamma distributions wherein only the first four moments $(m = 4)$ were used. This particular case was referred to as K-class of moderately non-gamma densities, given by

$$h_4(r) \approx g(r) = K(r)p(r|\alpha,\beta),\ r,\alpha,\beta > 0 \tag{2}$$

with

$$K(r) = \left(1 + \frac{\delta_3\sqrt{\alpha}}{6(\alpha+1)(\alpha+2)}\left(L_3(r) - \frac{3}{\alpha+3}L_4(r)\right) + \frac{\delta_4 \alpha}{24(\alpha+1)(\alpha+2)(\alpha+3)}L_4(r)\right).$$

The excess of skewness and kurtosis of K-class of non-gamma densities $g(r)$ over gamma density $p(r|\alpha,\beta)$ are measured by the parameters $\delta_3$ and $\delta_4$, respectively.

**Remark 1:** In particular, if we take $\alpha = 4, \beta = 1, \delta_3 = 0.15, \delta_4 = 2$, then skewness of gamma $p(r|\alpha,\beta) = \frac{4}{\sqrt{(\alpha)}} = 2$ and kurtosis of gamma $p(r|\alpha,\beta) = 3 + \frac{6}{\alpha} = 4.5$. Hence, skewness and kurtosis of K-prior $g(r)$ are 2.15 and 6.5, respectively.

## 2.3. Posterior distribution

The joint prior for θ and $r$ is

$$g(\theta, r) = g(\theta|r)g(r)$$

where $g(\theta|r)$ is $N(\mu, \tau r)$, and $g(r)$ is K-prior given in (2).

Using Bayes Theorem, the posterior distribution of $\theta$ and $r$, given observed sample **X** = **x**, is

$$g(\theta, r|\mathbf{x}) = g(\theta|r,\mathbf{x})g(r|\mathbf{x})$$

where

$$g(\theta|r,\mathbf{x}) = \left(\frac{(n+\tau)r}{2\pi}\right)^{n/2} exp\left(-\frac{r}{2}(n+\tau)(\theta-\mu^*)^2\right) \equiv N(\mu^*,(n+\tau)r), \quad (3)$$

and $$g(r|\mathbf{x}) = \left(\frac{K(r)g(r|\alpha^*,\beta^*,\mathbf{x})}{G(\delta_3,\delta_4)}\right), \quad (4)$$

with

$$\alpha^* = \alpha + \frac{n}{2}, \beta^* = \beta + \frac{1}{2}\sum_{i=1}^{n}(x_i-\bar{x})^2 + \frac{1}{2}\frac{n\tau}{n+\tau}(\mu-\bar{x})^2, \mu^* = \frac{n\bar{x}+\tau\mu}{n+\tau},$$

$$G(\delta_3,\delta_4) = 1 - \delta_3\frac{\alpha^{\frac{3}{2}}}{6}C_1(\alpha^*) + \delta_4\frac{\alpha^2}{24}C_2(\alpha^*),$$

$$C_1(\alpha^*) = 3R_4 - 13R_3 + 21R_2 - 15R_1 + 4R_0,$$

$$C_2(\alpha^*) = R_4 - 4R_3 + 6R_2 - 4R_1 + R_0,$$

and

$$R_j = \left(\frac{\Gamma(\alpha^*+j)}{\Gamma(\alpha^*)\beta^{*j}}\right)\Big/\left(\frac{\Gamma(\alpha+j)}{\Gamma(\alpha)\beta^j}\right), j = 0,1,\ldots,4.$$

(See Appendix A.1 for the details of derivation)

## 2.4. Bayes estimate using Zellner's MELO approach

Zellner (1978) pointed out that the usual Bayes estimate of the reciprocal of normal mean often fails to exist. He recommended MELO estimate as a solution to overcome the problem of non-existence. Following him, consider $\hat{a}$ as the estimate of CV $a = \sigma/\theta$ , $\left(\sigma^2 = \frac{1}{r}\right)$. Upon minimizing posterior expected loss $E((\hat{a}\theta-\sigma)^2|\mathbf{x}) = E(\theta^2(\hat{a}-a)^2|\mathbf{x})$, the MELO estimate is given by

$$\hat{a}_{MELO} = \frac{E(\theta^2 a|\mathbf{x})}{E(\theta^2|\mathbf{x})} = \frac{E\left(\frac{\theta}{\sqrt{r}}\middle|\mathbf{x}\right)}{E(\theta^2|\mathbf{x})} \quad (5)$$

where the expectations are with respect to posterior distribution and are given by

$$E(\theta^2|\mathbf{x}) = \mu^{*2} + \frac{\beta^*}{(\alpha^*-1)(\tau+n)}\frac{G_1(\delta_3,\delta_4)}{G(\delta_3,\delta_4)}, \quad (6)$$

$$E\left(\frac{\theta}{\sqrt{r}}\middle|\mathbf{x}\right) = \mu^*\frac{\sqrt{\beta^*}\Gamma\left(\alpha^*-\frac{1}{2}\right)}{\Gamma(\alpha^*)}\frac{G_2(\delta_3,\delta_4)}{G(\delta_3,\delta_4)}, \quad (7)$$

with

$$G_1(\delta_3,\delta_4) = 1 - \delta_3\frac{\alpha^{\frac{3}{2}}}{6}C_1(\alpha^*-1) + \delta_4\frac{\alpha^2}{24}C_2(\alpha^*-1),$$

and

$$G_2(\delta_3,\delta_4) = 1 - \delta_3\frac{\alpha^{\frac{3}{2}}}{6}C_1\left(\alpha^*-\frac{1}{2}\right) + \delta_4\frac{\alpha^2}{24}C_2\left(\alpha^*-\frac{1}{2}\right).$$

(See Appendix A.2 for the details of derivation of the posterior expectations (6) and (7))

**Remark 2:** The value of $\hat{a}_{MELO}$ in (5) depends on the observed sample values.

**Remark 3:** If we consider gamma prior for $r$, that is $\delta_3 = \delta_4 = 0$, then the MELO estimate reduces to $\mu^*\frac{\sqrt{\beta^*}\Gamma\left(\alpha^*-\frac{1}{2}\right)}{\Gamma(\alpha^*)}\Big/\left(\mu^{*2} + \frac{\beta^*}{(\alpha^*-1)(\tau+n)}\right)$ .

**Remark 4:** For non-informative prior, that is $g(\theta,r) \propto \frac{1}{r}$, the MELO estimate can be obtained by letting $\alpha \to -\frac{1}{2}, \beta \to 0, \tau \to 0$ (See De Groot (1970), page 195) and is given by

$$\frac{\bar{x}\sqrt{\frac{\sum(x_i-\bar{x})^2}{2}\frac{\Gamma\left(\frac{n-1}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}}}{\bar{x}^2+\frac{\sum(x_i-\bar{x})^2}{2n\left(\frac{n-1}{2}-1\right)}} = \left(\sqrt{\frac{n}{2}}\frac{\Gamma\left(\frac{n}{2}-1\right)}{\Gamma\left(\frac{n-1}{2}\right)}\right)\frac{\hat{\sigma}}{\bar{x}}\left(1+\frac{\hat{\sigma}^2}{\bar{x}^2}\frac{1}{n-3}\right)^{-1}, \quad (8)$$

where $\hat{\sigma}^2 = \frac{\sum(x_i-\bar{x})^2}{n}$. This conforms with the result obtained by Bansal (2007).

**Remark 5:** If we further use the result $\lim_{m\to\infty} m^{b-a}\frac{\Gamma(m+a)}{\Gamma(m+b)} = 1$, (see Abramowitz and Stegun (1964), formula 6.1.46, page 257), then on taking $m = \frac{n}{2}, a = -1$ and $b = -\frac{1}{2}$, the first factor on the right-hand side of (8) tends to one for large samples. Hence, it is seen that the MELO Bayes estimate of CV reduces to the product of the usual estimate, $\hat{\sigma}/\bar{x}$, of CV and the shrinkage factor $\left(1+\frac{\hat{\sigma}^2}{\bar{x}^2}\frac{1}{n-3}\right)^{-1}$ which has a value between zero and one. Thus, we may expect that the MELO Bayes estimate of CV to be smaller than the corresponding classical estimate for large samples and moderately non-gamma prior densities of the precision.

In the next Section, we obtain the regions in which $g(r)$ is non-negative and unimodal so that the above obtained results can be illustrated numerically using hypothetical data.

## 3. Positive Definite and Unimodal Region for Khamis' Class of Non-gamma Distributions

Figure 1 below exhibits the graphs of $g(r)$ for various combinations of $\delta_3$ and $\delta_4$ with $\alpha = 4, \beta = 1$. The Graph 1 of Figure 1 represents Gamma Distribution. Graphs 2, 3 and 4 of Figure 1 shows that the graphs change in shape and peakedness with change in $\delta_3$ and $\delta_4$. It may be noticed that there are combinations of $\delta_3$ and $\delta_4$ for which $g(r)$ is negative and multimodal. For example, for $(\delta_3, \delta_4) = (3, 4)$ and $(\delta_3, \delta_4) = (0.1, 15)$, $g(r)$ is negative and multimodal respectively as shown in Graph 5 and 6 of Figure 1 below. Thus, there is a need to obtain the regions in which $g(r)$ is non-negative and unimodal.

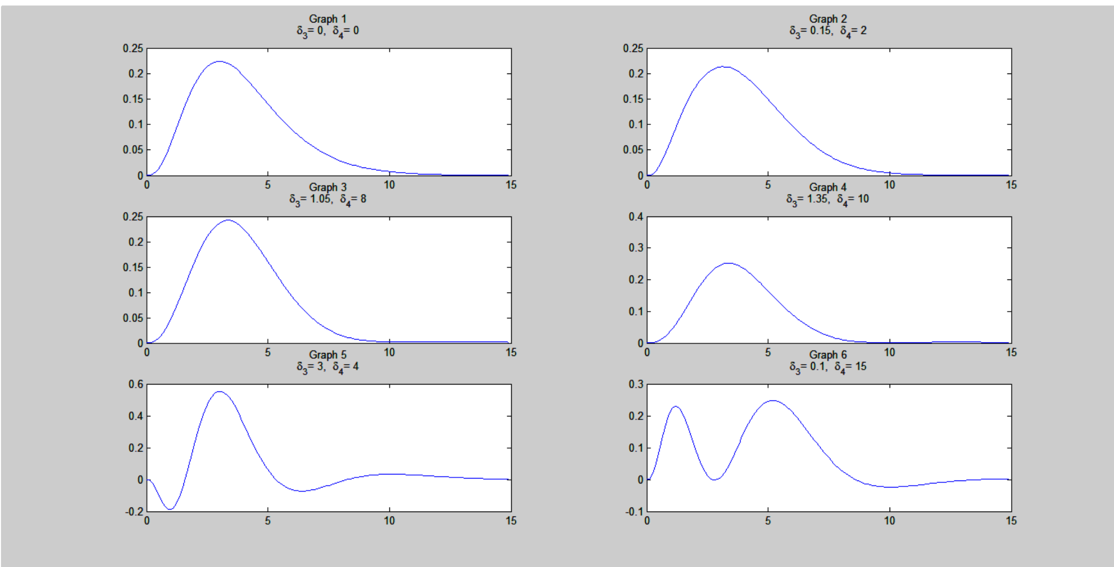


**Figure 1: Graphs of $g(r)$ for various combinations of $\delta_3$ and $\delta_4$ with $\alpha = 4, \beta = 1$**

We now determine the region where $g(r)$ is non-negative for a specific range of $r$. We have tabulated the combinations of $(\delta_3, \delta_4)$ where $g(r)$ is non-negative for $\alpha = 4, \beta = 1$ and the boundaries of the positive regions are provided in Table 1. Figure 2 exhibits the plot of the boundary points in Table 1. This positive region is then checked for unimodality using second derivative test. It is found that the unimodality exists throughout in this positive region. For the region beyond the boundary values given in Table 1, $g(r)$ may be unimodal but not positive. Thus, such regions are not considered. It may be noted that we are providing regions only for $\alpha = 4, \beta = 1$. The entire work is done using Mathematica. The same procedure may be employed as discussed above, to obtain positive and unimodal regions for other choices of $\alpha$ and $\beta$.

**Table 1: Positive and unimodal boundary points ($\delta_4$, L < $\delta_3$< U) for $g(r|4,1)$**

| $\delta_4$ | L | U |
|---|---|---|
| 0 | 0 | 0.18 |
| 0.1 | 0 | 0.2 |
| 0.2 | 0 | 0.21 |
| 0.3 | 0 | 0.22 |
| 0.4 | 0 | 0.23 |
| 0.5 | 0 | 0.25 |
| 0.6 | 0 | 0.26 |
| 0.7 | 0 | 0.27 |
| 0.8 | 0 | 0.28 |
| 0.9 | 0 | 0.3 |
| 1 | 0 | 0.31 |
| 1.1 | 0 | 0.32 |
| 1.2 | 0 | 0.33 |
| 1.3 | 0 | 0.35 |
| 1.4 | 0 | 0.36 |
| 1.5 | 0 | 0.37 |
| 1.6 | 0 | 0.38 |
| 1.7 | 0 | 0.4 |
| 1.8 | 0 | 0.41 |
| 1.9 | 0 | 0.42 |
| 2 | 0 | 0.43 |
| 2.1 | 0.01 | 0.45 |
| 2.2 | 0.02 | 0.46 |
| 2.3 | 0.03 | 0.47 |
| 2.4 | 0.04 | 0.48 |
| 2.5 | 0.06 | 0.5 |
| 2.6 | 0.07 | 0.51 |
| 2.7 | 0.08 | 0.52 |
| 2.8 | 0.1 | 0.53 |
| 2.9 | 0.11 | 0.55 |
| 3 | 0.12 | 0.56 |
| 3.1 | 0.14 | 0.57 |
| 3.2 | 0.15 | 0.58 |
| 3.3 | 0.16 | 0.6 |
| 3.4 | 0.18 | 0.61 |
| 3.5 | 0.19 | 0.62 |
| 3.6 | 0.2 | 0.63 |
| 3.7 | 0.22 | 0.65 |
| 3.8 | 0.23 | 0.66 |
| 3.9 | 0.25 | 0.67 |
| 4 | 0.26 | 0.68 |
| 4.1 | 0.27 | 0.7 |
| 4.2 | 0.29 | 0.71 |
| 4.3 | 0.3 | 0.72 |
| 4.4 | 0.32 | 0.73 |
| 4.5 | 0.33 | 0.75 |
| 4.6 | 0.35 | 0.76 |
| 4.7 | 0.36 | 0.77 |
| 4.8 | 0.38 | 0.78 |
| 4.9 | 0.39 | 0.8 |
| 5 | 0.41 | 0.81 |
| 5.1 | 0.42 | 0.82 |
| 5.2 | 0.44 | 0.83 |
| 5.3 | 0.45 | 0.85 |
| 5.4 | 0.47 | 0.86 |
| 5.5 | 0.48 | 0.87 |
| 5.6 | 0.5 | 0.88 |
| 5.7 | 0.5 | 0.9 |
| 5.8 | 0.53 | 0.91 |
| 5.9 | 0.54 | 0.92 |
| 6 | 0.56 | 0.93 |
| 6.1 | 0.57 | 0.95 |
| 6.2 | 0.59 | 0.96 |
| 6.3 | 0.61 | 0.97 |
| 6.4 | 0.62 | 0.98 |
| 6.5 | 0.64 | 1 |
| 6.6 | 0.65 | 1.01 |
| 6.7 | 0.67 | 1.02 |
| 6.8 | 0.69 | 1.03 |
| 6.9 | 0.7 | 1.05 |
| 7 | 0.72 | 1.06 |
| 7.1 | 0.74 | 1.07 |
| 7.2 | 0.75 | 1.08 |
| 7.3 | 0.77 | 1.1 |
| 7.4 | 0.78 | 1.11 |
| 7.5 | 0.8 | 1.12 |
| 7.6 | 0.82 | 1.13 |
| 7.7 | 0.84 | 1.15 |
| 7.8 | 0.85 | 1.16 |
| 7.9 | 0.87 | 1.17 |
| 8 | 0.89 | 1.18 |
| 8.1 | 0.9 | 1.2 |
| 8.2 | 0.92 | 1.21 |
| 8.3 | 0.94 | 1.22 |
| 8.4 | 0.96 | 1.23 |
| 8.5 | 0.97 | 1.25 |
| 8.6 | 0.99 | 1.26 |
| 8.7 | 1.01 | 1.27 |
| 8.8 | 1.03 | 1.28 |
| 8.9 | 1.04 | 1.3 |
| 9 | 1.06 | 1.31 |
| 9.1 | 1.08 | 1.32 |
| 9.2 | 1.1 | 1.33 |
| 9.3 | 1.12 | 1.35 |
| 9.4 | 1.14 | 1.36 |
| 9.5 | 1.15 | 1.37 |
| 9.6 | 1.17 | 1.38 |
| 9.7 | 1.19 | 1.4 |
| 9.8 | 1.21 | 1.41 |
| 9.9 | 1.23 | 1.42 |
| 10 | 1.25 | 1.43 |
| 10.1 | 1.27 | 1.45 |
| 10.2 | 1.29 | 1.46 |
| 10.3 | 1.31 | 1.47 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 10.4 | 1.33 | 1.48 | 11 | 1.45 | 1.56 | 11.6 | 1.58 | 1.63 |
| 10.5 | 1.35 | 1.5 | 11.1 | 1.47 | 1.57 | 11.7 | 1.61 | 1.65 |
| 10.6 | 1.37 | 1.51 | 11.2 | 1.49 | 1.58 | 11.8 | 1.63 | 1.66 |
| 10.7 | 1.39 | 1.52 | 11.3 | 1.52 | 1.6 | 11.9 | 1.65 | 1.67 |
| 10.8 | 1.41 | 1.53 | 11.4 | 1.54 | 1.61 | 12 | 1.68 | 1.68 |
| 10.9 | 1.43 | 1.55 | 11.5 | 1.56 | 1.62 | 12.1 | 1.7 | 1.7 |

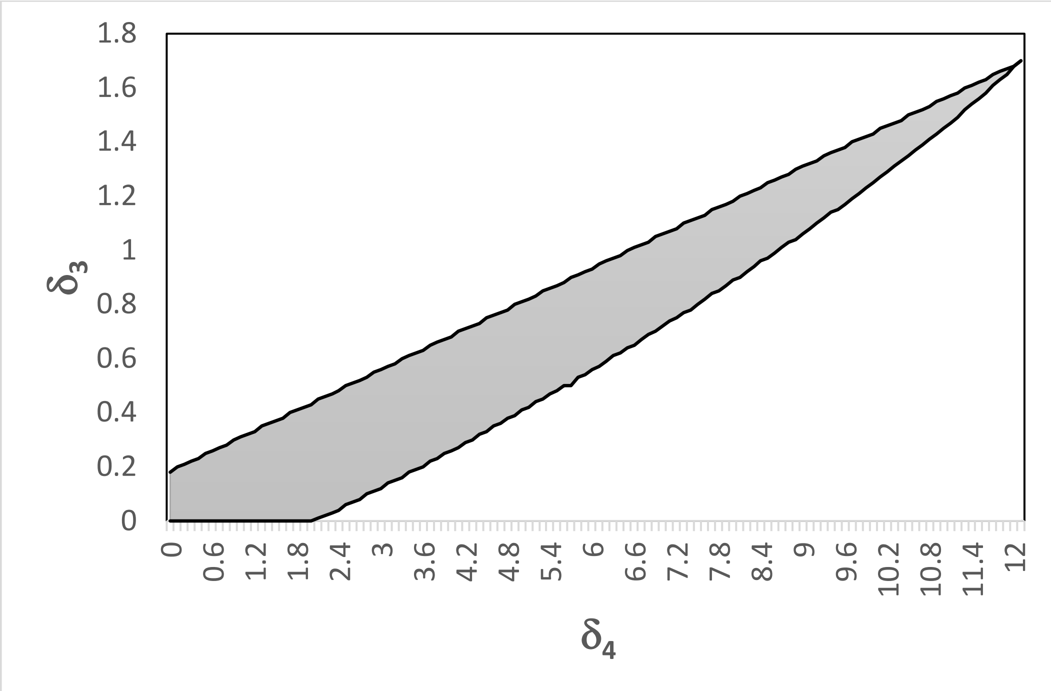


**Figure 2: Plot of Positive and unimodal region for $g(r)$ $with\ \alpha = 4, \beta = 1$**

## 4. Quantitative Robustness using Kullback-Leibler Divergence (KLD) Measure

By virtue of the significance of Gamma distribution in problem of statistical estimation, it is deemed necessary to study the sensitivity of the estimates to its possible misspecification. In this direction, we make an effort to study the quantitative robustness employing Kullback-Leibler divergence (KLD) measure.

To examine quantitative robustness with respect to the K-class of moderately non-gamma densities $g(r)$, we compute its distance from gamma $p(r|\alpha,\beta)$ using KLD as

$$I(p,g) = \int_0^\infty \log\left(\frac{p(r|\alpha,\beta)}{g(r)}\right) p(r|\alpha,\beta)\,dr = E\left(\log\left(\frac{p(r|\alpha,\beta)}{g(r)}\right)\right). \qquad (9)$$

The expectation is taken with respect to $p(r|\alpha,\beta)$. Observe that $I(p,g)$ is not a symmetric distance.

Aggarwal and Bansal (2010) used KLD to evaluate the distance between Normal and Edgeworth distributions for some selected values of $\lambda_3\ (= \delta_3)$ and $\lambda_4\ (= \delta_4)$ lying in region given by Barton and Dennis (1952). Aggarwal and Bansal (2017) computed $I(p,g)$ and it is found that there is an error in its computation. Thus, we extend the study on quantitative robustness using corrected $I(p,g)$ while considering $KL_{min} = \min\{I(p,g), I(g,p)\}$ as a measure to find distance between $p(r|\alpha,\beta)$ and $g(r)$. It may be observed that the distance $KL_{min}$ is a symmetric distance as specified in Bernardo and Rueda (2002).

Table 2 provides computed values of $KL_{min}$ for arbitrarily chosen $\alpha = 4, \beta = 1$ and some selected values of $\delta_3$ and $\delta_4$. The chosen values of $\delta_3$ and $\delta_4$ are those in which $g(r)$ is unimodal and non-negative.

**Table 2: Values of $KL_{min}$ for $\alpha = 4, \beta = 1$ and some selected values of $\delta_3$ and $\delta_4$**

| $\delta_3$ | $\delta_4$ | $KL_{min}$ | $\delta_3$ | $\delta_4$ | $KL_{min}$ | $\delta_3$ | $\delta_4$ | $KL_{min}$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0.4 | 2 | 0.0061 | 0.9 | 6 | 0.016 |
| | 2 | 0.0144 | | 4 | 0.005 | | 8 | 0.0209 |
| 0.15 | 0 | 0.0049 | 0.6 | 4 | 0.0064 | 1.05 | 7 | 0.0226 |
| | 2 | 0.0023 | | 6 | 0.0139 | | 8 | 0.0192 |
| 0.3 | 2 | 0.0011 | 0.75 | 5 | 0.0107 | 1.35 | 9.5 | 0.0358 |
| | 4 | 0.0118 | | 7 | 0.0167 | | 10.3 | 0.0354 |

From Table 2, it may be observed that

(1) Out of the chosen combinations of ($\delta_3$, $\delta_4$), $KL_{min}$ is minimum for (0, 0) as it corresponds to Gamma distribution, and is maximum for (1.35, 9.5).

(2) $KL_{min}$ could be approximately same for different choices for ($\delta_3$, $\delta_4$). In particular, for the combinations (0, 2) and (0.6, 6), $KL_{min}$ is approximately 0.014. However, the graphs of $g(r)$ for these values of ($\delta_3$, $\delta_4$) are different as shown in Figure 3.

(3) For (0.6, 4), $KL_{min} = 0.0064$, and for (0.15, 2), $KL_{min} = 0.0023$. So, $g(r)$ corresponding to (0.6, 4) is more non-gamma than gamma distribution as compared to the $g(r)$ corresponding to (0.15, 2).

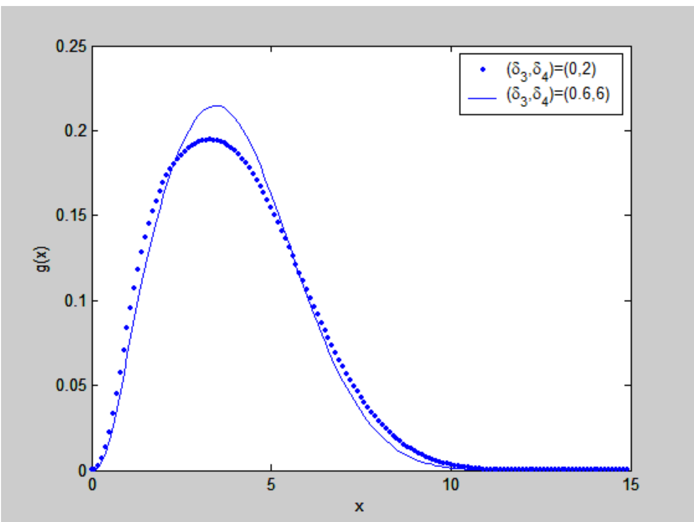

**Figure 3: Graph of $g(r)$**

In the next section, Bayes estimates of CV, obtained in Section (2.4), will now be calculated for hypothetical data using some values of $\delta_3$ and $\delta_4$ selected based on $KL_{min}$ discussed in this section.

## 5. Numerical Illustration

To study the effect of non-gamma prior, we generate a hypothetical data of size $n = 10$ from $N(4,4)$ distribution given by 0.0660, 5.2140, 3.7548, 5.4743, 6.2490, 2.0363, 4.8134, 9.4950, 6.6342, 4.4920. It is clear that the true CV is 0.5 whereas the classical estimate of CV, the ratio of observed standard deviation and observed mean, is 0.505187. The MELO estimate under non-informative prior is 0.5622.

The Bayes estimates of CV, with $\alpha = 4, \beta = 1, \mu = 0, \tau = 1$, and various values of $\delta_3$ and $\delta_4$ selected using Table 2, are tabulated in Table 3.

**Table 3: Bayes estimate of CV for various values of $\delta_3$ and $\delta_4$**

| $\delta_3$ | $\delta_4$ | $KL_{min}$ | $\hat{a}_{MELO}$ |
|---|---|---|---|
| 0 | 0 | 0 | 0.4978 |
| 0.15 | 2 | 0.0023 | 0.4998 |
| 0.40 | 4 | 0.0050 | 0.5001 |
| 0.75 | 5 | 0.0107 | 0.4924 |
| 1.05 | 8 | 0.0192 | 0.4980 |
| 1.35 | 9.5 | 0.0358 | 0.4871 |

From Table 3, one may observe that the Bayes estimates $\hat{a}_{MELO}$ of CV are close to the Bayes estimate of CV under gamma prior for all chosen combination of $\delta_3$ and $\delta_4$. The difference in the maximum and minimum value of $\hat{a}_{MELO}$ is 0.013 which is insignificant and hence, we may say that the moderate deviation from gamma prior may not significantly affect Bayes estimate of coefficient of variation under MELO. We may, therefore, conclude that the Bayes estimate is robust with respect to misspecification of the prior distribution for precision in our illustration.

## 6. Conclusion

In this paper, Bayes estimate of coefficient of variation is derived for normal model with both mean and variance unknown. The normal conditional prior for unknown mean and K-prior for the unknown precision of the normal distribution are considered. The positive and unimodal regions for K-class of non-gamma densities are obtained for $\alpha = 4$ and $\beta = 1$. The boundary values of $\delta_3$ and $\delta_4$ where the pdf of non-gamma distribution changes from the positive definite to non-positive definite are provided. It is seen that in the region bounded by the above values, pdf is unimodal as well. For other values of $\alpha$ and $\beta$, one may find region where pdf is positive and unimodal using the same procedure. It is found that for two or more members of K-class of non-gamma distributions, $KL_{min}$ could be approximately same which means that these members are equally non-gamma as compared to the gamma distribution. A numerical illustration is also discussed and therein, it is observed that Bayes estimate of coefficient of variation under K-prior distributions are very close to that based on gamma prior distribution for all chosen combinations of $\delta_3$ and $\delta_4$. We may also conclude that the Bayes estimate of CV under MELO is reasonably insensitive to moderate deviation from generally assumed gamma prior distribution.

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**References**

Abramowitz, M. and Stegun, I. A. (1964). *Handbook of Mathematical Functions*. Dover Publications, Tenth printing (1972), New York.

Aggarwal, P. and Bansal, A. K. (2010). Robustness of Bayes prediction under error-in-variables superpopulation model. *Communications in Statistics: Theory and Methods*, **39**(**2**), 327-339.

Aggarwal, P. and Bansal, A. K. (2017). Bayes prediction of Poisson regression superpopulation mean with a non-gamma prior. *Communications in Statistics: Theory and Methods*, **46**(**11**), 5531-5543.

Bansal, A. K. (2007). *Bayesian Parametric Inference*. Narosa Publishing House, India.

Bansal, A. K. and Singh, R. C. (1999). Bayes estimation with a non-gamma prior. *Ganita*, **50**(**2**), 93-102.

Barton, D. E. and Dennis, K. E. (1952). The conditions under which Gram-Charlier and Edgeworth curves are positive definite and unimodal. *Biometrika*, **39**, 425-427.

Berger, J. O. (1984). The robust Bayesian view-point. In: Kadane, J. B., Ed. *Robustness of Bayesian Statistics*. Amsterdam: North-Holland, 63-124.

Bernardo, J. M. and Rueda, R. (2002). Bayesian hypothesis testing: A reference approach. *International Statistical Review*, **70**, 351-372.

De Groot, M. H. (1970). *Optimal Statistical Decisions*. McGraw-Hill, New York.

Draper, N. and Tierney, D. (1972). Regions of positive and unimodal series expansion of the Edgeworth and Gram-Charlier approximation. *Biometrika*, **59**, 463-465.

Khamis, S. H. (1960). Incomplete gamma functions expansions of statistical distribution functions. *Bulletin of International Statistical Institute*, **37**, 385-396.

Tiku, M. L. and Tan W. Y. (1999). *Sampling Distributions in Terms of Laguerre Polynomials With Applications*. Wiley Eastern, New Age International, New Delhi.

Spiring, F. (2011). The refined positive definite and unimodal regions for the Gram-Charlier and Edgeworth Series Expansion. *Advances in Decision Sciences*, Article ID 463097, 18 pages.

Zellner, A. (1978). Estimation of functions of population means and regression coefficients including structural coefficients (A minimum expected loss (MELO) approach). *Journal of Econometrics*, **8**, 127-158.

## Appendix

### A.1. (Derivation of posterior distribution given in Section 4.1)

It is known that

$$g(\theta,r|\mathbf{x}) = \frac{\ell(\theta,r\,|\,\mathbf{x})g(\theta,r)}{\int_{-\infty}^{\infty}\int_0^{\infty}\ell(\theta,r|\mathbf{x})g(\theta,r)d\theta dr}$$

Using

$$\ell(\theta,r|\mathbf{x}) = \left(\frac{r}{2\pi}\right)^{n/2} exp\left(-\frac{r}{2}\sum_{i=1}^{n}(x_i-\bar{x})^2 - \frac{nr}{2}(\bar{x}-\theta)^2\right),$$

and

$$g(\theta,r) = g(\theta|r)g(r) = \left(\sqrt{\frac{r}{2\pi}}exp\left(-\frac{\tau r}{2}(\theta-\mu)^2\right)\right)(K(r)\frac{\beta^\alpha}{\Gamma(\alpha)}\exp(-\beta r)\,r^{\alpha-1},$$

we get

$$g(\theta,r|\mathbf{x}) = \frac{r^{\frac{n}{2}+\frac{1}{2}+\alpha-1}K(r)\exp\left(-r\left(\beta+\frac{1}{2}\sum_{i=1}^{n}(x_i-\bar{x})^2\right)\right)\exp\left(-\frac{r}{2}(n(\theta-\bar{x})^2+\tau(\theta-\mu)^2)\right)}{\int_{-\infty}^{\infty}\int_0^{\infty} r^{\frac{n}{2}+\frac{1}{2}+\alpha-1}K(r)\exp\left(-r\left(\beta+\frac{1}{2}\sum_{i=1}^{n}(x_i-\bar{x})^2\right)\right)\exp\left(-\frac{r}{2}(n(\theta-\bar{x})^2+\tau(\theta-\mu)^2)\right)d\theta dr}.$$

Using a result that

$$A(z-a)^2 + B(z-b)^2 = (A+B)(z-c)^2 + \frac{AB}{A+B}(a-b)^2, c = \frac{Aa+Bb}{A+B},$$

we can write

$$n(\theta-\bar{x})^2 + \tau(\theta-\mu)^2 = (n+\tau)(\theta-\mu^*)^2 + \frac{n\tau}{n+\tau}(\mu-\bar{x})^2, \mu^* = \frac{n\bar{x}+\tau\mu}{n+\tau}.$$

Thus,

$$g(\theta,r|\mathbf{x})$$
$$= \frac{r^{\frac{n}{2}+\frac{1}{2}+\alpha-1}K(r)\exp\left(-r\left(\beta+\frac{1}{2}\sum_{i=1}^{n}(x_i-\bar{x})^2+\frac{1}{2}\frac{n\tau}{n+\tau}(\mu-\bar{x})^2\right)\right)\exp\left(-\frac{r}{2}((n+\tau)(\theta-\mu^*)^2)\right)}{\int_{-\infty}^{\infty}\int_0^{\infty} r^{\frac{n}{2}+\frac{1}{2}+\alpha-1}K(r)\exp\left(-r\left(\beta+\frac{1}{2}\sum_{i=1}^{n}(x_i-\bar{x})^2+\frac{1}{2}\frac{n\tau}{n+\tau}(\mu-\bar{x})^2\right)\right)\exp\left(-\frac{r}{2}((n+\tau)(\theta-\mu^*)^2)\right)d\theta dr}$$

$$= \frac{r^{\frac{n}{2}+\frac{1}{2}+\alpha-1}K(r)\exp\left(-r\left(\beta+\frac{1}{2}\sum_{i=1}^{n}(x_i-\bar{x})^2+\frac{1}{2}\frac{n\tau}{n+\tau}(\mu-\bar{x})^2\right)\right)\exp\left(-\frac{r}{2}((n+\tau)(\theta-\mu^*)^2)\right)}{\int_0^{\infty} r^{\frac{n}{2}+\alpha-1}K(r)\exp\left(-r\left(\beta+\frac{1}{2}\sum_{i=1}^{n}(x_i-\bar{x})^2+\frac{1}{2}\frac{n\tau}{n+\tau}(\mu-\bar{x})^2\right)\right)\sqrt{\frac{2\pi}{n+\tau}}dr}.$$

Writing $\alpha^* = \alpha + \frac{n}{2}, \beta^* = \beta + \frac{1}{2}\sum_{i=1}^{n}(x_i-\bar{x})^2 + \frac{1}{2}\frac{n\tau}{n+\tau}(\mu-\bar{x})^2$, we can write

$$g(\theta,r|\mathbf{x}) = \left(\sqrt{\frac{(n+\tau)r}{2\pi}}\exp\left(-\frac{r}{2}((n+\tau)(\theta-\mu^*)^2)\right)\right)\frac{\left(r^{\alpha^*-1}K(r)\exp(-r\beta^*)\right)}{\int_0^{\infty} r^{\alpha^*-1}K(r)\exp(-r\beta^*)\,dr}$$
$$= \left(\sqrt{\frac{(n+\tau)r}{2\pi}}\exp\left(-\frac{r}{2}((n+\tau)(\theta-\mu^*)^2)\right)\right)\frac{K(r)p(r|\alpha^*,\beta^*,\mathbf{x})}{G(\delta_3,\delta_4)}$$

where

$$G(\delta_3,\delta_4) = 1 + A_1\frac{\Gamma(\alpha+3)}{\Gamma(\alpha)}\sum_{i=0}^{3}(-1)^i\binom{3}{i}R_{3-i} + A_2\frac{\Gamma(\alpha+4)}{\Gamma(\alpha)}\sum_{i=0}^{4}(-1)^i\binom{4}{i}R_{4-i}$$

$$= 1 - \delta_3 \frac{\alpha^{\frac{3}{2}}}{6} C_1(\alpha^*) + \delta_4 \frac{\alpha^2}{24} C_2(\alpha^*),$$
$$C_1(\alpha^*) = 3R_4 - 13R_3 + 21R_2 - 15R_1 + 4R_0,$$
$$C_2(\alpha^*) = R_4 - 4R_3 + 6R_2 - 4R_1 + R_0,$$

and

$$R_j = \left(\frac{\Gamma(\alpha^* + j)}{\Gamma(\alpha^*)\beta^{*j}}\right) \Big/ \left(\frac{\Gamma(\alpha + j)}{\Gamma(\alpha)\beta^j}\right).$$

$\nu_j$ and $\mu_j$ are the moments about origin of order $j$ of gamma prior and posterior gamma, respectively.

**A.2. (Derivation of the posterior expectations given in Section 4.2)**

(1) $E(\theta^2|\mathbf{x}) = \int_0^\infty \frac{K(r)p(r|\alpha^*,\beta^*,\mathbf{x})}{G(\delta_3,\delta_4)} \left(\int_{-\infty}^\infty \theta^2 \sqrt{\frac{(n+\tau)r}{2\pi}} \exp\left(-\frac{r}{2}\left((n+\tau)(\theta-\mu^*)^2\right)\right) d\theta\right) dr$

$$= \int_0^\infty \frac{K(r)p(r|\alpha^*,\beta^*,\mathbf{x})}{G(\delta_3,\delta_4)} \left(\mu^{*2} + \frac{1}{r(\tau+n)}\right) dr$$
$$= \mu^{*2} + \frac{G_1(\delta_3,\delta_4)}{G(\delta_3,\delta_4)} \frac{\beta^{*\alpha^*}}{\Gamma(\alpha^*)} \frac{\Gamma(\alpha^*-1)}{\beta^{*\alpha^*-1}} \frac{1}{\tau+n}$$

where

$$G_1(\delta_3,\delta_4) = 1 - \delta_3 \frac{\alpha^{\frac{3}{2}}}{6} C_1(\alpha^*-1) + \delta_4 \frac{\alpha^2}{24} C_2(\alpha^*-1),$$
$$\therefore E\left(\theta^2\middle|\mathbf{x}\right) = \mu^{*2} + \frac{\beta^*}{(\alpha^*-1)(\tau+n)} \frac{G_1(\delta_3,\delta_4)}{G(\delta_3,\delta_4)}.$$

(2) $E\left(\frac{\theta}{\sqrt{r}}\middle|\mathbf{x}\right) = \int_0^\infty r^{-\frac{1}{2}} \frac{K(r)p(r|\alpha^*,\beta^*,\mathbf{x})}{G(\delta_3,\delta_4)} \left(\int_{-\infty}^\infty \theta \sqrt{\frac{(n+\tau)r}{2\pi}} \exp\left(-\frac{r}{2}\left((n+\tau)(\theta-\mu^*)^2\right)\right) d\theta\right) dr$

$$= \int_0^\infty r^{-\frac{1}{2}} \frac{K(r)p(r|\alpha^*,\beta^*,\mathbf{x})}{G(\delta_3,\delta_4)} \mu^* dr$$
$$= \mu^* \frac{G_2(\delta_3,\delta_4)}{G(\delta_3,\delta_4)} \frac{\beta^{*\alpha^*}}{\Gamma(\alpha^*)} \frac{\Gamma\left(\alpha^* - \frac{1}{2}\right)}{\beta^{*\alpha^*-\frac{1}{2}}}$$

where

$$G_2(\delta_3,\delta_4) = 1 - \delta_3 \frac{\alpha^{\frac{3}{2}}}{6} C_1\left(\alpha^* - \frac{1}{2}\right) + \delta_4 \frac{\alpha^2}{24} C_2\left(\alpha^* - \frac{1}{2}\right)$$
$$\therefore E\left(\frac{\theta}{\sqrt{r}}\middle|\mathbf{x}\right) = \mu^* \frac{\sqrt{\beta^*}\,\Gamma\left(\alpha^* - \frac{1}{2}\right)}{\Gamma(\alpha^*)} \frac{G_2(\delta_3,\delta_4)}{G(\delta_3,\delta_4)}.$$