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Advancements in Power Inverse Rayleigh Modeling: Exploring Applications in Environmental and Medical Domains

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Abstract

In this manuscript, a novel probability distribution, known as the *MTI* inverse power Rayleigh distribution (*MTI-IPRD*), is obtained through the use of the *MTI* transformation. This new distribution fits data better than many existing models. A number of statistical features are examined in detail. The maximum likelihood estimation (*MLE*) procedure is employed to estimate the unknown parameters. An extensive simulation study is carried out to illustrate the behaviour of *MLE*s on the basis of mean square errors. In addition, the suggested distribution's flexibility and importance are assessed in comparison to existing probability distributions using three real data sets.

Key words: MTI transformation; Rayleigh distribution; Moments; Renyi entropy; Stress strength reliability; Maximum likelihood estimation.

AMS Subject Classifications: 62K05, 05B05

1. Introduction

Lifetime phenomena modelling and analysis are essential components of statistical work in many scientific and technological fields. Lifetime data analysis has had tremendous growth and progress in terms of theory, applications, and technique. Various generalisation or transformation techniques together with continuous probability distributions have been presented to simulate real-life situations. The model's acceptability and applicability are enhanced by these generalisations for a variety of uses. These can be achieved by altering the distribution's functional form or by including one or more shape factors.

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One popular lifetime probability model for explaining skewed data is the Rayleigh distribution (*RD*). Lord Rayleigh (1880) is the inspiration behind its name. The statistical literature contains numerous proposals for extending the Rayleigh distribution (*RD*). Surles and Padgett (2001) studied exponentiated *RD* (*ERD*). Using various estimating techniques, Kundu and Raqab (2005) investigated and evaluated the parameters of the generalised *RD*. Abd Elfattah *et al.* (2006) investigated the effectiveness of *MLE* under different censored sampling strategies for the *RD*. The odd Lindley power *RD* was extended by Bhat *et al.* (2023), who also examined its characteristics and assessed parameter estimate approaches using Bayesian and classical approaches. Using the power transformation technique, Bhat and Ahmad (2020) presented a new generalisation of the *RD*. The sine power *RD* was presented and its features and applications were explored by Mir and Ahmad (2024). By Voda (1972), the inverse *RD* (*IRD*) was proposed. Additionally, the parameters were estimated by Gharraph (1993) and Soliman *et al.* (2010) utilising various classical and Bayesian estimation techniques. For instance, Merovci (2013) and Merovci (2014) created transmuted inverse *RD* and transmuted *RD*, respectively, using the *QRTM*.

This paper's primary goal is to suggest and investigate a new lifetime model based on the *MTI* approach, known as the *MTI* inverse power Rayleigh distribution (*MTI-IPRD*). The main goal of the new model is to give the hazard and density functions more flexibility and other desirable qualities through the addition of a parameter. Furthermore, when applied to three real data sets, the suggested model performs better than a few well-known models. This article's remaining sections are organised as follows. The *MTI-IPRD* is introduced in Section 2. The reliability analysis is covered in Section 3. Its statistical properties are discussed in Section 4. The objective of Section 5 is to estimate the unknown parameters using the maximum likelihood technique. The outcomes of the simulation are shown in Section 6. In Section 7, the model's applicability is illustrated. Lastly, Section 8 presents conclusions.

2. MTI inverse power Rayleigh distribution (*MTI-IPRD*)

The CDF and PDF of the *MTI* transformation method proposed by Lone *et al.* (2022) are defined by

$$F_{MTI}(y) = \frac{\eta F(y)}{\eta - \log \eta \, \bar{F}(y)} \quad ; \qquad y \in \mathbb{R}, \eta \in \mathbb{R}^+ \tag{1}$$

where $\bar{F}(y) = 1 - F(y)$.

$$f_{MTI}(y) = \frac{\eta(\eta - \log \eta) f(y)}{(\eta - \log \eta \, \bar{F}(y))^2} \quad ; \qquad y \in \mathbb{R}, \eta \in \mathbb{R}^+ \tag{2}$$

Where the CDF and PDF of the baseline distribution are denoted by $F(y)$ and $f(y)$ in the Eqs. (1) and (2) respectively.

Bhat *et al.* (2022) proposed the inverse power Rayleigh distribution (*IPRD*) with PDF and CDF, respectively, given by

$$g(y;\zeta,\delta)=\frac{\zeta}{\delta^2}y^{-(2\zeta+1)}\exp\left(-\frac{y^{-2\zeta}}{2\delta^2}\right);\quad \zeta,\delta>0 \tag{3}$$

$$G(y;\zeta,\delta)=\exp\left(-\frac{y^{-2\zeta}}{2\delta^2}\right);\quad \zeta,\delta>0 \tag{4}$$

We now present the *MTI* technique. The CDF of the *MTI-IPRD* can be obtained by using (4) in (1), where $G(y;\zeta,\delta)$ is considered to represent the CDF of the *IPRD* and is given by

$$F(y;\eta,\zeta,\delta)=\begin{cases}\dfrac{\eta\, e^{-\frac{y^{-2\zeta}}{2\delta^2}}}{\eta-\log(\eta)\left(1-e^{-\frac{y^{-2\zeta}}{2\delta^2}}\right)} & ;\, y>0,\eta\neq 1,\eta,\zeta,\delta>0\\ \exp\left(-\frac{y^{-2\zeta}}{2\delta^2}\right) & ;\, \eta=1,\zeta,\delta>0\end{cases} \tag{5}$$

The corresponding PDF of *MTI-IPRD* is obtained as:

$$f(y;\eta,\zeta,\delta)=\begin{cases}\frac{\eta\zeta}{\delta^2}y^{-(2\zeta+1)}\dfrac{(\eta-\log(\eta))e^{-\frac{y^{-2\zeta}}{2\delta^2}}}{\left(\eta-\log(\eta)\left(1-e^{-\frac{y^{-2\zeta}}{2\delta^2}}\right)\right)^2} & ;\, y>0,\eta\neq 1,\eta,\zeta,\delta>0\\ \frac{\zeta}{\delta^2}y^{-(2\zeta+1)}\exp\left(-\frac{y^{-2\zeta}}{2\delta^2}\right) & ;\, \eta=1,\zeta,\delta>0\end{cases} \tag{6}$$

Figure 1 shows the PDF of the *MTI-IPRD* for various parameter values of η, ζ, and δ. This example highlights the versatility of the *MTI-IPRD* by showing PDFs that can have increasing and decreasing density functions, be symmetric, or be right-skewed. These variants demonstrate how well the model may represent a variety of data patterns found in lifespan distributions.

3. Reliability analysis of *MTI-IPRD*

3.1. Reliability function

The Reliability function for *MTI-IPRD* is given as

$$R(y;\eta,\zeta,\delta)=\frac{(\eta-\log(\eta))\left(1-e^{-\frac{y^{-2\zeta}}{2\delta^2}}\right)}{\eta-\log(\eta)\left(1-e^{-\frac{y^{-2\zeta}}{2\delta^2}}\right)} \tag{7}$$

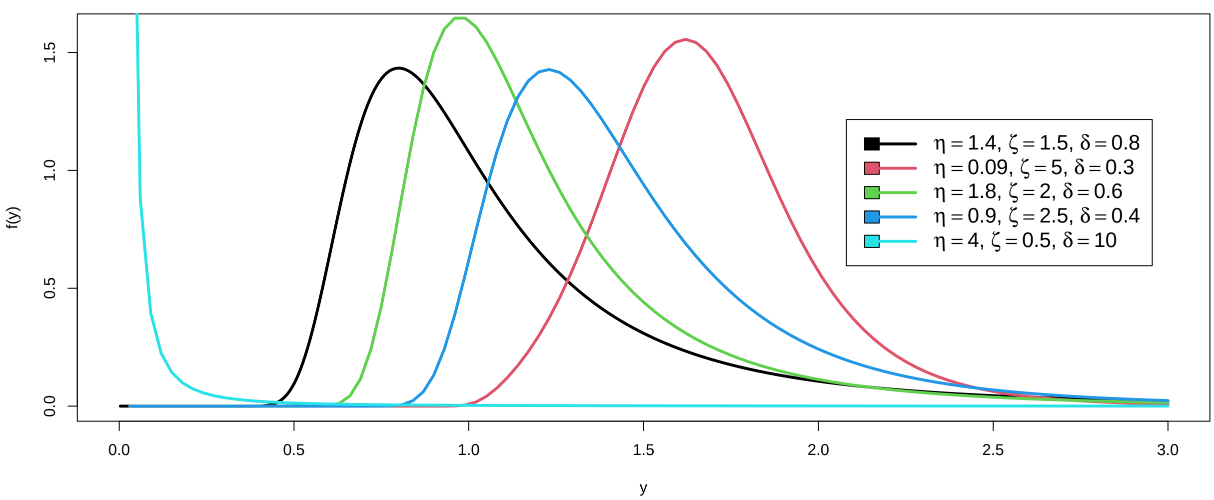

Figure 1: Plots of the pdf of the *MTI-IPRD*

3.2. Hazard rate

The hazard rate for *MTI-IPRD* is obtained as

$$h(y;\eta,\zeta,\delta)=\frac{\eta\zeta\ y^{-(2\zeta+1)}\ e^{-\frac{y^{-2\zeta}}{2\delta^2}}}{\delta^2\left(\eta-\log(\eta)\ \left(1-e^{-\frac{y^{-2\zeta}}{2\delta^2}}\right)\right)\left(1-e^{-\frac{y^{-2\zeta}}{2\delta^2}}\right)} \tag{8}$$

The hazard rate graphs for the *MTI-IPRD* for various parameter values are shown in Figure 2. The proposed distribution is extremely flexible and can take on a number of shapes, as shown in Figure 2.

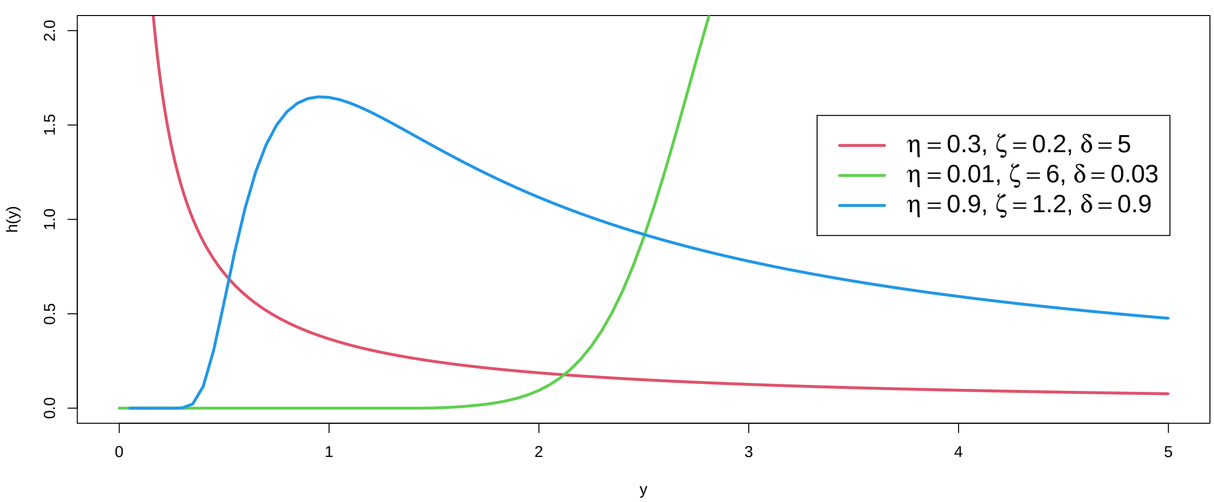

Figure 2: Plots of the hazard rate of the model *MTI-IPRD*

3.3. Reverse hazard rate

The reverse hazard rate for *MTI-IPRD* is obtained as

$$h_r(\eta,\zeta,\delta)=\frac{\zeta(\eta-\log(\eta))\ y^{-(2\zeta+1)}}{\delta^2\left(\eta-\log(\eta)\ \left(1-e^{-\frac{y^{-2\zeta}}{2\delta^2}}\right)\right)} \tag{9}$$

3.4. Cumulative hazard function

The cumulative hazard function for *MTI-IPRD* is is determined as

$$\Lambda(y;\eta,\zeta,\delta)=\log\left\{\frac{\eta-\log(\eta)\left(1-e^{-\frac{y^{-2\zeta}}{2\delta^2}}\right)}{\eta\left(\eta-\log(\eta)\right)\ \left(1-e^{-\frac{y^{-2\zeta}}{2\delta^2}}\right)}\right\} \tag{10}$$

3.5. Mills ratio

The mills ratio for *MTI-IPRD* is obtained as

$$M.R=\frac{\eta\ e^{-\frac{y^{-2\zeta}}{2\delta^2}}}{\eta-\log(\eta)\ \left(1-e^{-\frac{y^{-2\zeta}}{2\delta^2}}\right)} \tag{11}$$

4. Statistical properties of *MTI-IPRD*

The main statistical and distributional aspects of the suggested model are examined in this Section.

4.1. Quantile function

Theorem 1: If $Y\sim$ *MTI-IPRD*(η,ζ,δ), then the quantile function of Y is given as

$$y=\left[-2\delta^2\log\left(\frac{m(\eta-\log(\eta))}{\eta-m\log(\eta)}\right)\right]^{\frac{-1}{2\zeta}} \tag{12}$$

where M is a uniform random variable, $0<m<1$.

Proof: Let $F(y;\eta,\zeta,\delta)=m$. The quantile function of the *MTI-IPRD* can be obtained by using (4) in (1) as follows:

$$\frac{\eta\ e^{-\frac{y^{-2\zeta}}{2\delta^2}}}{\eta-\log(\eta)\left(1-e^{-\frac{y^{-2\zeta}}{2\delta^2}}\right)}=m$$

Taking the logarithm on both sides and simplifying further, we obtain the required quantile function as

$$y = \left[-2\delta^2 \log\left(\frac{m(\eta - \log(\eta))}{\eta - m\log(\eta)}\right)\right]^{\frac{-1}{2\zeta}} \tag{13}$$

By setting $m = \frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ in (13), respectively, one can derive the first quartile (Q_1), median (Q_2), and third quartile (Q_3). □

4.2. Moments

Theorem 2: If $Y \sim$ *MTI-IPRD*(η, ζ, δ), then the r^{th} moment of the *MTI-IPRD* about the origin is given as

$$\mu_r' = \frac{(\eta - \log(\eta))(2\delta^2)^{\frac{-r}{2\zeta}}}{\eta} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^q \binom{p}{q}(p+1)\left(\frac{\log(\eta)}{\eta}\right)^p \frac{\Gamma\left(1-\frac{r}{2\eta}\right)}{(q+1)^{1-\frac{r}{2\eta}}}; \quad r < 2\eta \tag{14}$$

Proof: Utilising the subsequent series representation, the r^{th} moment of the *MTI-IPRD* is found.

$$(1-v)^{-2} = \sum_{q=0}^{\infty}(q+1)v^q \ ; \quad |v| < 1, \tag{15}$$

$$(1-v)^p = \sum_{q=0}^{\infty}(-1)^q\binom{p}{q}v^q \tag{16}$$

The r^{th} moment of Y is given by

$$\mu_r' = \int_0^{\infty} y^r \frac{\eta\zeta}{\delta^2} y^{-(2\zeta+1)} \frac{(\eta - \log(\eta))\ e^{-\frac{y^{-2\zeta}}{2\delta^2}}}{\left(\eta - \log(\eta)\left(1 - e^{-\frac{y^{-2\zeta}}{2\delta^2}}\right)\right)^2} dy. \tag{17}$$

By substituting $\frac{y^{-2\zeta}}{2\delta^2} = z$ in (17), we get

$$\mu_r' = \frac{(\eta - \log(\eta))(2\delta^2)^{\frac{-r}{2\zeta}}}{\eta} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^q \binom{p}{q}(p+1)\left(\frac{\log(\eta)}{\eta}\right)^p \left(\int_0^{\infty} z^{1-\frac{r}{2\eta}-1} e^{-(q+1)z} dz\right) \tag{18}$$

The final expression is

$$\mu_r' = \frac{(\eta - \log(\eta))(2\delta^2)^{\frac{-r}{2\zeta}}}{\eta} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^q \binom{p}{q}(p+1)\left(\frac{\log(\eta)}{\eta}\right)^p \frac{\Gamma\left(1-\frac{r}{2\eta}\right)}{(q+1)^{1-\frac{r}{2\eta}}}; \quad r < 2\eta \tag{19}$$

Setting $r = 1$ in (19), the mean of the *MTI-IPRD* is computed as

$$\mu_1' = \frac{(\eta - \log(\eta))(2\delta^2)^{-\frac{1}{2\zeta}}}{\eta} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^q \binom{p}{q}(p+1)\left(\frac{\log(\eta)}{\eta}\right)^p \frac{\Gamma\left(1-\frac{1}{2\eta}\right)}{(q+1)^{1-\frac{1}{2\eta}}}; \quad \eta > \frac{1}{2} \tag{20}$$

Similarly, substituting $r = 2, 3, 4$ in (19), the second, third, and fourth moments about the origin of the *MTI-IPRD* are obtained, respectively. □

4.3. Moment generating function of *MTI-IPRD*

Theorem 3: If $Y \sim$ *MTI-IPRD*(η, ζ, δ), then the moment generating function, $M_Y(t)$, is

$$M_Y(t) = \frac{(\eta - \log(\eta))}{\eta} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \frac{t^r}{r!} (-1)^q \binom{p}{q} (p+1) (2\delta^2)^{\frac{-r}{2\zeta}} \left(\frac{\log(\eta)}{\eta} \right)^p \frac{\Gamma\left(1 - \frac{r}{2\eta}\right)}{(q+1)^{1-\frac{r}{2\eta}}}; \quad r < 2\eta \tag{21}$$

Proof: The *MTI-IPRD* moment generating function is described as

$$M_Y(t) = \int_0^\infty e^{ty} f(y; \eta, \zeta, \delta)\, dy \tag{22}$$

By utilizing the series representation of e^{ty}, we have

$$M_Y(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(Y^r) \tag{23}$$

Substituting (19) in (23), we get

$$M_Y(t) = \frac{(\eta - \log(\eta))}{\eta} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \frac{t^r}{r!} (-1)^q \binom{p}{q} (p+1) (2\delta^2)^{\frac{-r}{2\zeta}} \left(\frac{\log(\eta)}{\eta} \right)^p \frac{\Gamma\left(1 - \frac{r}{2\eta}\right)}{(q+1)^{1-\frac{r}{2\eta}}}; \quad r < 2\eta \tag{24}$$

□

4.4. Conditional moments and associated measures

Lemma 1: Assume we have a random variable $Y \sim$ *MTI-IPRD*(η, ζ, δ) with PDF given in (6), and let $\phi_r(z) = \int_0^z y^r f(y; \eta, \zeta, \delta)\, dy$ denote the r^{th} incomplete moment, then we have

$$\phi_r(z) = \frac{(\eta - \log(\eta))(2\delta^2)^{\frac{-r}{2\zeta}}}{\eta} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^q \binom{p}{q} (p+1) \left(\frac{\log(\eta)}{\eta} \right)^p \frac{\gamma\left(\left(1 - \frac{r}{2\eta}\right), \frac{q+1}{2\delta^2} z^{-2\zeta}\right)}{(q+1)^{1-\frac{r}{2\eta}}}\ ; r < 2\eta \tag{25}$$

where $\gamma(m, n) = \int_0^n z^{m-1} e^{-z} dz$ represents the lower incomplete gamma function.

Proof: By utilizing the PDF of *MTI-IPRD* provided by (6), we have

$$\phi_r(z) = \int_0^z y^r f(y; \eta, \zeta, \delta)\, dy = \frac{\eta\zeta(\eta - \log(\eta))}{\delta^2} \int_0^z y^r y^{-(2\zeta+1)} \frac{e^{-\frac{y^{-2\zeta}}{2\delta^2}}}{\left(\eta - \log(\eta)\left(1 - e^{-\frac{y^{-2\zeta}}{2\delta^2}}\right)\right)^2}\, dy \tag{26}$$

On simplification, we obtain the r^{th} incomplete moment as

$$\phi_r(z) = \frac{(\eta - \log(\eta))(2\delta^2)^{\frac{-r}{2\zeta}}}{\eta} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^q \binom{p}{q} (p+1) \left(\frac{\log(\eta)}{\eta} \right)^p \frac{\gamma\left(\left(1 - \frac{r}{2\eta}\right), \frac{q+1}{2\delta^2} z^{-2\zeta}\right)}{(p+1)^{1-\frac{r}{2\eta}}}; r < 2\eta \tag{27}$$

Setting $r = 1$ in (27) will provide the first incomplete moment as given by

$$\phi_1(z) = \frac{(\eta - \log(\eta))(2\delta^2)^{\frac{-1}{2\zeta}}}{\eta} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^q \binom{p}{q}(p+1)\left(\frac{\log(\eta)}{\eta}\right)^p \frac{\gamma\left(\left(1-\frac{1}{2\eta}\right), \frac{q+1}{2\delta^2} z^{-2\zeta}\right)}{(q+1)^{1-\frac{1}{2\eta}}} \; ; \eta > \frac{1}{2} \tag{28}$$

□

4.4.1. Lorenz and Bonferroni inequality curves

An important use of the first incomplete moment is found in the curves of Lorenz and Bonferroni inequality. For a specific probability distribution, they are defined as follows:

$$L_w = \frac{1}{E(Y)} \int_0^t y f(y; \eta, \zeta, \delta)\, dy = \frac{\phi_1(t)}{E(Y)}$$

$$L_w = \frac{\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^q \binom{p}{q}(p+1)\left(\frac{\log(\eta)}{\eta}\right)^p \gamma\left(\left(1-\frac{1}{2\zeta}\right), \frac{q+1}{2\delta^2} t^{-2\zeta}\right)}{\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^q \binom{p}{q}(p+1)\left(\frac{\log(\eta)}{\eta}\right)^p \Gamma\left(1-\frac{1}{2\zeta}\right)} \; ; \quad \eta > \frac{1}{2}$$

Similarly,

$$B_w = \frac{1}{wE(Y)} \int_0^t y f(y; \eta, \zeta, \delta)\, dy = \frac{\phi_1(t)}{wE(Y)}$$

$$B_w = \frac{\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^q \binom{p}{q}(p+1)\left(\frac{\log(\eta)}{\eta}\right)^p \gamma\left(\left(1-\frac{1}{2\zeta}\right), \frac{q+1}{2\delta^2} t^{-2\zeta}\right)}{w\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^q \binom{p}{q}(p+1)\left(\frac{\log(\eta)}{\eta}\right)^p \Gamma\left(1-\frac{1}{2\zeta}\right)} \; ; \quad \eta > \frac{1}{2}$$

4.4.2. r^{th} Conditional moment and r^{th} reversed conditional moment of *MTI-IPRD*

The r^{th} conditional moment of the *MTI-IPRD* is calculated by

$$E\left[Y^r \mid y > t\right] = \frac{1}{R(t)} \int_t^{\infty} y^r f(y; \eta, \zeta, \delta)\, dy = \frac{1}{R(t)}\left[E(Y^r) - \phi_r(t)\right]$$

Inserting the value's of (7), (19), and (27), we obtain

$$E\left[Y^r \mid y > t\right] = \frac{\left(\eta - \log(\eta)\left(1 - e^{-\frac{t^{-2\zeta}}{2\delta^2}}\right)\right)}{\eta\left(1 - e^{-\frac{t^{-2\zeta}}{2\delta^2}}\right)} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^q \binom{p}{q}(p+1)\left(\frac{\log(\eta)}{\eta}\right)^p \frac{(2\delta^2)^{\frac{-r}{2\zeta}}}{(q+1)^{1-\frac{r}{2\zeta}}} \times \left[\Gamma\left(1 - \frac{r}{2\zeta}\right) - \gamma\left(\left(1 - \frac{r}{2\zeta}\right), \frac{q+1}{2\delta^2} t^{-2\zeta}\right)\right] \; ; \quad r < 2\eta \tag{29}$$

Likewise, the *MTI-IPRD's* r^{th} reversed conditional moment is determined by

$$E\left[Y^r \mid y \leq t\right] = \frac{1}{F(t)} \int_0^t y^r f(y;\eta,\zeta,\delta)\,dy = \frac{\phi_r(t)}{F(t)}$$

$$E\left[Y^r \mid y \leq t\right] = \frac{(\eta-\log(\eta))\left(\eta-\log(\eta)\left(1-e^{-\frac{t^{-2\zeta}}{2\delta^2}}\right)\right)}{\eta^2\ e^{-\frac{t^{-2\zeta}}{2\delta^2}}}\times$$
$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^q\binom{p}{q}(p+1)\left(\frac{\log(\eta)}{\eta}\right)^p\frac{(2\delta^2)^{\frac{-r}{2\zeta}}}{(q+1)^{1-\frac{r}{2\zeta}}}\gamma\left(\left(1-\frac{r}{2\zeta}\right),\frac{q+1}{2\delta^2}t^{-2\zeta}\right);r<2\eta$$

4.4.3. Mean residual life (textitMRL) and mean waiting time (*MWT*)

The *MRL*, say $\mu(t)$, of *MTI-IPRD* can be acquired in the manner described below:

$$\mu(t) = \frac{1}{R(t)}\left[E(Y) - \int_0^t y f(y;\eta,\zeta,\delta)\,dy\right] - t = \frac{1}{R(t)}\left[E(Y)-\phi_1(t)\right]-t$$

Following the insertion of the value's of (19), (20), and (28), we derive the necessary mean residual life expression as

$$\mu(t) = \frac{\left(\eta-\log(\eta)\left(1-e^{-\frac{t^{-2\zeta}}{2\delta^2}}\right)\right)}{\eta\left(1-e^{-\frac{t^{-2\zeta}}{2\delta^2}}\right)}\times$$
$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^q\binom{p}{q}(p+1)\left(\frac{\log(\eta)}{\eta}\right)^p\frac{(2\delta^2)^{\frac{-1}{2\zeta}}}{(q+1)^{1-\frac{1}{2\zeta}}}\left[\Gamma\left(1-\frac{1}{2\zeta}\right)-\gamma\left(\left(1-\frac{1}{2\zeta}\right),\frac{q+1}{2\delta^2}t^{-2\zeta}\right)\right]-t \tag{30}$$

The *MWT* of Y, say $\overline{\mu}(t)$, is determined by

$$\overline{\mu}(t) = t - \frac{1}{F(t)}\int_0^t y f(y;\eta,\zeta,\delta)\,dy = t - \frac{\phi_1(t)}{F(t)}$$

$$\overline{\mu}(t) = t - \frac{(\eta-\log(\eta))\left(\eta-\log(\eta)\left(1-e^{-\frac{t^{-2\zeta}}{2\delta^2}}\right)\right)}{\eta^2\ e^{-\frac{t^{-2\zeta}}{2\delta^2}}}$$
$$\times\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^q\binom{p}{q}(p+1)\left(\frac{\log(\eta)}{\eta}\right)^p\frac{(2\delta^2)^{\frac{-1}{2\zeta}}}{(q+1)^{1-\frac{1}{2\zeta}}}\gamma\left(\left(1-\frac{1}{2\zeta}\right),\frac{q+1}{2\delta^2}t^{-2\zeta}\right)$$

4.5. Renyi Entropy

Theorem 4: If $X \sim$ *MTI-IPRD*(η, ζ, δ), then the *MTI-IPRD's* Renyi entropy is provided by

$$RE_X(v) = \frac{1}{1-v}\log\left\{\left(\frac{\zeta(\eta-\log(\eta))}{\eta\delta^2}\right)^v \frac{1}{2\eta}\right\} + \frac{1}{1-v}\log\left\{\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^q\binom{2v}{p}\binom{p}{q}\left(\frac{\log(\eta)}{\eta}\right)^p\left(\frac{2\delta^2}{v+q}\right)^{\frac{(2\eta+1)v-1}{2\eta}}\Gamma\left(\frac{(2\eta+1)v-1}{2\eta}\right)\right\}$$

Proof: The Rényi entropy, say $RE_X(v)$, of *MTI-IPRD* can be defined as

$$RE_X(v) = \frac{1}{1-v}\log\left(\int_0^{\infty} f^v(y)\,dy\right); \quad v>0, \quad v\neq 1 \tag{31}$$

Substituting (6) in (31), we get

$$RE_X(v) = \frac{1}{1-v}\log\left\{\left(\frac{\zeta(\eta-\log(\eta))}{\eta\delta^2}\right)^v \int_0^{\infty} y^{-v(2\zeta+1)}\frac{e^{-v\frac{y^{-2\zeta}}{2\delta^2}}}{\left(\eta-\log(\eta)\left(1-e^{-\frac{y^{-2\zeta}}{2\delta^2}}\right)\right)^{2v}}\,dy\right\} \tag{32}$$

Using (16), the Renyi entropy of *MTI-IPRD* is given by

$$RE_X(v) = \frac{1}{1-v}\log\left\{\left(\frac{\zeta(\eta-\log(\eta))}{\eta\delta^2}\right)^v \frac{1}{2\eta}\right\} + \frac{1}{1-v}\log\left\{\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^q\binom{2v}{p}\binom{p}{q}\left(\frac{\log(\eta)}{\eta}\right)^p\left(\frac{2\delta^2}{v+q}\right)^{\frac{(2\eta+1)v-1}{2\eta}}\Gamma\left(\frac{(2\eta+1)v-1}{2\eta}\right)\right\}$$

□

4.6. Stress Strength Reliability

Theorem 5: When $Y_1 \sim$ *MTI-IPRD*$(\eta_1, \zeta, \delta_1)$ and $Y_2 \sim$ *MTI-IPRD*$(\eta_2, \zeta, \delta_2)$, where Y_1 and Y_2 denote the independent random variables for stress and strength, respectively, then the MTI-IPRD's stress strength reliability $P(Y_1 > Y_2)$ is

$$SSR = \frac{\delta_2^2(\eta_1-\log(\eta_1))}{\eta_1}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\binom{p}{q}\binom{q}{s}\left(\frac{\log(\eta_1)}{\eta_1}\right)^p\left(\frac{\log(\eta_2)}{\eta_2}\right)^q \frac{(p+1)}{(s+1)\delta_1^2+(r+1)\delta_2^2}$$

Proof: The stress strength reliability $P(Y_1 > Y_2)$ of the *MTI-IPRD*, say *SSR*, can be acquired as

$$SSR = \int_0^\infty f_1(y)F_2(y)\,dy. \tag{33}$$

Using (5) and (6) in (33), *SSR* can be acquired as

$$\begin{aligned} SSR =& \frac{\zeta(\eta_1 - \log(\eta_1))}{\delta_1^2\eta_1} \int_0^\infty y^{-(2\zeta+1)} e^{-\frac{y^{-2\zeta}}{2\delta_1^2}} e^{-\frac{y^{-2\zeta}}{2\delta_2^2}} \left(1 - \frac{\log(\eta_1)}{\eta_1}\left(1 - e^{-\frac{y^{-2\zeta}}{2\delta_1^2}}\right)\right)^{-2} \\ &\times \left(1 - \frac{\log(\eta_2)}{\eta_2}\left(1 - e^{-\frac{y^{-2\zeta}}{2\delta_2^2}}\right)\right)^{-1} dy. \end{aligned}$$

Using the same method as in (19), we obtain the stress strength reliability for *MTI-IPRD* in its final expression as

$$\begin{aligned} SSR =& \frac{\delta_2^2(\eta_1 - \log(\eta_1))}{\eta_1} \sum_{p=0}^\infty \sum_{q=0}^\infty \sum_{r=0}^\infty \sum_{s=0}^\infty (-1)^{r+s} \binom{p}{q}\binom{q}{s} \left(\frac{\log(\eta_1)}{\eta_1}\right)^p \left(\frac{\log(\eta_2)}{\eta_2}\right)^q \\ & \frac{(p+1)}{(s+1)\delta_1^2 + (r+1)\delta_2^2} \end{aligned}$$

□

5. Estimation

5.1. Maximum likelihood estimation

Assuming a random sample $y_1, y_2, \ldots, y_n$ from *MTI-IPRD* with parameters $\eta, \zeta, \delta > 0$, the likelihood function of *MTI-IPRD* may be expressed as follows:

$$\begin{aligned} l =& n\log(\zeta) + n\log(\eta(\eta - \log(\eta))) - 2n\log(\delta) - (2\zeta+1)\sum_{a=1}^n \log(y_a) - \frac{1}{2\delta^2}\sum_{a=1}^n y_a^{-2\zeta} \\ & - 2\sum_{a=1}^n \log\left(\eta - \log(\eta)\left(1 - e^{-\frac{y_a^{-2\zeta}}{2\delta^2}}\right)\right) \end{aligned} \tag{34}$$

The partial differentiation (34) of η, ζ, and δ leads to their *MLEs* by equating to zero and partially differentiating with regard to the required parameters. As we have

$$\frac{\partial l}{\partial \eta} = \frac{n(2\eta - \log(\eta) - 1)}{\eta(\eta - \log(\eta))} - \frac{2}{\eta}\sum_{a=1}^{n}\left[\frac{e^{-\frac{y_a^{-2\zeta}}{2\delta^2}} + \eta - 1}{\eta - \log(\eta)\left(1 - e^{-\frac{y_a^{-2\zeta}}{2\delta^2}}\right)}\right] = 0 \tag{35}$$

$$\frac{\partial l}{\partial \zeta} = \frac{n}{\zeta} + \frac{\sum_{a=1}^{n} y_a^{-2\zeta}\log(y_a)}{\delta^2} - 2\sum_{a=1}^{n}\log(y_i) - 2\log(\eta)\sum_{a=1}^{n}\left[\frac{y_a^{-2\zeta}\log(y_a)e^{-\frac{y_a^{-2\zeta}}{2\delta^2}}}{\eta - \log(\eta)\left(1 - e^{-\frac{y_a^{-2\zeta}}{2\delta^2}}\right)}\right] = 0 \tag{36}$$

$$\frac{\partial l}{\partial \delta} = \frac{1}{\delta^3}\sum_{a=1}^{n} y_a^{-2\zeta} - \frac{2n}{\delta} + \frac{2\log(\eta)}{\delta^3}\sum_{a=1}^{n}\left[\frac{y_a^{-2\zeta}e^{-\frac{y_a^{-2\zeta}}{2\delta^2}}}{\eta - \log(\eta)\left(1 - e^{-\frac{y_a^{-2\zeta}}{2\delta^2}}\right)}\right] = 0 \tag{37}$$

The solutions to the nonlinear equations 35, 36, and 37 are not in closed form. To estimate the parameters, these equations will be solved using the *R* software.

6. Simulation study

Here, *R* software is utilised to perform a simulation study to examine how *MLE*s behave for various sample sizes. We generate random samples in the following sizes: 25, 75, 150, 300, and 500 using *MTI-IPRD*. The procedure is then carried out a thousand times using *R* software. Various combinations of parameters are chosen in relation to the standard order (η, ζ, δ), such as $(0.55, 0.35, 0.25)$ and $(0.25, 0.85, 0.65)$. For every scenario, mean squared errors (*MSE*s), bias, and average *MLE* values were calculated. Tables 1 and 2 display the outcomes. The estimations are consistent and fairly close to the real parameter values, as these Tables show. Furthermore, in every scenario, the *MSE* falls as sample size increases.

Table 1: *MLE*, *Bias*, and *MSE* for the parameters η, ζ, and δ

Sample size	Parameters			MLE			Bias			MSE		
n	η	ζ	δ	$\hat{\eta}$	$\hat{\zeta}$	$\hat{\delta}$	$\hat{\eta}$	$\hat{\zeta}$	$\hat{\delta}$	$\hat{\eta}$	$\hat{\zeta}$	$\hat{\delta}$
25	0.55	0.35	0.25	1.00576	0.37845	0.25182	0.70285	0.07355	0.05670	1.10630	0.00898	0.00710
75				0.79346	0.36203	0.25491	0.45981	0.04886	0.03320	0.58870	0.00370	0.00233
150				0.69982	0.35489	0.25209	0.31398	0.03863	0.02263	0.30394	0.00221	0.00088
300				0.63625	0.35171	0.25137	0.21134	0.02780	0.01704	0.13579	0.00121	0.00052
500				0.58196	0.35025	0.25055	0.13613	0.02123	0.01277	0.04363	0.00072	0.00028

Table 2: *MLE*, *Bias*, and *MSE* for the parameters η, ζ, and δ

Sample size	Parameters			MLE			Bias			MSE		
n	η	ζ	δ	$\hat{\eta}$	$\hat{\zeta}$	$\hat{\delta}$	$\hat{\eta}$	$\hat{\zeta}$	$\hat{\delta}$	$\hat{\eta}$	$\hat{\zeta}$	$\hat{\delta}$
25	0.25	0.85	0.65	0.62927	0.89377	0.68999	0.46417	0.14567	0.24876	0.69750	0.03404	0.12298
75				0.37205	0.88736	0.66497	0.20624	0.10320	0.17098	0.15497	0.01667	0.05673
150				0.29298	0.86108	0.66084	0.11332	0.07061	0.12175	0.03297	0.00797	0.02595
300				0.27159	0.85840	0.65621	0.07223	0.04908	0.08378	0.00980	0.00388	0.01193
500				0.25904	0.85237	0.65178	0.05570	0.03895	0.06778	0.00541	0.00238	0.00802

7. Applications to real life data

In this section, three actual data sets have been used to assess the newly created distribution's efficacy. As the new distribution is compared to alpha-power exponentiated inverse *RD* (*APEIRD*) Ali *et al.* (2021), inverse *RD* (*IPRD*) Bhat *et al.* (2022), modified *IRD* (*MIRD*) Khan (2014), and transmuted *IRD* (*TIRD*) Ahmad *et al.* (2014), it turns out that the newly created distribution provides a suitable fit. *AIC*, *BIC*, *AICC*, and *KS* are only a few of the metrics used to compare the fitted models. The *p*-value for each model is also given. It is considered better to have a distribution with a high *p*-value and lower *AIC*, *BIC*, *AICC*, and *KS*.

Data set 1 : The first data set consists of thirty observations for the rainfall (in inches) of March in Minneapolis/St. Paul Hinkley (1977). The data are as follows:
0.77 , 1.74 ,0.81, 1.20, 1.95 ,1.20, 0.47, 1.43 ,3.37 ,2.20 ,3.00 ,3.09 ,1.51 ,2.10 ,0.52, 1.62, 1.31, 0.32 ,0.59, 0.81, 2.81, 1.87, 1.18 ,1.35, 4.75 ,2.48, 0.96 ,1.89, 0.90 and 2.05

Data set 2 : The second data set consists of average annual percent change in private health insurance premiums. The data has been previously used by Malik and Ahmad (2024). and is given as follows:
14.4, 14.0, 15.4, 9.4, 11.7, 15.0, 24.9, 20.7, 12.5,14.9, 12.6, 16.7, 13.8, 11.0, 12.9, 10.1, 1.9, 8.5, 16.5, 15.3, 13.3, 9.8, 8.4, 7.9, 3.7,5.1, 4.6, 4.4, 5.4, 6.1, 8.0, 10.0, 11.2, 10.1, 6.4, 6.7, 5.7, 5.8.

Data set 3: The third data set consists of 150 observations and is related to the Reddit advertising data. The data has been previously used by Shen *et al.* (2022). and is given as follows:
11.340, 6.296, 5.136, 7.292, 6.700, 3.648, 7.272, 5.140, 2.980, 6.336, 5.128, 7.564, 6.180, 5.588, 8.760, 11.560, 5.192, 10.032, 6.660, 7.932, 7.536, 4.436, 17.580, 9.836, 5.580, 6.092, 7.912, 6.760, 10.096, 5.948, 11.156, 6.936, 5.424, 8.532, 4.980, 6.536, 13.012, 6.668, 5.540, 11.184, 7.052, 9.336, 10.624, 7.148, 4.892, 6.584, 4.436, 11.696, 7.868, 4.040, 6.748, 5.336, 9.056, 11.496, 11.548, 12.392, 3.636, 7.380, 8.940, 9.068, 4.536, 13.060, 8.940, 7.900, 9.972, 5.740, 5.560, 6.136, 10.904, 7.960, 9.380, 6.484, 3.512, 4.792, 8.980, 5.512, 2.392, 5.336, 3.440, 4.580, 6.704, 7.296, 4.600, 5.592, 9.292, 7.816, 7.068, 8.492,7.556, 8.836, 5.960, 3.696, 9.816, 10.908, 5.636, 8.536, 6.260, 7.912, 12.492, 8.880, 6.188, 11.900, 7.692, 7.496, 10.340, 9.636, 3.784, 5.068, 2.940, 9.992, 7.252, 10.956, 7.512, 8.108, 7.796, 6.928, 6.236, 4.924, 8.056, 3.468, 7.904, 3.780, 5.912, 7.756, 9.900, 5.472, 3.956, 5.044, 12.676, 5.376.

When compared to other competing models, the *MTI-IPRD* achieves the lowest values for *AIC*, *BIC*, and *AICC*, as shown in Tables 6, 7, and 8. As a result, the *MTI-IPRD* performs better than the other competing models listed as well as the base model. Additional evidence for these conclusions may be found in Figures 3, 4 and 5.

8. Conclusion

In this paper, we present a new model for data analysis with real support, which extends the *IPRD* and we call it *MTI-IPRD*. More modelling flexibility for real-world data is the primary objective of generalising a standard distribution. Key statistical properties of the proposed model have been inferred. The new distribution's hazard rate function exhibits more flexibility and complex forms. The *MLE* performs well, as demonstrated by a simulation analysis that shows it to be accurate and consistent. To estimate the parameters, the *MLE* approach is applied. In comparison to other competing distributions, the suggested model might offer a superior fit given its application. In the field of statistics, we believe that the suggested method will find wide usage.

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Conflict of interest

The authors do not have any financial or non-financial conflict of interest to declare for the research work included in this article.

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ANNEXURE

Table 3: MLEs of *MTI-IPRD* and competing models with corresponding SE (given in parentheses) for data Set 1

Model	$\hat{\eta}$	$\hat{\zeta}$	$\hat{\delta}$	$\hat{\lambda}$
MTI-IPRD	0.05897	1.35728	3.04861	
	(0.10150)	(0.22783)	(3.23500)	
APEIRD	35.35513	0.59188	0.61802	
	(67.35165)	(43.43911)	(45.35775)	
IPRD	-	0.77479	0.69834	
		(0.10132)	(0.06737)	
MIRD	0.35977	0.58811		
	(0.37454)	(0.29747)		
TIRD	-	-	0.85914	0.00100
			(0.21365)	(0.40164)

Table 4: MLEs of *MTI-IPRD* and competing models with corresponding SE (given in parentheses) for data Set 2

Model	$\hat{\eta}$	$\hat{\zeta}$	$\hat{\delta}$	$\hat{\lambda}$
MTI-IPRD	0.01321	1.68518	0.27327	-
	(0.01977)	(0.22194)	(0.25462)	
APEIRD	168.36874	4.11696	3.12876	-
	(375.38918)	(1498.48126)	(1138.8015)	
IPRD	-	0.82199	0.14022	-
		(0.08939)	(0.02346)	
MIRD	0.00100	44.25819		
	(2.949399)	(17.271966)		
TIRD	-	-	44.28455	0.00100
			(13.91961)	(0.56670)

Table 5: MLEs of *MTI-IPRD* and competing models with corresponding SE (given in parentheses) for data Set 3

Model	$\hat{\eta}$	$\hat{\zeta}$	$\hat{\delta}$	$\hat{\lambda}$
MTI-IPRD	0.00266	2.09448	0.57909	
	(0.00040)	(0.19230)	(0.23767)	
APEIRD	0.06374	8.10089	7.90391	
	(0.03970)	(107.03792)	(104.43518)	
IPRD	-	1.30989	0.07160	
		(0.08066)	(0.00950)	
MIRD	0.00100	36.17014		
	(2.41996)	(9.48393)		
TIRD	-	-	54.68699	1.00000
			(11.3787766)	(0.69597)

Table 6: Comparison of *MTI-IPRD* and competitive models for data set 1

Model	$-2ll$	AIC	BIC	AICC	K-S	p-value
MTI-IPRD	77.5967	83.5967	87.8003	84.5198	0.0737	0.9968
APEIRD	80.8098	86.8098	91.0134	87.7329	0.1224	0.7593
IPRD	83.8340	87.8340	90.6364	88.2784	0.1523	0.4893
MIRD	87.2598	91.2598	94.0622	91.7043	0.1832	0.2662
TIRD	88.28373	92.28373	95.08613	92.72818	0.2398	0.0634

Table 7: Comparison of *MTI-IPRD* and competitive models for data set 2

Model	$-2ll$	AIC	BIC	AICC	K-S	p-value
MTI-IPRD	228.8583	234.8583	239.7711	235.5642	0.0901	0.9172
APEIRD	236.2966	242.2966	247.2094	243.0025	0.1551	0.3196
IPRD	243.6060	247.6060	250.8812	247.9488	0.1673	0.2377
MIRD	247.3570	251.3570	254.6322	251.6999	0.2111	0.0674
TIRD	247.3682	251.3682	254.6434	251.7110	0.2113	0.0671

Table 8: Comparison of *MTI-IPRD* and competitive models for data set 3

Model	$-2ll$	AIC	BIC	AICC	K-S	p-value
MTI-IPRD	616.3947	622.3947	630.9973	622.5851	0.0577	0.7781
APEIRD	639.0906	645.0906	653.6932	645.2810	0.0875	0.2724
IPRD	640.2807	644.2807	650.0158	644.3752	0.1020	0.1331
MIRD	656.4957	660.4957	666.2308	660.5902	0.1834	0.0003
TIRD	625.8180	629.8180	635.5531	629.9125	0.0748	0.4599

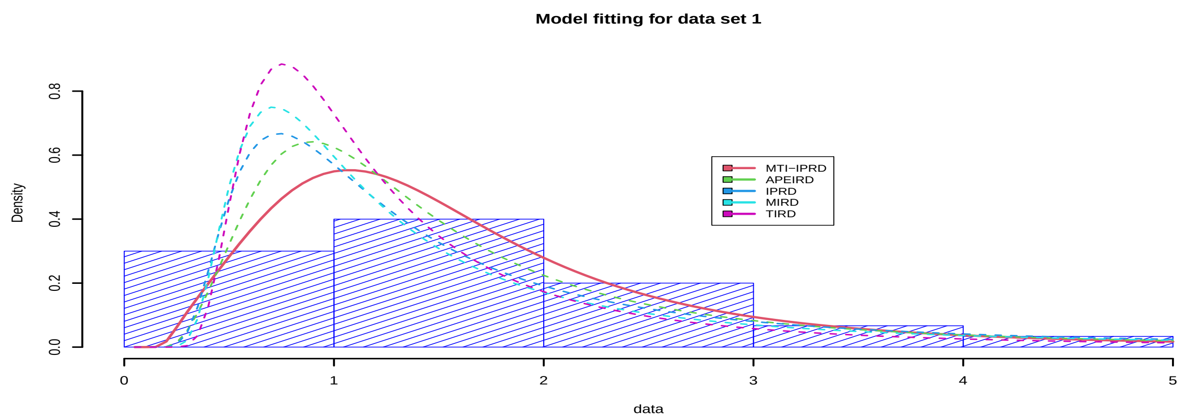

Figure 3: Fitted density plots for data set 1

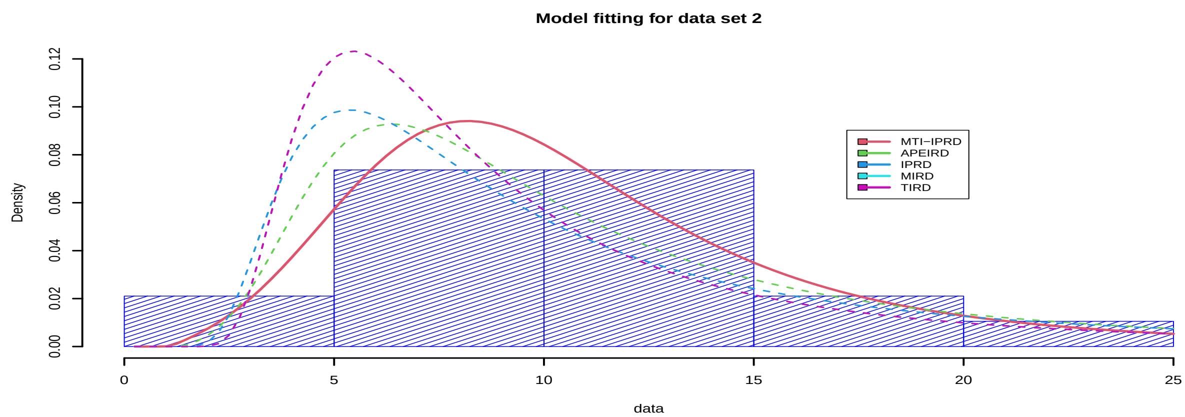

Figure 4: Fitted density plots for data set 2

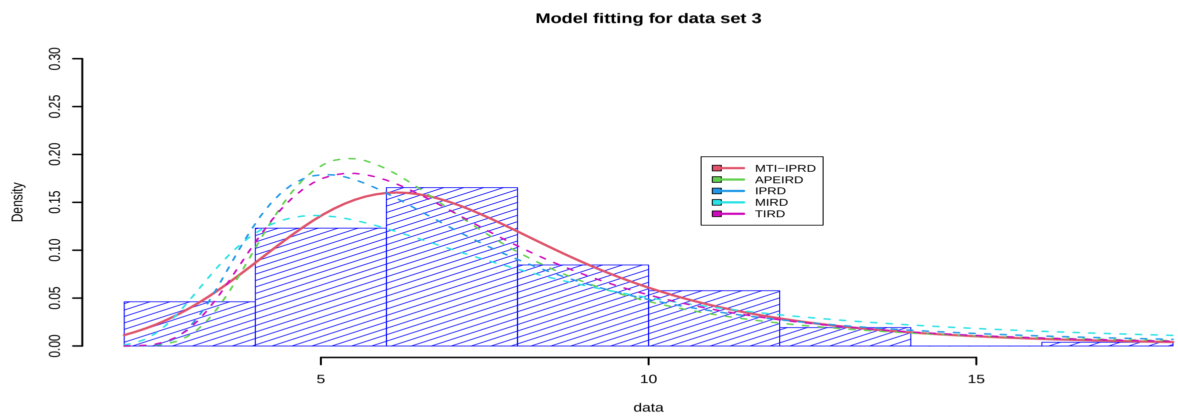

Figure 5: Fitted density plots for data set 3