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Modelling of Stochastic Volatility Using Birnbaum-Saunders Markov Sequence

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Abstract

This paper analyzes a stochastic volatility model generated by first order Birnbaum-Saunders Markov sequence. The model parameters are estimated by the efficient importance sampling method. A simulation experiment is conducted to check the performance of the estimates. The model is used to analyze two sets of data and found that it captures the stylised factors of the financial return series.

Key words: Autoregressive process; Birnbaum-Saunders distribution; Efficient importance sampling; Volatility model.

1 Introduction

Modelling of stochastic volatility is an important aspect of analyzing financial time series. One way of quantifying the heteroscedastcity in the series is the identification of a suitable model for generating the conditional variances. A class of models employed to describe the evolution of conditional variance is the celebrated class of autoregressive conditional heteroscedastic (ARCH) models introduced by Engle (1982) in which the volatility evolves as a function of the squares of the past observations. This model was later generalized to GARCH model by Bollerslev (1986). One may refer Shephard (1996) and Tsay (2005) for other generalizations and more details. These models are simple to handle as one can easily apply the likelihood based inference methods for analysis. However, the model has many shortcomings. For example, in practice, the past data may have positive or negative effects on the future volatilities, but the ARCH/GARCH models always result in positive effect. The stationarity condition of the process makes the parameter space very restrictive. Another class of models used to generate conditional variances is the stochastic volatility (SV) model introduced by Taylor (1986), in which the volatilities are assumed to be generated by an unobservable latent model. Even though this is a better model, the presence of latent variables, makes the inference problems more difficult. In the SV model introduced by Taylor (1986) and developed by several other authors assume that the the conditional distribution of a financial return follows a normal distribution and its log-variance follows a stationary first

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order Gaussian autoregressive (AR(1)) model. One may refer Shephard (1996) for more details on variety of generalizations of ARCH models and the SV models along with the related inference problems. Tsay (2005) gives some more updated developments of these topics.

One may describe the standard form of an SV model as follows. Let $\{r_t\}$ be a sequence of returns on certain asset and $\{\varepsilon_t\}$ be a sequence of independent and identically distributed (iid) random variables (rvs) which are distributed symmetrically around zero. An SV model may be defined by

$$r_t = \varepsilon_t e^{h_t/2}, \quad \text{with} \quad h_t = \theta_0 + \theta_1 h_{t-1} + e_t, \ \ |\theta_1| < 1, \ t = 1, 2, ... \tag{1.1}$$

A large body of the literature in this area deals with the models and their inference procedure by assuming that $\{\varepsilon_t\}$ is a sequence of iid standard normal variates and $\{h_t\}$ as a Gaussian AR(1) sequence. Such models are referred to as normal - lognormal SV models. The methods of estimation for these models are based on Bayesian set up and are mostly computation intensive. For example, Kim et al. (2015) and Chib et al. (2002) discuss the Markov Chain Monte Carlo method of estimation for stochastic volatilty models. Jacquier et al. (1994, 2004) study the Bayesian analysis of such models. Ghosh et al. (2015) proposed Kalman filter based forecasting methods when volatilities are generated by linear threshold models.

The empirical studies on financial assets show that the return distribtions are more peaked around the mean and have fatter tails than the one implied by the normal distribution. These empirical observations have led to models in which the volatility of returns follow non-Gaussian distributions. To account for heavy tails observed in returns series, some researchers have developed SV models by assuming non-normal conditional distributions for the return series. For recent work on such models with conditional Student-t distribution one may refer Nakajima and Omori (2012), Delatola and Griffin (2013) and the references therin. The other studies on non-normal SV models include the Normal Inverse Gaussian distribution for ε by Barndorff-Nielson (1997), Anderson (2001), generalized error distribution by Liesenfeld and Jung (2000), generalized student-t distribution by Wang et al. (2013) and so on. An alternative class of SV models was generated by assuming normal distribution for ε_t whereas the non-negative volatility sequences $\{h_t\}$ were generated by some non-Gaussian AR(1) models . For example, Abraham et al. (2006) proposed an SV model generated by gamma AR(1) model, Balakrishna and Shiji (2014) developed an SV model generated by a first order extreme value autoregressive model.

This paper proposes a stationary Markov sequence of Birnbaum- Saunders(BS) rvs for modelling volatilities under the assumption that ε follows standard normal distribution. The BS distribution introduced by Birnbaum and Saunders (1969) has received great attention in recent years in the context of lifetime modelling. The attractive properties of the BS distribution makes it a useful alternative to the normal model under positive skewness and non-negative support. Recently, much work has been carried out on BS distribution while modelling non-negative lifetime data; see the recent text book by Leiva (2016) and the references therein for more details. However, applications of this model for modelling volatility in financial time series context have not received much attention except Bhatti (2010). With an idea to introduce SV models induced by non-Gaussian volatility sequences, Rahul et al. (2018) developed a stationary Markov sequence with BS marginal distribution. In the present paper we study different aspects of SV model generated by BS Markov sequence.

This paper is organized as follows. In Section 2, we describe the stochastic volatility model generated by a stationary Markov sequence of BS random variables and study its second order properties. In Section 3, we estimate the unknown parameters of the model by Method of Moments and Efficient Importance Sampling (EIS). Section 4 presents some numerical results on the estimators via simulation. In Section 5, we apply our model to two daily returns data. Finally, some conclusions are given in section 6.

2 The Model

A rv X is said to follow a Birnbaum-Saunders distribution and is denoted by $BS(\alpha,\beta)$ if its probability density function is given by

$$f(x;\alpha,\beta)=\frac{1}{2\alpha\beta\sqrt{2\pi}}\Big[\Big(\frac{\beta}{x}\Big)^{1/2}+\Big(\frac{\beta}{x}\Big)^{3/2}\Big]\ \exp\Big(-\frac{1}{2\alpha^2}\Big[\frac{x}{\beta}+\frac{\beta}{x}-2\Big]\Big), \tag{2.1}$$

where $x>0$, $\alpha,\beta>0$. The corresponding cumulative distribution function may be expressed as

$$F(x;\alpha,\beta)=\Phi\Big(\frac{1}{\alpha}\Big[\Big(\frac{x}{\beta}\Big)^{1/2}-\Big(\frac{\beta}{x}\Big)^{1/2}\Big]\Big), \tag{2.2}$$

where $\Phi(.)$ is the cumulative distribution function of the standard normal variable. A key point useful in developing the theory of BS distribution is its relation with normal distribution. That is, if Z is a standard normal rv then the rv X defined by

$$X=\beta\Big[\frac{1}{2}\alpha Z+\sqrt{\big(\frac{1}{2}\alpha Z\big)^2+1}\Big]^2 \tag{2.3}$$

follows a $BS(\alpha,\beta)$ distribution. For detailed study on the properties of this distribution, one may refer Johnson et al. (1995). Rahul et al. (2018) proposed a method of constructing stationary autoregressive moving average (ARMA) models with BS marginal distribution and studied their properties. That article also provides an up to date survey of the developments on the BS distribution. In the present paper, we use stationary Markov sequence of BS rvs defined by Rahul et al. (2018) to generate a stochastic volatility model and discuss its properties and applications. The basic idea in constructing a BS AR(1) model is the link between normal and BS ditribution specified by the relation (2.3).

The SV model we are going to study in this paper is described below. Let r_t be the return of an asset at time t. Define the SV model

$$r_t = \sqrt{h_t}\varepsilon_t, \tag{2.4}$$

$$h_t=\beta\left[\frac{1}{2}\alpha X_t+\sqrt{\left(\frac{1}{2}\alpha X_t\right)^2+1}\right]^2, \tag{2.5}$$

$$X_t=\rho X_{t-1}+\eta_t\ ;\ |\rho|<1,\ t=1,2,..., \tag{2.6}$$

where $\{X_t\}$ is a stationary Gaussian AR(1) sequence such that each X_t follows standard normal distribution and $\{\eta_t\}$ is a sequence of iid normal rvs with mean 0 and variance $1-\rho^2$. In this case $\{\varepsilon_t\}$ is a sequence of independent and identically distributed standard normal random variables. We assume that the sequence $\{\varepsilon_t\}$ is independent of h_t and η_t for every t. This is a SV model for the return series $\{r_t\}$ whose volatilities are generated by a stationary Markov sequence $\{h_t\}$ of $BS(\alpha,\beta)$ random variables with marginal probability density function:

$$f\left(h_t;\alpha,\beta\right)=\frac{1}{2\alpha\beta\sqrt{2\pi}}\left[\left(\frac{\beta}{h_t}\right)^{1/2}+\left(\frac{\beta}{h_t}\right)^{3/2}\right]\,\exp\left(-\frac{1}{2\alpha^2}\left[\frac{h_t}{\beta}+\frac{\beta}{h_t}-2\right]\right), \qquad (2.7)$$

where $h_t>0\,,\;\alpha,\beta>0$.

The r^{th} raw moment about origin of $\{h_t\}$ is given by (cf Rieck (1999)):

$$E\left(h_t^r\right)=\frac{\beta^r\left[K_{r+\frac{1}{2}}(\alpha^{-2})+K_{r-\frac{1}{2}}(\alpha^{-2})\right]}{2K_{\frac{1}{2}}(\alpha^{-2})},$$

where $K_\nu(z)$ is the modified Bessel function of the third kind with ν representing its order and z the coefficient of argument. That is,

$$K_\nu(z)=\frac{1}{2}\int_{-\infty}^{\infty}\exp\left[-z\cosh(t)-\nu t\right]\,dt.$$

Since the sequence $\{\varepsilon_t\}$ follows standard normal distribution, the odd moments of $\{r_t\}$ are zero and its even moments are given by

$$E\left(r_t^{2r}\right)=\frac{\beta^r\left[K_{r+\frac{1}{2}}(\alpha^{-2})+K_{r-\frac{1}{2}}(\alpha^{-2})\right]}{2K_{\frac{1}{2}}(\alpha^{-2})}\prod_{j=1}^{r}(2j-1),\;\; r=1,2,....$$

Then $Var\left(r_t\right)=\beta\left(1+\frac{\alpha^2}{2}\right)$ and the kurtosis of r_t becomes

$$K=3+\frac{3\left(\alpha^2+\frac{5}{4}\alpha^4\right)}{\left(1+\alpha^2+\frac{1}{4}\alpha^4\right)}>3.$$

The structure of the model (2.4) implies that the ACF of $\{r_t\}$ is zero and that of $\{r_t^2\}$ is significant. The variance of the squared return series is obtained as

$$Var\left(r_t^2\right)=\beta^2\left(2+5\alpha^2+\frac{17}{4}\alpha^4\right).$$

The lag k autocovariance function of $\{r_t^2\}$ is given by

$$Cov(r_t^2,\,r_{t-k}^2)=\beta^2\left(\frac{1}{2}\alpha^4\rho^{2k}+\alpha^2 I_1\right).$$

Hence the lag k autocorrelation of the squared sequence $\{r_t^2\}$ is

$$\rho_k\left(r_t^2\right) = \frac{\left(\frac{1}{2}\alpha^4\rho^{2k} + \alpha^2 I_1\right)}{\left(2 + 5\alpha^2 + \frac{17}{4}\alpha^4\right)},$$

where I_1 is the expression given by

$$\begin{aligned} I_1 &= E\left[X_t X_{t+k}\left(\sqrt{\left(\frac{\alpha X_t}{2}\right)^2 + 1}\right)\left(\sqrt{\left(\frac{\alpha X_{t+k}}{2}\right)^2 + 1}\right)\right] \\ &= E\left[\left\{X_t + \frac{1}{2^3}\ \alpha^2 X_t^3 + \sum_{i=2}^{\infty}(-1)^{i-1}\ \frac{1\cdot 3\ldots(2i-3)}{2^{3i}\ i!}\alpha^{2i}X_t^{2i+1}\right\}\right. \\ &\quad \left.\times\left\{X_{t+k} + \frac{1}{2^3}\ \alpha^2 X_{t+k}^3 + \sum_{j=2}^{\infty}(-1)^{j-1}\ \frac{1\cdot 3\ldots(2j-3)}{2^{3j}\ j!}\alpha^{2j}X_t^{2j+1}\right\}\right]. \end{aligned}$$

For non-negative integers m and n, we have [see Kotz et al. (2000), pp 261]

$$E\left(X_t^{2m+1}X_{t+k}^{2n+1}\right) = \frac{(2m+1)!(2n+1)!}{2^{m+n}}\sum_{i=0}^{\min(m,n)}\frac{(2\rho_X)^{2i+1}}{(m-i)!\,(n-i)!\,(2i+1)!} = a_{m,n}\ (say).$$

Therefore,

$$\begin{aligned} I_1 &= a_{0,0} + \frac{1}{2^2}\alpha^2 a_{0,1} + \frac{1}{2^6}\alpha^4 a_{1,1} + \sum_{i=2}^{\infty}(-1)^{i-1}\frac{1\cdot 3\ldots(2i-3)}{2^{3i-1}\ i!}\ \alpha^{2i}a_{0,i} \qquad (2.8) \\ &\quad + \sum_{i=2}^{\infty}(-1)^{i-1}\frac{1\cdot 3\ldots(2i-3)}{2^{3i+2}\ i!}\ \alpha^{2i+2}a_{1,i} \\ &\quad + \sum_{i=2}^{\infty}\sum_{j=2}^{\infty}(-1)^{i+j}\ \frac{1\cdot 3\ldots(2i-3)}{2^{3i}\ i!}\ \times\ \frac{1\cdot 3\ldots(2j-3)}{2^{3j}\ j!}\alpha^{2i+2j}a_{i,j} \end{aligned}$$

The ACF is an exponentially decreasing function of the lag, k for different values of the parameters, as can be seen in the right panel of Figure 1. By choosing different values for α, one can get a distribution with larger kurtosis as shown in the left panel of Figure 1.

3 Estimation of Parameters

The statistical inference based on likelihood function is not of much help to analyze the SV models as the likelihood function is an $n - fold$ integral, where n is the sample size. Several algorithms are available for computing estimates based on Bayesian and McMC methods. Simulation-based maximum likelihood method when the innovations have some specified distribution has been used by Danielsson and Richard (1993), Danielsson (1994), Kim et al. (1998), Sandmann and Koopman (1998),Liesenfeld and Richard (2003, 2006), Richard and Zhang (2007). The other methods include the Generalized Method of Moments (GMM) (Melino and Turnbull

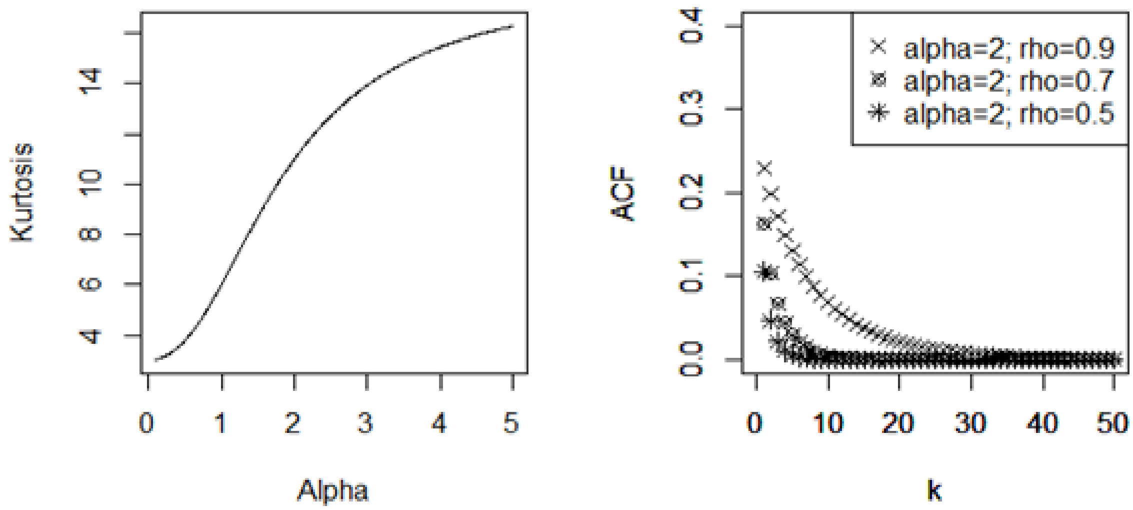

Figure 1: The plot of kurtosis of return and the ACF of squared return

(1990)), Quasi Maximum Likelihood (QML) approach (Harvey et al. (1994) and Ruiz (1994)), Efficient Method of Moments (EMM) (Gallant et al. (1997)). For an overview of such estimation methods of SV models, see Shephard (1996), Broto and Ruiz (2004) and Tsay (2005). We use the Method of Moment estimates as initial values to implement the Efficient Importance Sampling (EIS) method proposed by Richard and Zhang (2007).

3.1 Parameter Estimation by the Method of Moments

In this sub-section we summarize the method of moment estimates of the paramters, whose consistency and asymptotic normality properties may be established using the results of Hansen (1982). Let $(r_1, r_2, \ldots, r_T)$ be a realization of length T from the SV model (2.4), $\Theta = (\alpha, \beta, \rho)$ be the parameter vector to be estimated. We use the moments

$$E(r_t^2) = \beta\left(1 + \frac{\alpha^2}{2}\right), \qquad E(r_t^4) = 3\beta^2\left(1 + 2\alpha^2 + \frac{3}{2}\alpha^4\right),$$
$$E(r_t^2 r_{t-1}^2) = \beta^2\left(1 + \alpha^2 + \frac{\alpha^4}{4}(1 + 2\rho^2) + \alpha^2 I_1\right)$$

to estimate the parameters. Comparing with Hansen (1982) we use the moment function:

$$f(r_t, r_{t-1}, \Theta) = \begin{pmatrix} r_t^2 - \beta\left(1 + \frac{\alpha^2}{2}\right) \\ r_t^4 - 3\beta^2\left(1 + 2\alpha^2 + \frac{3}{2}\alpha^4\right) \\ r_t^2 r_{t-1}^2 - \beta^2\left(1 + \alpha^2 + \frac{\alpha^4}{4}(1 + 2\rho^2) + \alpha^2 I_1\right) \end{pmatrix}, \tag{3.1}$$

where I_1 is given by (2.8), and then the moment estimators are obtained by solving the equation:

$$\frac{1}{T}\sum_{t=1}^{T} f(r_t,\, r_{t-1}, \Theta) = 0.$$

The resulting moment equations for α, β and ρ may be expressed as

$$\frac{\bar{R}_2^2}{\bar{R}_4} = \frac{\left(1+\frac{\hat{\alpha}^2}{2}\right)^2}{3\left(1+2\hat{\alpha}^2+\frac{3}{2}\hat{\alpha}^4\right)}; \qquad \hat{\beta} = \frac{\bar{R}_2}{\left(1+\frac{\hat{\alpha}^2}{2}\right)}$$

and

$$\bar{R}_{22} = \hat{\beta}\left(1+\hat{\alpha}^2+\frac{\hat{\alpha}^4}{4}(1+2\hat{\rho}^2)+\hat{\alpha}^2\hat{I}_1\right),$$

where

$$\bar{R}_2 = (1/T)\sum_{t=1}^{T} r_t^2, \qquad \bar{R}_{22} = (1/T)\sum_{t=1}^{T} r_t^2 r_{t-1}^2, \qquad \bar{R}_4 = (1/T)\sum_{t=1}^{T} r_t^4.$$

These equations have to be solved by numerical methods and are illustrated using simulated samples in Section 4. In our further analysis, we will use these estimates as initial values for iterative procedure in the next sub-section.

3.2 Estimation by Efficient Importance Sampling (EIS)

As noted earlier, in order to utilize the likelihood function for estimation, one has to eliminate the latent variables by integration. Let $R = (r_1, r_2, ..., r_T)$ be a vector of observations from the model and $H = (h_1, h_2, ..., h_T)$ be the vector of associated latent variables. If we denote the joint density function of (R, H) by $f(R, H; \Theta)$, then the likelihood function of the parameter vector $\Theta = (\alpha, \beta, \rho)$ based on the observations is given by

$$L(\Theta; R) = \int f(R, H; \Theta)\, dH = \iint ... \int f(r_1, r_2, ..., r_T, h_1, h_2, ..., h_T)\, dh_1 dh_2 ... dh_T. \tag{3.2}$$

Thus the maximum likelihood method of estimation involves the computation of the likelihood function by evaluating this multiple integral and then maximizing the resulting function with respect to the parameters. One of the common methods used in such situations is to obtain the Monte Carlo (MC) estimates of the likelihood function based on the observations simulated from the auxiliary variables. However, such procedures lead to inefficient estimators. Richard and Zhang (2007) proposed EIS to overcome this problem. To obtain an MC estimate of $L(\Theta; R)$, we need to decompose the above joint density function $f(R, H; \Theta)$ sequentially as

$$f(r_t, h_t; \Theta) = \prod_{t=1}^{T} g(r_t|h_t)p(h_t|h_{t-1}). \tag{3.3}$$

In our case, $g(.)$ and $p(.)$ are respectively given by

$$g(r_t|h_t) = \frac{1}{\sqrt{2\pi h_t}}\; exp\left\{-\frac{r_t^2}{2h_t}\right\} \tag{3.4}$$

and

$$p(h_t|h_{t-1}) = \frac{1}{2\alpha\beta\sqrt{2\pi}\sqrt{1-\rho^2}}\left(\left(\frac{\beta}{h_t}\right)^{1/2}+\left(\frac{\beta}{h_t}\right)^{3/2}\right) \times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{1}{\alpha}\left(\sqrt{\frac{h_t}{\beta}}-\sqrt{\frac{\beta}{h_t}}\right)-\frac{\rho}{\alpha}\left(\sqrt{\frac{h_{t-1}}{\beta}}-\sqrt{\frac{\beta}{h_{t-1}}}\right)\right]^2\right\}. \tag{3.5}$$

A natural MC estimate of the likelihood function is given by

$$\hat{L}(\Theta;R) = \frac{1}{S}\sum_{j=1}^{S}\left[\prod_{t=1}^{T} g(r_t|\tilde{h}_t^{(j)})\right], \tag{3.6}$$

where $\tilde{h}_t^{(j)}$ denotes a draw from the density $p(h_t|h_{t-1}^{(j)})$, which we shall refer to as natural samplers. This estimator is highly inefficient since MC variance increases with the sample size.

The EIS procedure constructs a sequence of samplers that exploits the sample information on the $\{h_t\}$as conveyed by $\{r_t\}$. Let $\{m(h_t|h_{t-1}, a_t)\}_{t=1}^T$ denote such a sequence of auxiliary samplers, typically a straightforward parametric extension of the natural samplers $p(h_t|h_{t-1})$, indexed by the auxiliary parameters $\{a_t = (a_{1,t}, a_{2,t})\}_{t=1}^T$.

For any given values of the auxiliary parameters, the likelihood function is rewritten as

$$L(\Theta;R) = \int\left[\prod_{t=1}^{T}\frac{f(r_t,h_t|r_{t-1},h_{t-1},\Theta)}{m(h_t|h_{t-1},a_t)}\prod_{t=1}^{T}m(h_t|h_{t-1},a_t)\right]\,dh, \tag{3.7}$$

and the corresponding importance sampling MC estimate of the likelihood is given by

$$\tilde{L}(\Theta;R) = \frac{1}{S}\sum_{j=1}^{S}\left\{\prod_{t=1}^{T}\left[\frac{f(r_t,\tilde{h}_t^{(j)}(a_t)|r_{t-1},\tilde{h}_{t-1}^{(j)}(a_{t-1}),\Theta)}{m(\tilde{h}_t^{(j)}(a_t)|\tilde{h}_{t-1}^{(j)}(a_{t-1}),a_t)}\right]\right\}, \tag{3.8}$$

where $\left\{\tilde{h}_t^{(j)}(a_t)\right\}_{t=1}^T$ are trajectories drawn from the auxiliary samplers.

The EIS aims at selecting values of the auxiliary parameters $\{a_t\}_{t=1}^T$ which provide a good match between the product in the numerator and that in the denominator of (3.8) in order to minimize the MC sampling variance of $\tilde{L}$. This minimization problem can be decomposed in to a sequence of sub-problems for each element t of the sequence of observations, provided that the elements depending on the lagged values h_{t-1} are transferred back to the $(t-1)^{th}$ minimization sub-problem. More precisely, if we decompose $m(.|.)$ in the product of a function of h_t and h_{t-1} and one of h_{t-1} only, such that

$$m\left(h_t|h_{t-1},a_t\right) = \frac{k\left(h_t,a_t\right)}{\chi\left(h_{t-1},a_t\right)},$$

where $\chi\left(h_{t-1},a_t\right) = \int k\left(h_t,a_t\right)dh_t$.

Now the EIS requires solving a back-recursive sequence of low-dimensional least-squares problems of the form:

$$\hat{a}_t(\Theta) = \arg\ \min_{a_t} \sum_{j=1}^{S} \left\{ \ln\left[f(r_t, \tilde{h}_t^{(j)} | r_{t-1}, \tilde{h}_{t-1}^{(j)}, \Theta) \chi(\tilde{h}_t^{(j)}, \hat{a}_{t+1}) \right] - c_t - \ln(k(\tilde{h}_t^{(j)}, a_t)) \right\}^2, \quad (3.9)$$

where c_t are unknown constants to be estimated jointly with a_t. If the density kernel $k(h_t, a_t)$ is chosen within the exponential family of distributions, the EIS least-squares problems become linear in a_t. Following Liesenfeld and Richard (2003), we can start by the following specification of the function $k(h_t, a_t)$:

$$k(h_t, a_t) = p(h_t | h_{t-1}) \zeta(h_t, a_t),$$

where $\zeta(h_t, a_t) = exp(a_{1,t}h_t + a_{2,t}h_t^2)$ and $a_t = (a_{1,t}, a_{2,t})$. Finally, the EIS estimate of the likelihood function for a given value of Θ is obtained by substituting $\hat{a}_t$ for a_t using the following algorithm (cf. Richard and Zhang, 2007).

Step 1: Use the natural sampler $m(h_t | h_{t-1}, a_t)$ to draw S trajectories of the latent variable $\{\tilde{h}_t^{(j)}\}_{t=1}^T$.

Step 2: The draws obtained in Step 1 are used to solve the least squares problems described in (3.9) for each t (in the order from T to 1), which takes the form of the auxiliary linear regression:

$$-\frac{1}{2}\log h_t - \frac{1}{2}\log(2\pi) - \frac{r_t^2}{2h_t} + \ln \chi(\tilde{h}_t^{(j)}, \hat{a}_{t+1})$$
$$= a_{0,t} + a_{1,t}\tilde{h}_t^{(j)} + a_{2,t}(\tilde{h}_t^{(j)})^2 + v_t^{(j)}, \quad j = 1, 2, ..., S,$$

where $v_t^{(j)}$ is the error term.

Step 3: Use the estimated auxiliary parameters $\hat{a}_t$ to obtain S trajectories $\left\{ \tilde{h}_t^{(j)}(\hat{a}_t) \right\}_{t=1}^T$ from the auxiliary sampler $m(h_t | h_{t-1}, \hat{a}_t)$.

Step 4: Return to Step 2, this time using the draws obtained with the auxiliary sampler. Steps 2, 3 and 4 are usually iterated a small number of times (from 3 to 5), until a reasonable convergence of the parameters $\hat{a}_t$ is obtained.

Once the auxiliary trajectories have attained a reasonable degree of convergence, the simulated samples can be plugged in formula (3.8) to obtain an EIS estimate of the likelihood function. This procedure is embedded in a numerical maximization algorithm that converges to a maximum of the likelihood function. The same random numbers were also employed for each of the likelihood evaluations required by the maximization algorithm. The number of draws used (S in Eq. (3.8)) for all computations in this section is equal to 100. EIS-ML estimates are finally obtained by maximizing $\tilde{L}(\theta; R, a)$ with respect to θ.

4 Simulation Study

This section illustrates our estimation procedure using the simulated data from BS-SV model. We conducted several repeated simulation experiments with different ρ-values, by fixing α= 2 and

β= 1. The trajectories of 500, 1000 and 3000 observations from a SV data generating process were simulated 100 times and the EIS-ML estimates were obtained using moment estimates as initial values.

Table 1: The average estimates and the corresponding mean square error for the Method of Moment Estimates, when α=2, β=1 and for different ρ's.

n	ρ	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\rho}$
500	0.90	2.0989(0.6558)	1.3152(0.6268)	0.9450(0.6622)
	0.70	1.9268(0.6563)	1.1559(0.4945)	0.7275(0.5952)
	0.50	1.7638(0.6915)	1.2145(0.4897)	0.4847(0.4621)
	0.25	1.9093(0.6732)	1.1194(0.4325)	0.2978(0.4433)
	0.00	1.9115(0.6219)	1.2234(0.4203)	0.1234(0.3314)
	-0.25	1.9186(0.6654)	1.1245(0.4896)	-0.3051(0.4476)
	-0.50	1.7834(0.7651)	1.1521(0.5123)	-0.5689(0.6123)
	-0.70	1.9167(0.6754)	1.1656(0.4987)	-0.7934(0.5123)
	-0.90	2.0862(0.6578)	1.2565(0.5051)	-0.9562(0.6340)
1000	0.90	1.9367(0.6334)	1.1808(0.4872)	0.9312(0.6622)
	0.70	1.9398(0.5892)	1.1672(0.4092)	0.7287(0.5952)
	0.50	1.8873(0.4907)	1.1783(0.4367)	0.4769(0.4621)
	0.25	1.9462(0.6234)	1.1098(0.4469)	0.2765(0.4433)
	0.00	1.9346(0.6092)	1.1456(0.4064)	0.0997(0.3314)
	-0.25	1.9419(0.6571)	1.1273(0.4764)	-0.2876(0.4476)
	-0.50	1.8995(0.7075)	1.1183(0.5082)	-0.5561(0.6123)
	-0.70	1.9319(0.5767)	1.1519(0.4337)	-0.7409(0.5123)
	-0.90	2.0510(0.5976)	1.1190(0.4278)	-0.9420(0.6340)
3000	0.90	1.9545(0.6066)	1.0967(0.4562)	0.9267(0.5901)
	0.70	1.9581(0.5906)	1.0877(0.4278)	0.7261(0.5783)
	0.50	1.9256(0.4893)	1.1980(0.4084)	0.4729(0.4563)
	0.25	1.9686(0.6024)	1.0835(0.5011)	0.2710(0.4419)
	0.00	1.9382(0.6063)	1.1052(0.4178)	0.0884(0.3882)
	-0.25	1.9576(0.6327)	1.0728(0.4267)	-0.2765(0.4370)
	-0.50	1.9124(0.6529)	1.0639(0.4271)	-0.5394(0.5922)
	-0.70	1.9412(0.5672)	1.1092(0.4093)	-0.7337(0.5092)
	-0.90	1.9610(0.5571)	1.1076(0.4124)	-0.9374(0.5955)

The Method of Moments estimates are presented in Table 1 and EIS-ML estimates are in Table 2 with corresponding mean square error in parentheses. From the simulation results, we observe that the Method of Moment estimates are slightly biased. When the sample size is large, the estimates perform reasonably well and there is a marginal reduction in bias and root mean square errors. But EIS-ML method provides estimates which are closer to the true parameter values and mean square error of estimates are remarkably small.

Table 2: The average estimates and the corresponding mean square error for the EIS-ML Estimates, when α=2, β=1 and for different ρ's.

n	ρ	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\rho}$
500	0.90	1.9429(0.0776)	0.9493(0.1125)	0.8580(0.0576)
	0.70	1.9501(0.0774)	0.9462(0.1277)	0.6590(0.0522)
	0.50	1.9545(0.0873)	0.9420(0.1126)	0.4820(0.0614)
	0.25	1.9513(0.0793)	0.9521(0.1161)	0.2160(0.0507)
	0.00	1.9468(0.0808)	0.9520(0.1195)	-0.0440(0.0709)
	-0.25	1.9420(0.0942)	0.9524(0.1026)	-0.2820(0.0650)
	-0.50	1.9581(0.0915)	0.9480(0.0950)	-0.5640(0.0578)
	-0.70	1.9400(0.0874)	0.9540(0.1043)	-0.7240(0.0656)
	-0.90	1.9525(0.0727)	0.9559(0.0985)	-0.9355(0.0558)
1000	0.90	1.9626(0.0601)	0.9722(0.0815)	0.8850(0.0471)
	0.70	1.9759(0.0662)	0.9729(0.0866)	0.6812(0.0403)
	0.50	1.9812(0.0712)	0.9810(0.0796)	0.4901(0.0521)
	0.25	1.9788(0.0608)	0.9789(0.0811)	0.2396(0.0488)
	0.00	1.9821(0.0699)	0.9809(0.0785)	-0.0221(0.0532)
	-0.25	1.9760(0.0671)	0.9755(0.0711)	-0.2718(0.0519)
	-0.50	1.9789(0.0615)	0.9801(0.0762)	-0.5355(0.0477)
	-0.70	1.9810(0.0711)	0.9882(0.0815)	-0.7188(0.0452)
	-0.90	1.9729(0.0655)	0.9759(0.0795)	-0.9128(0.0410)
3000	0.90	1.9829(0.0521)	0.9923(0.0802)	0.8906(0.0424)
	0.70	1.9876(0.0556)	0.9847(0.0716)	0.6890(0.0475)
	0.50	1.9890(0.0579)	0.9907(0.0689)	0.4955(0.0545)
	0.25	1.9835(0.0600)	0.9844(0.0709)	0.2431(0.0388)
	0.00	1.9901(0.0628)	0.9879(0.0691)	-0.0560(0.0532)
	-0.25	1.9859(0.0571)	0.9937(0.0669)	-0.2517(0.0519)
	-0.50	1.9914(0.0550)	0.9965(0.0681)	-0.5188(0.0427)
	-0.70	1.9945(0.0609)	0.9899(0.0679)	-0.7054(0.0400)
	-0.90	1.9899(0.0533)	0.9902(0.0602)	-0.9059(0.0379)

5 Data Analysis

We apply the BS-SV model to analyse the daily returns for (1) the rate of exchange on the Rupee/Dollar from January 1, 1999 to March 28, 2018 obtained from Database on Indian Economy, Reserve Bank of India and (2) the opening index of Standard and Poors 500 (S&P 500) from January 02, 2009 to March 29, 2018 obtained from Bombay Stock Exchange. The time series plots of these data are given in Figure 2.

Denoting the daily price index by p_t, the returns are transformed into continuously compounded rates centred around their sample mean:

$$r_t = 100\left[\ln\left(\frac{p_t}{p_{t-1}}\right) - \left(\frac{1}{T}\right)\sum_{t=1}^{T}\ln\left(\frac{p_t}{p_{t-1}}\right)\right],\ t = 1, 2, ..., T.$$

The left panels show the plots of actual data series whereas the continuously compounded return series are on the right panels.

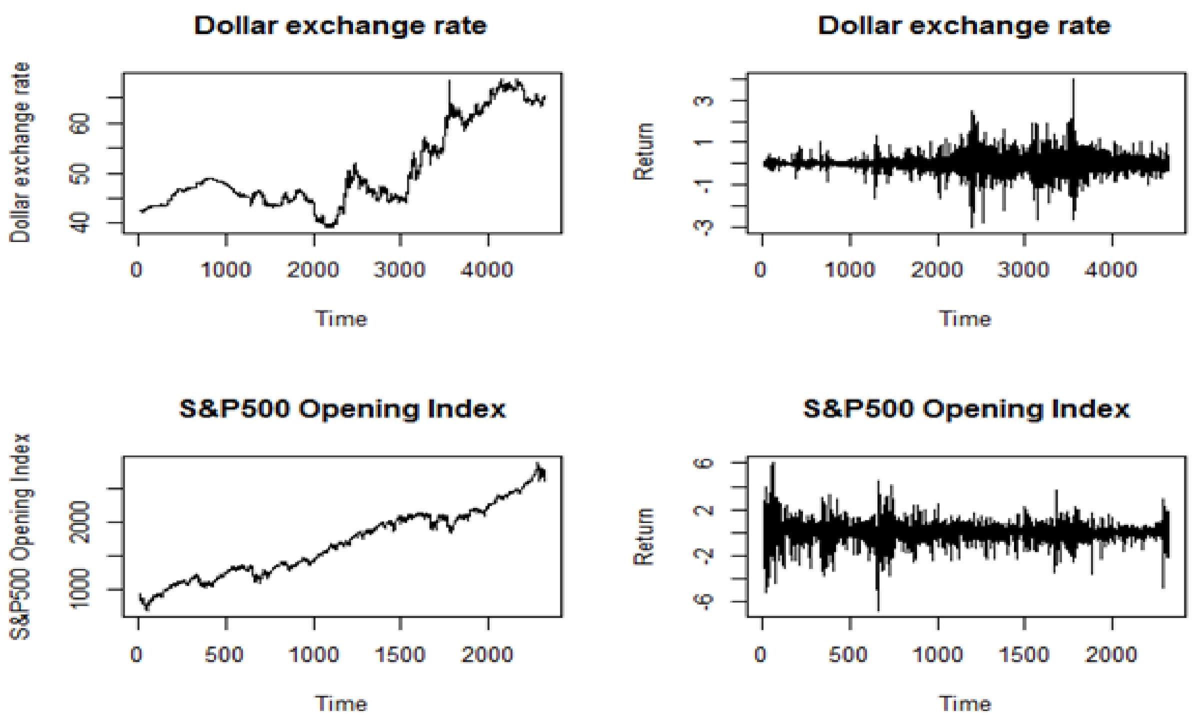

Figure 2: Time series plot the data and the return

The descriptive statistics of the return series are reported in Table 3, where $Q(20)$ and $Q^2(20)$ are respectively the Ljung-Box statistics for return and squared return series with lag 20. The corresponding χ^2 table value at 5% significance level is 10.117. Hence the test suggests that the return series are serially uncorrelated whereas the squared return series have significant serial correlation. The kurtosis of the returns for both the series are greater than three which implies that the distributions of the returns are leptokurtic in nature.

Table 3: Descriptive statistics of the return series

Statistic	Dollar Exchange rate	S&P500 Opening Index
Sample size	4660	2326
Minimum	-3.0156	-6.7977
Maximum	4.0108	6.0192
Std. Dev.	0.4085	0.9905
Kurtosis	11.4238	8.0453
$Q(20)$	1.6744	3.7623
$Q^2(20)$	101.4591	113.0604

From the ACF of the returns plotted in Figure 3, it is observed that serial correlations in the return series are insignificant where as the ACF of the squared returns in the bottom panel remains positive and decay very slowly. In Table 4, we present the parameter estimates for both the return series.

Table 4: Estimates of the model parameters

Parameters	Dollar Exchange rate	S&P 500 Opening Index
α	2.8501	2.2011
β	0.1425	0.4302
ρ	0.8821	0.7344

5.1 Remark

The simulation study confirms that our method of estimation performs well for the proposed model. However, it is important to check the goodness of fit of the model in a given situation. That is, we need to perform diagnosis check for the model. Note that the model (2.4) is in terms of the unobservable volatilities h_t, which makes the diagnosis problem difficult. One of the methods suggested in such cases is to employ Kalman filtering by rewriting the model (2.4) in the state-space form to generate residuals from the fitted model. This method works well for linear normal-lognormal SV models as discussed in Jacquier et al. (1994) and Tsay (2005). In our case the BS Markov sequence is generated by a non-linear transformation and we have to deal with nonlinear filters if we want to use their methods. This will require detailed analysis of nonlinear Kalman filters, which is beyond the scope of this paper.

6 Conclusions

The present paper proposes a stationary Markov sequence of Birnbaum Saunders rvs to model stochastic volatility for analyzing financial time series. This could be a potential alternative

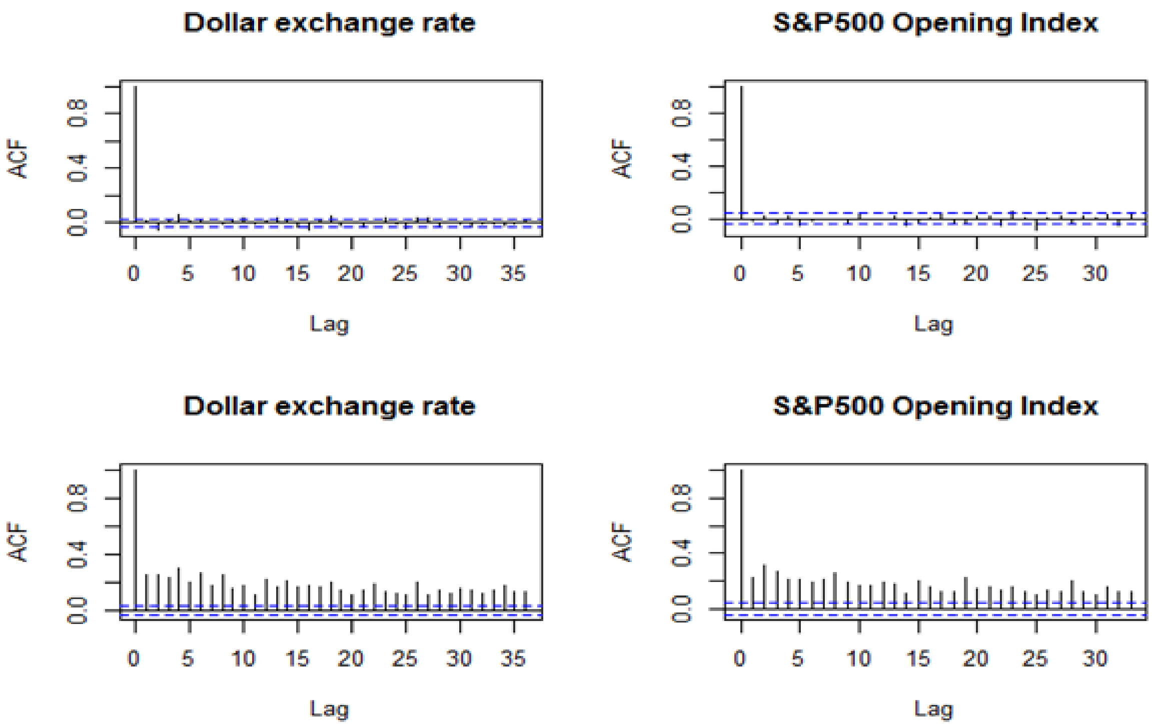

Figure 3: ACF of the returns (upper panel) and the squared returns(lower panel)

to the commonly used normal-lognormal SV model. The simulation study demonstrates that the present model works well for the return series considered in the paper. The model diagnosis requires an effective method of residual analysis. The volatilities in the returns are generated by a nonlinear model and application of suitable non-linear filters may help in extracting the residuals, which could be used for model diagnosis.

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