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Compatibility of BIB designs

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Dedicated to Professor Aloke Dey on his retirement.

Abstract

The purpose of this paper is to develop the results on additive BIB designs by Colbourn and Rosa (1999) and Matsubara et al. (2006) and Sawa et al. (2007). Additive Steiner triple systems (STS) of order 3^n, which include additive STS(9) constructed by Colbourn and Rosa (1999), are presented in relation to affine geometry AG$(n, 3)$. The concept of additivity is generalized for BIB designs with the number of points not divisible by their blocksize. The merit of additive BIB designs is also discussed. First, it is shown that for any odd integer s, a B$(s^2k,\ sk,\ \lambda(sk-1)/(k-1))$ can be constructed from additive B(sk, k, λ). Applying this result to additive STS(3^{2m-1}) mentioned above yields a B$(3^{2m-1}, 3^m, (3^m-1)/2)$ nonisomorphic to those constructed through affine groups. Secondly, a close relationship between multiply nested BIB designs and additive BIB designs is shown. As a by-product two infinite families of multiply nested BIB designs are constructed.

Key words: Additive structure; Compatible minimal partition; Compatibly nested minimal partition; Incidence matrix; Balanced incomplete block (BIB) design; Multiply nested BIB (MNB) design; Steiner triple system (STS).

1 Introduction

A balanced incomplete block (BIB) design is a system with v points and b blocks each containing k different points, each point appearing in r different blocks and any two distinct points appearing in exactly λ different blocks (for example, see Colbourn and Dinitz (2007)). This is denoted by $\mathrm{B}(v, b, r, k, \lambda)$, or shortly by $\mathrm{B}(v, k, \lambda)$ if there is no need to express the values of b and r. The incidence matrix $\boldsymbol{N} = (n_{ij})$ of a BIB design is a $v \times b$ matrix such that $n_{ij} = 1$ or 0 for all i, j, according as the ith point occurs in the jth block or otherwise. Thus the incidence matrix $\boldsymbol{N}$ satisfies the following conditions.

1. $\sum_{j=1}^{b} n_{ij} = r$ for all $i = 1, \ldots, v$.
2. $\sum_{i=1}^{v} n_{ij} = k$ for all $j = 1, \ldots, b$.
3. $\sum_{j=1}^{b} n_{ij} n_{i'j} = \lambda$ for all $i, i' = 1, \ldots, v,\ i \neq i'$.

Necessary conditions for the existence of a $\mathrm{B}(v, k, \lambda)$ are that

(a) $\lambda(v-1) \equiv 0 \pmod{k-1}$,

(b) $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$.

It is known (see Wilson (1975)) that for given k and λ, the above two conditions are also sufficient for sufficiently large v.

A BIB design with v points is called a *Steiner triple system of order* v (STS(v)) if $k = 3$ and $\lambda = 1$. It is well known that an STS(v) exists if and only if $v \equiv 1, 3 \pmod 6$; for example, see Colbourn and Dinitz (2007). Consider an STS(v) with a set of points V and a set of blocks $\mathcal{B} = \{B_j \mid j = 1, \ldots, b\}$, where $b = v(v-1)/6$ is the number of triples. A BIB design with points V and blocks $\mathcal{C} = \{C_j \mid j = 1, \ldots, b\}$, of size k, is called a $(k, \lambda; 3, 1)$-*nesting* of $(V, \mathcal{B})$ if $B_j \subseteq C_j$ for $j = 1, \ldots, b$. By using (a) and (b), it is easy to see that a $(k, \lambda; 3, 1)$-nesting exists only if $\lambda = k(k-1)/6$ and $k \equiv 0, 1 \pmod 3$.

The simplest nontrivial case arises when $k = 4$, that is, a $(4, 2; 3, 1)$-nesting of an STS. A $(4, 2; 3, 1)$-nesting is closely related to a block colouring (see Colbourn and Rosa (1999)), and is also useful to construct optimal optical orthogonal codes of weight 4, as revealed in Yin (1998). The next small block size to be considered is $k = 6$. Let $\mathcal{D}_i = (V, \mathcal{B}_i),\ i = 1, 2$, be STS($v$) with $\mathcal{B}_i = \{B_j^{(i)} \mid j = 1, \ldots, b\}$. Then $\mathcal{D}_1$ and $\mathcal{D}_2$ are said to be *compatible* if $\mathcal{D}_{\{1,2\}}$ is a $(6, 5; 3, 1)$-nesting of $\mathcal{D}_i$, where $\mathcal{D}_{\{1,2\}} = (V, \mathcal{B}_{\{1,2\}})$ with $\mathcal{B}_{\{1,2\}} = \{B_j^{(1)} \cup B_j^{(2)} \mid j = 1, \ldots, b\}$. A merit of compatible STS(v) is to yield a nested $\mathrm{B}(v; v(v-1)/6, v(v-1)/3; 6, 3)$. In Section 5 more results about nested BIB designs will be mentioned in relation to the notion of compatibility of designs.

The concept of compatibility for two STS is naturally extended to that for more than two STS.

Definition 1.1 *(Colbourn and Rosa (1999)) Let* v, ℓ *be positive integers such that* $v \equiv 3 \pmod 6$ *and* $2 \leq \ell \leq v/3$. *A set of* ℓ *STS*(v), *say* $\{\mathcal{D}_i \mid i = 1, \ldots, \ell\}$, *is said to be* ℓ *pairwise compatible if there is an ordering of triples of each STS satisfying the following condition:*

(i) for distinct $i, i' = 1, \ldots, \ell$, $\mathcal{D}_{\{i,i'\}}$ *is a* $(6, 5; 3, 1)$*-nesting of* $\mathcal{D}_i$ *and* $\mathcal{D}_{i'}$.

In particular, this set is called a compatible minimal partition for v *if* $\ell = v/3$.

Definition 1.2 *(Colbourn and Rosa (1999)) Let* v, ℓ *be positive integers such that* $v \equiv 1 \pmod 6$ *and* $2 \leq \ell \leq (v-1)/3$. *A set of* ℓ *STS*(v), *say* $\{\mathcal{D}_i \mid i = 1, \ldots, \ell\}$, *is said to be* ℓ *pairwise compatible if there is an ordering of triples of each STS satisfying the condition (i) given in Definition 1.1. This is called a compatibly nested minimal partition for* v *if* $\ell = (v-1)/3$.

When v is divisible by k, the concept of compatibility for STS has been extended to that for BIB designs in general; see Matsubara et al. (2006), Sawa et al. (2007). Let $(V, \mathcal{B})$ be a $\mathrm{B}(v, b, r, k, \lambda)$ with $\mathcal{B} = \{B_j \mid j = 1, \ldots, b\}$. Then a $\mathrm{B}(v, b, r', k', \lambda')$, say $(V, \mathcal{C})$, with $\mathcal{C} = \{C_j \mid j = 1, \ldots, b\}$ is called a $(k', \lambda'; k, \lambda)$-*nesting* of $(V, \mathcal{B})$ if $B_j \subseteq C_j$ for $j = 1, \ldots, b$. For two BIB designs, say $\mathcal{D}_i$, $i = 1, 2$, with the same parameters, $\mathcal{D}_1$ and $\mathcal{D}_2$ are said to be *compatible* if $\mathcal{D}_{\{1,2\}}$ is a $(2k, \lambda'; k, \lambda)$-nesting of $\mathcal{D}_i$.

Definition 1.3 *(Matsubara et al.(2006), Sawa et al. (2007)) Let* k, ℓ, s *be positive integers with* $2 \leq \ell \leq s$. *A set of* ℓ $B(sk, k, \lambda)$, *say* $\{\mathcal{D}_i \mid i = 1, \ldots, \ell\}$, *is said to be* ℓ *pairwise additive if there is an ordering of blocks of each BIB design satisfying the following condition:*

(i) for distinct $i, i' = 1, \ldots, \ell$, $\mathcal{D}_{\{i,i'\}}$ *is a* $(2k, \lambda'; k, \lambda)$*-nesting of* $\mathcal{D}_i$ *and* $\mathcal{D}_{i'}$.

In particular, such designs are said to be additive if $\ell = s$, *that is,* *(ii) for any $j = 1, \ldots, b$, it holds that $\bigcup_{i=1}^{s} B_j^{(i)} = V$.*

In Definition 1.3, it was noted (see Sawa et al. (2007)) that for ℓ pairwise additive $\mathrm{B}(sk, k, \lambda)$,

$$\lambda \geq \begin{cases} (k-1)/2 & \text{if } k \text{ is an odd integer,} \\ k-1 & \text{otherwise,} \end{cases} \tag{1}$$

which implies that the class of BIB designs with pairwise additivity and coincidence number 1 yields the STS or the unreduced designs. The introductory results on the additive BIB designs appeared in Matsubara et al.(2006). Since then, such BIB designs were largely investigated in Sawa et al. (2007) where the following important theorem for design theory was proved.

Theorem 1.4 *(Sawa et al. (2007)) Let s be an odd prime power. Assume that there exist additive $B(sk, sr, r, k, \lambda)$. Then, (i) there exist additive BIB designs with parameters $v^* = s^2k$, $k^* = sk$, $\lambda^* = r$, and (ii) for any positive integer $t \leq s$, there exists a BIB design with parameters $v_t = s^2k$, $k_t = stk$, $\lambda_t = t[(s+1)r - s\lambda](stk-1)/(s^2k-1)$.*

By use of Theorem 1.4, infinitely many BIB designs, whose existence was in doubt in literature, are constructed in [8]. Also, observe that for given v and k, if $\lambda = (k-1)/2$ and $t \in \{1, 2\}$, designs constructed from Theorem 1.4 (ii) are minimal possible.

The purpose of this paper is to develop the results on additive BIB designs appearing in Colbourn and Rosa (1999), Matsubara et al. (2006) and Sawa et al. (2007). In Section 2, it will be proved that for any positive integer n, there exists a compatible minimal partition for $v = 3^n$. As far as the authors know, this is the first infinite family of compatible minimal partitions, which includes a compatible minimal partition for STS(9) constructed in Colbourn and Rosa (1999). In Section 3, Theorem 1.4 (i) will be improved so as to be applicable for any odd integer s. Applying this result to a compatible minimal partition for 3^{2m-1} mentioned above gives a $\mathrm{B}(3^{2m-1}, 3^m, (3^m-1)/2)$ non-isomorphic to those constructed through affine groups, and then this construction is different from a classical method using finite groups. In Section 4, the concept of compatibly nested minimal partitions for STS will be extended to that for additive BIB designs in general. Two infinite families of additive designs are presented. On the other hand, it is expected that results obtained in Colbourn and Rosa (1999), Matsubara et al. (2006) and Sawa et al. (2007) might hold without the restriction that v is not divisible by k. Particularly in Section 5, a result which mentions a relationship between nested BIB designs and additive BIB designs (see [Sawa et al. (2007), Theorem 2.3]) will be generalized for v not divisible by k. As a by-product, two infinite families of nested BIB designs are constructed.

2 Compatible minimal partitions

Let V be an n-dimensional vector space over $\mathbb{F}_q$. Each vector of V is regarded as a point of $\mathrm{AG}(n,q)$. A t-dimensional subspace or its coset in $\mathbb{F}_{q^n}$ is called a t-flat of $\mathrm{AG}(n,q)$. In particular 0-flats, 1-flats, 2-flats and $(n-1)$-flats are called points, lines, planes and hyperplanes, respectively. For a primitive element α of $\mathbb{F}_{q^n}$ each nonzero point in $\mathrm{AG}(n,q)$ is represented by α^j. Let $d=(q^n-1)/(q-1)$. Then $B_0=\{0\}\cup\{\alpha^{id}\mid i=0,\ldots,q-2\}$ is a line of $\mathrm{AG}(n,q)$. Given $a,b\in\mathbb{F}_{q^n}$ with $a\neq 0$, consider a mapping $T_{a,b}:V\to V$ defined by $T_{a,b}(x)=ax+b$. It is well known that the group of all such transformations, denoted by $G=\mathrm{AGL}(1,q^n)$, acts sharply doubly-transitively on $\mathbb{F}_{q^n}$, and then for any k-subset H of $\mathbb{F}_{q^n}$, (V,H^G) forms a $\mathrm{B}(q^n,k,k(k-1))$, where H^G denotes the image of H under G. If we take H to be B_0, the stabilizer K of B_0 is isomorphic to $\mathrm{AGL}(1,q)$ since B_0 is a subfield of $\mathbb{F}_{q^n}$. Hence, $(V,\mathrm{Orb}_G(B_0))$ is a $\mathrm{B}(q^n,q,1)$, the well-known design derived from the set of lines of $\mathrm{AG}(n,q)$, where $\mathrm{Orb}_G(B_0)$ denotes the G-orbit of B_0. Choose a plane π_0 and a line ℓ_0 both through the origin such that $\pi_0=B_0\oplus\ell_0$, and let $\bar{B}_0=\pi_0\backslash B_0$. Then, observing that $\bar{B}_0$ is fixed by K, we have the following.

Lemma 2.1 *The design* $(V,\mathrm{Orb}_G(\bar{B}_0))$ *is a* $B(q^n,q(q-1),q^2-q-1)$.

By virtue of Lemma 2.1, the following result is obtained.

Theorem 2.2 *There exists a compatible minimal partition for* $v=3^n$.

Proof. Let H_0 be a hyperplane of $\mathrm{AG}(n,3)$ through the origin such that $V=H_0\oplus B_0$. For any $x,y\in H_0$ with $x\neq y$, let $B=(B_0+x)\cup(B_0+y)$ and ℓ_0 be a line passing through x,y. Then $(B_0\oplus\ell_0)\backslash B$ can be written as a translate of B_0, that is, $(B_0\oplus\ell_0)\backslash B=B_0+(2x-y)$.

Thus, by Lemma 2.1, $\{(V, \mathrm{Orb}_G(B_0+x)) \mid x \in H_0\}$ forms a compatible minimal partition for $v = 3^n$. $\square$

Example 2.3 *Let α be a primitive element of $\mathbb{F}_{3^3}$ such that $\alpha^3+2\alpha+1 = 0$. Then $B_0 = \{\infty, \alpha^0, \alpha^{13}\}$ is a line passing through the origin. Each point α^j is abbreviated by its exponent j, and the notation ∞ means the origin. Here a set of lines parallel to B_0 is considered as follows:*

$$\begin{array}{lll} L_0 = \{\infty, 0, 13\}, & L_1 = \{1, 9, 3\}, & L_2 = \{14, 16, 22\}, \\ L_3 = \{4, 18, 7\}, & L_4 = \{17, 20, 5\}, & L_5 = \{2, 21, 12\}, \\ L_6 = \{15, 25, 8\}, & L_7 = \{10, 6, 11\}, & L_8 = \{23, 24, 19\}. \end{array}$$

A compatible minimal partition for $v = 27$ is obtained by developing the base blocks in Table 1 by modulo 13.

Table 1: A compatible minimal partition for $v = 27$

Designs	Base blocks								
$(\mathbb{F}_{27}, \mathcal{B}_1)$	L_0	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8
$(\mathbb{F}_{27}, \mathcal{B}_2)$	L_1	L_2	L_0	L_5	L_8	L_7	L_4	L_3	L_6
$(\mathbb{F}_{27}, \mathcal{B}_3)$	L_2	L_0	L_1	L_7	L_6	L_3	L_8	L_5	L_4
$(\mathbb{F}_{27}, \mathcal{B}_4)$	L_3	L_5	L_7	L_4	L_0	L_8	L_2	L_6	L_1
$(\mathbb{F}_{27}, \mathcal{B}_5)$	L_4	L_8	L_6	L_0	L_3	L_1	L_7	L_2	L_5
$(\mathbb{F}_{27}, \mathcal{B}_6)$	L_5	L_7	L_3	L_8	L_1	L_6	L_0	L_4	L_2
$(\mathbb{F}_{27}, \mathcal{B}_7)$	L_6	L_4	L_8	L_2	L_7	L_0	L_5	L_1	L_3
$(\mathbb{F}_{27}, \mathcal{B}_8)$	L_7	L_3	L_5	L_6	L_2	L_4	L_1	L_8	L_0
$(\mathbb{F}_{27}, \mathcal{B}_9)$	L_8	L_6	L_4	L_1	L_5	L_2	L_3	L_0	L_7

The following theorem was proved in [Sawa et al. (2007), Lemma 2.1] under the restriction that v is divisible by k. For further discussion, its proof is given here; in fact the result follows without the above restriction.

Theorem 2.4 *([Sawa et al. (2007)]) Let v, k, ℓ, s be positive integers with $v = sk$ and $2 \leq \ell \leq s$. Assume that there exist ℓ pairwise additive $B(v, k, \lambda)$, say $(V, \mathcal{B}_i)$, $i = 1, \ldots, \ell$. Then, for any p-subset $R \subseteq \{1, \ldots, \ell\}$, $(V, \mathcal{B}_R)$ forms a $B(v, pk, \Lambda_p)$, where*

$$\Lambda_p = \frac{\lambda p(pk-1)}{k-1}. \tag{2}$$

Proof. Let R be a p-subset of $\{1,\dots,\ell\}$. For each $h\in R$, let $\boldsymbol{N}_h$ be the incidence matrix of a BIB design in ℓ pairwise additive $\mathrm{B}(v,k,\lambda)$. Since

$$\begin{aligned}
&\Big(\sum_{h\in R}\boldsymbol{N}_h\Big)\Big(\sum_{h\in R}\boldsymbol{N}_h\Big)^T\\
&=\sum_{\substack{i,j\in R,\\ i<j}}\Big(\sum_{h=i,j}\boldsymbol{N}_h\Big)\Big(\sum_{h=i,j}\boldsymbol{N}_h\Big)^T-(p-2)\sum_{h\in R}\boldsymbol{N}_h\boldsymbol{N}_h^T\\
&=\sum_{\substack{i,j\in R,\\ i<j}}\Big((2r-\lambda')\boldsymbol{I}+\lambda'\boldsymbol{J}\Big)-(p-2)\sum_{h\in R}\Big((r-\lambda)\boldsymbol{I}+\lambda\boldsymbol{J}\Big),
\end{aligned}$$

where λ' is the coincidence number of a nesting of $(V,\mathcal{B}_i)$, I or J is respectively the identity matrix or the all-one matrix of order v, $\sum_{h\in R}\boldsymbol{N}_h$ is the incidence matrix of a BIB design. The coefficient of the matrix $\boldsymbol{J}$ contributes the coincidence number of the design. By (i) in Definition 1.3, we have

$$\lambda'=\frac{2\lambda(2k-1)}{k-1}, \tag{3}$$

and thus

$$\Big(\sum_{\substack{i,j\in R,\\ i<j}}\frac{2\lambda(2k-1)}{k-1}-(p-2)\sum_{h\in R}\lambda\Big)\boldsymbol{J}=\frac{\lambda p(pk-1)}{k-1}\boldsymbol{J},$$

which completes the proof.

Corollary 2.5 *There exists a* $(3s,s(3s-1)/2;3,1)$*-nesting for all* $s=2,\dots,3^n$.

Proof. The result follows from Theorems 2.2 and 2.4.

It was shown (see [Sawa et al. (2007)]) that additive $\mathrm{B}(q^2,q,(q-1)/2)$ exist for any odd prime power. When $q=3$, such designs form a compatible minimal partition for $v=9$ which is seen to be equivalent to that constructed in [Colbourn and Rosa (1999)]. In Theorem 2.2, another generalization is given for values of n. It remains unsolved whether or not there exist additive $\mathrm{B}(q^n,q,(q-1)/2)$ for a positive integer $n>2$ and an odd prime power $q>3$.

3 A construction of BIB designs through pairwise additivity

A construction of BIB designs is presented here by using additive BIB designs.

The pairwise additivity can also be described in terms of matrices, that is, ℓ pairwise additive $\mathrm{B}(v = sk, b, r, k, \lambda)$ are regarded as ℓ incidence matrices $\boldsymbol{N}_i,\ i = 1, \ldots, \ell$, such that

(i') for distinct $i, i' = 1, \ldots, \ell$, $\boldsymbol{N}_i + \boldsymbol{N}_{i'}$ is the incidence matrix of a $\mathrm{B}(sk, 2k, \lambda')$.

In particular, the additivity requires that

(ii') $$\textstyle\sum_{i=1}^{s} \boldsymbol{N}_i = \boldsymbol{J},$$

where $\boldsymbol{J}$ is of size $v \times b$. Using such matrix-expression of additive BIB designs enables us to develop a construction of BIB designs.

Construction 3.1 *Let s be an odd integer, say $s = 2m + 1$, and for a positive integer $\ell \geq (s + 1)/2$, assume that there exist ℓ pairwise additive $B(sk, sr, r, k, \lambda)$. Denote by $\boldsymbol{N}_i,\ i = 1, \ldots, \ell$, the ℓ incidence matrices of the designs. Moreover, let $\boldsymbol{C} = (c_{ij})$ be a circulant matrix of order s defined by*

$$c_{ij} = \begin{cases} 1 & \text{if } j = i + 1, \\ 0 & \text{otherwise} \end{cases}$$

and a matrix $\boldsymbol{A} = (a_{ij})$ be

$$\boldsymbol{A} = \sum_{\ell=1}^{m+1} \ell \boldsymbol{C}^{\ell} + \sum_{\ell=m+2}^{s} (2m + 2 - \ell) \boldsymbol{C}^{\ell}. \tag{4}$$

Then we consider the matrix $\boldsymbol{N}^*$ defined by

$$\boldsymbol{N}^* = [(\boldsymbol{N}_{ij}) : (\boldsymbol{J}_{ij})],$$

where $\boldsymbol{N}_{ij} = \boldsymbol{N}_{a_{ij}}$, $\boldsymbol{J}_{ij} = \delta_{ij} \boldsymbol{J}$ (subscripts being reduced modulo s), $\boldsymbol{J}$ is of size $sk \times (r - s\lambda)$ and δ is the Kronecker delta.

It will be shown that the matrix $\boldsymbol{N}^*$ constructed above is the incidence matrix of a BIB design. In order to prove this, we need a lemma. Let $(V, \mathcal{B})$ be a group divisible design of type $(sk)^s$ which corresponds to the submatrix $(\boldsymbol{N}_{ij})$ of $\boldsymbol{N}^*$. Denote the point set V by $\{(\mu, \nu) \mid \mu = 1, \ldots, sk,\ \nu = 1, \ldots, s\}$, where (μ, ν) is regarded as the μth point of the νth group on the design.

Lemma 3.2 *Let $\boldsymbol{N}_i$ be the two incidence matrices of $B(sk, sr, r, k, \lambda)$, say $(V, \mathcal{B}_i)$, having pairwise additive structure, and let λ' be the coincidence number of a nesting of $(V, \mathcal{B}_i)$, $i = 1, 2$. Then for $\mu, \mu' = 1, \ldots, sk$ and $\nu, \nu' = 1, \ldots, s$,*

$$|\{B \in \mathcal{B} \mid (\mu, \nu), (\mu', \nu') \in B\}| = \begin{cases} s\lambda & \text{if } \nu = \nu' \text{ and } \mu \neq \mu', \\ r & \text{if } \nu \neq \nu' \text{ and } \mu = \mu', \\ \frac{(\lambda' - 2\lambda)(s-1)}{2} + \lambda & \text{if } \nu \neq \nu' \text{ and } \mu \neq \mu'. \end{cases}$$

Proof. Take a pair of points, say $P = \{\mu, \nu\}$. Then by Construction 3.1, if $\nu = \nu'$, the contribution from $(\boldsymbol{N}_{ij})$ to the coincidence number of P is obviously $s\lambda$. On the other hand, by noting (i) in Definition 1.3, we have

$$\boldsymbol{N}_1 \boldsymbol{N}_2^T + \boldsymbol{N}_2 \boldsymbol{N}_1^T = (\lambda' - 2\lambda)(\boldsymbol{J} - \boldsymbol{I}), \tag{5}$$

where $\boldsymbol{I}$ and $\boldsymbol{J}$ are of order sk and λ' is the coincidence number of a nesting of $(V, \mathcal{B}_i)$, $i = 1, 2$. Hence, by use of Construction 3.1 with (5), the contribution from $(\boldsymbol{N}_{ij})$ to the coincidence number of P is calculated for the case where $\nu \neq \nu'$.

Theorem 3.3 *The matrix $\boldsymbol{N}^*$ defined in Construction 3.1 is the incidence matrix of a BIB design with parameters*

$$v^* = s^2 k,\ b^* = s[(s+1)r - s\lambda],\ r^* = (s+1)r - s\lambda,\ k^* = sk,\ \lambda^* = r.$$

Proof. We use the same notations as those used in Lemma 3.2. By Construction 3.1, it can be seen that the $(\boldsymbol{J}_{ij})$-substitution part of $\boldsymbol{N}^*$ does not contribute to the coincidence number of P if and only if $\nu \neq \nu'$. If $\nu = \nu'$, the contribution equals $r - s\lambda$. Hence the result follows from Lemma 3.2 with (3).

Corollary 3.4 *(i) The existence of a resolvable $B(3k, k, \lambda)$ implies that of a resolvable $B(3^n k, 3^{n-1} k, (3^{n-1}k - 1)/2)$ for any positive integer n. (ii) There exists a $B(3^{2n-1}, 3^n, (3^n - 1)/2)$ for any positive integer n.*

Proof. (i) Let $\{B_{i1}, B_{i2}, B_{i3}\}$, $i = 1, \ldots, r$, be the ith resolution set of a resolvable $\mathrm{B}(3k, 3r, r, k, \lambda)$, say $\mathcal{D}$. Identify a k-subset B_{ij} with a zero-one column vector $\boldsymbol{B}_{ij}$ of length v such that $x \in B_{ij}$ if and only if the xth coordinate of $\boldsymbol{B}_{ij}$ equals 1. Then, the matrices $\boldsymbol{N}_t$ defined by arranging $\boldsymbol{B}_{i,j+t-1}$ in the $(3(i-1)+j)$th column, where the second subscript $j+t-1$ of $\boldsymbol{B}_{i,j+t-1}$ is reduced modulo 3, form the incidence matrices of additive $\mathrm{B}(3k, 3r, r, k, \lambda)$; see [Matsubara et al. (2006)]. Hence, by applying Theorem 3.3 inductively, a $\mathrm{B}(3^n k, 3^{n-1}k, (3^{n-1}k - 1)/2)$ can be obtained for any n. The resolvability follows from the ordering of columns in each $\boldsymbol{N}_t$.

(ii) Apply Theorem 3.3 to a compatible minimal partition given in Theorem 2.2.

Remark 1 *(i) The designs constructed by Theorem 3.3 have the same parameters as those constructed by [Sawa et al. (2007) Theorem 5.8 (I)], which, however, is largely improved on s. Theorem 5.8 (I) in [Sawa et al. (2007)] requires the very strict restriction that s is an odd prime power, whereas Theorem 3.3 is valid for any odd integer s.*

(ii) Consider a BIB design $\mathcal{D}$ given in Corollary 3.4 (ii). The circulant matrix substitution part of $\mathcal{D}$ forms a simple incidence structure, since the coincidence numbers of the starting designs are 1. Whereas, each block of the $\boldsymbol{J}$-substitution part of $\mathcal{D}$ occurs exactly $r - s\lambda$ times in the system. On the other hand, a BIB design $\mathcal{D}'$ with the same parameters as those of $\mathcal{D}$ is constructed by taking together the $AGL(1, 3^{2n-1})$-orbits of suitable 3^n-subsets of $\mathbb{F}_{3^{2n-1}}$. It should be noted that $\mathcal{D}$ and $\mathcal{D}'$ are nonisomorphic. Otherwise, the $\boldsymbol{J}$-substitution part consists of $AGL(1, 3^{2n-1})$-orbits. Since $AGL(1, 3^{2n-1})$ is doubly-transitive on points, the $\boldsymbol{J}$-substitution part shows a BIB design, which is a contradiction. We thus conclude that Construction 3.1 is different from a classical construction of BIB designs using the affine groups.

(iii) The designs constructed in Theorem 3.3 are minimal possible for given v and k. Let k, s be odd integers and $v = sk$. Assume that there exist additive $B(sk, k, (k-1)/2)$. Then by Theorem 3.3, a $B(s^2k, sk, (sk-1)/2)$ can be obtained. Given a BIB design with s^2k points and blocks of size sk, the coincidence number of this design is given as $r^*(sk-1)/(s^2k-1)$, where r^* is the replication number of the design. Therefore if $\gcd(s^2k-1, sk-1) = 2$, the design is minimal on coincidence numbers. For example, the coincidence number of a BIB design with 243 points and blocks of size 27 is at least 13 and thus, in this case, a $B(243, 27, 13)$ is minimal.

4 A generalization of compatibly nested minimal partitions

In this section, the concept of pairwise additivity of BIB designs is generalized for v not divisible by k. In particular, as a natural generalization of Definition 1.2, we consider only the case where $v \equiv 1 \pmod{k}$.

Definition 4.1 *Let k, ℓ, s, v be positive integers with $v = sk + 1$ and $2 \leq \ell \leq s$. A set of ℓ $B(v, b, r, k, \lambda)$, say $\{\mathcal{D}_i \mid i = 1, \ldots, \ell\}$, is said to be ℓ pairwise additive if there is an ordering of blocks of each BIB design satisfying the condition (i) given in Definition 1.3. In particular, such designs are said to be additive if $\ell = s$.*

Take a $\mathrm{B}(\nu = sk + 1, b, r, k, \lambda)$. Then, by use of (a) and (b) in Introduction, it holds that

$$r = \lambda sk/(k-1) \qquad \text{and} \qquad b = \lambda s(sk+1)/(k-1). \tag{6}$$

Let $\mathcal{B}_i = \{B_j^{(i)} \mid j = 1, \ldots, b\}$, $i = 1, \ldots, \ell$, be the sets of blocks of ℓ pairwise additive $\mathrm{B}(v = sk + 1, b, r, k, \lambda)$. Then, for distinct $i, i' = 1, \ldots, \ell$, the coincidence number of $(V, \mathcal{B}_{\{i,i'\}})$ counts

$$\frac{2r(2k-1)}{v-1} = \frac{2\lambda r(2k-1)}{\lambda(v-1)} = \frac{2\lambda r(2k-1)}{r(k-1)} = \frac{2\lambda(2k-1)}{k-1}.$$

Since $2k - 1$ and $k - 1$ are relatively prime,

$$2\lambda \equiv 0 \pmod{k-1} \tag{7}$$

holds. Hence it follows that

$$\lambda \geq \begin{cases} (k-1)/2 & \text{if } k \text{ is an odd integer,} \\ k-1 & \text{otherwise.} \end{cases} \tag{8}$$

A set of additive BIB designs which attain (8) plays a key role in constructing BIB designs minimal for given v and k, as the following theorem shows.

Theorem 4.2 *Let p, ℓ, s be positive integers such that $2 \leq p \leq \ell \leq s$. The existence of ℓ pairwise additive $B(v = sk + 1, b, r, k, \lambda)$ implies that of a $B(v, b, b - pr, v - pk, \Lambda_p)$, where*

$$\Lambda_p = \frac{\lambda(s-p)\{(s-p)k+1\}}{k-1}. \tag{9}$$

In particular, $\Lambda_p = \lambda(k+1)/(k-1)$ when $p = \ell - 1 = s - 1$.

Proof. Denote by $\mathcal{B}_i$, $i = 1, \ldots, \ell$, the sets of blocks of ℓ pairwise additive $\mathrm{B}(v = sk+1, b, r, k, \lambda)$. Observe that in the proof of Theorem 2.4 there is no need to restrict v being divisible by k. Thus for any $p = 2, \ldots, \ell$, the design $(V, \mathcal{B}_{\{1,\ldots,p\}})$ yields a $\mathrm{B}(v, b, pr, pk, \lambda p(pk-1)/(k-1))$. The complement of the design yields a BIB design with coincidence number as

$$\begin{aligned} b - 2pr + \frac{\lambda p(pk-1)}{k-1} &= \frac{\lambda s(sk+1)}{k-1} - 2p\frac{\lambda sk}{k-1} + \frac{\lambda p(pk-1)}{k-1} \\ &= \frac{\lambda(s-p)\{(s-p)k+1\}}{k-1} \end{aligned}$$

by (6).

Corollary 4.3 *Under the same notation as those used in Theorem 4.2, if $\ell = s = p+1$, then*

$$\Lambda_{s-1} \geq \begin{cases} (k+1)/2 & \text{if } k \text{ is an odd integer,} \\ k+1 & \text{otherwise.} \end{cases} \tag{10}$$

Proof. By Theorem 4.2, $\Lambda_{s-1} = \lambda(k+1)/(k-1)$. Hence the result follows by applying Euclidean algorithm to $k \pm 1$.

Remark 2 *A BIB design attaining (10) can be minimal possible for given s and $k \equiv 1 \pmod 2$. For example, if there is a set of additive $B(19, 3, 1)$, then the design constructed by Theorem 4.2 has coincidence number 2. Since the coincidence number of a BIB design with 19 points and blocks of size 4 is at least 2, this design is minimal. These observation provides a merit of additive BIB designs for $v \equiv 1 \pmod k$.*

We conclude this section by presenting two infinite families of additive BIB designs. Let $q = tk+1$ be a prime power and x be a primitive element of $\mathbb{F}_q$. Let H be the multiplicative subgroup of $\mathbb{F}_q$ of order k. For $i = 0, \ldots, tk-1$ let $H_i = \{x^i y \mid y \in H\}$.

Lemma 4.4 *(Sprott (1954)) Let $q = tk+1$ be a prime power and x be a primitive element of $\mathbb{F}_q$. Then, (i) the set of base blocks $\{H_i \mid i = 0, \ldots, t-1\}$ generates a $B(q, k, k-1)$, and (ii) if k and t are respectively an odd integer and an even integer, then the set of base blocks $\{H_i \mid i = 0, \ldots, t/2-1\}$ generates a $B(q, k, (k-1)/2)$.*

Theorem 4.5 *For a prime power $q = tk+1$ the following statements hold.*

(i) There exist additive $B(q, k, k-1)$.

(ii) If k and t are respectively an odd integer and an even integer, then there exist additive $B(q, k, (k-1)/2)$.

Proof. It suffices to show that the designs constructed in Lemma 4.4 are additive.

(i) It follows that for distinct $i, j = 0, \ldots, t-1$,

$$\begin{aligned}\bigcup_{l=0}^{t-1} \bigcup_{(a,b)\in H_{i+l}\times H_{j+l}} \{a-b\} &= \bigcup_{n=0}^{k-1} \left(\bigcup_{l=0}^{t-1} \bigcup_{m=0}^{k-1} \{x^{l+mt} \cdot x^{j}(x^{i-j-nt}-1)\} \right) \\ &= \bigcup_{n=0}^{k-1} (\mathbb{F}_q \setminus \{0\}), \end{aligned} \tag{11}$$

since $x^{i-j-nt} - 1 \neq 0$. Combining (11) with Lemma 4.4 implies that each nonzero element of $\mathbb{F}_q$ occurs exactly $2(k-1)+2k = 4k-2$ times as the differences arising from the set $\{H_{i+l} \cup H_{j+l} \mid l = 0, \ldots, t-1\}$. Hence BIB designs $(\mathbb{F}_q, \mathcal{B}_i)$ with $i = 1, \ldots, t$ are formed, where each $\mathcal{B}_i$ is obtained by developing H_j, $j = 0, \ldots, t-1$, according to the ordering $(H_i : H_{i+1} : \ldots : H_{i-1})$ (subscripts reduced modulo t).

(ii) The result follows from an argument similar to that used in the proof of (i).

Corollary 4.6 *([Colbourn and Rosa (1999), Theorem 22.12]) There exists a compatibly nested minimal partition for a prime power $q = 6m+1$.*

Proof. The result follows from Theorem 4.5 (ii) with $k = 3$ and $t = 2m$.

Example 4.7 *Put $m = 2$ in Corollary 4.6. Then a set of additive STS(13) is obtained by developing the base blocks modulo 13 given in Table 2.*

Table 2: Additive STS(13)

Designs	Base blocks	
$(\mathbb{F}_{13}, \mathcal{B}_1)$	$\{1, 3, 9\}$	$\{2, 6, 5\}$
$(\mathbb{F}_{13}, \mathcal{B}_2)$	$\{2, 6, 5\}$	$\{4, 12, 10\}$
$(\mathbb{F}_{13}, \mathcal{B}_3)$	$\{4, 12, 10\}$	$\{8, 11, 7\}$
$(\mathbb{F}_{13}, \mathcal{B}_4)$	$\{8, 11, 7\}$	$\{1, 3, 9\}$

5 Multiply nested designs from pairwise additive designs

In this section a relationship between pairwise additive BIB designs and nested BIB designs is discussed.

A nested BIB design, say NB$(v; b_1, b_2; k_1, k_2)$, is a nested block design $(V, \mathcal{B}_1, \mathcal{B}_2)$ which satisfies the following conditions:

(i) the first system is nested within the second, that is, each block in $\mathcal{B}_2$ is partitioned into l subblocks of size k_1 which form $\mathcal{B}_1$, say, $b_1 = lb_2$ and $k_2 = lk_1$,

(ii) $(V, \mathcal{B}_i)$ is a BIB design with v points and b_i blocks of k_i points each, $i = 1, 2$.

There are many relationships between nested BIB designs and other designs. For example, a resolvable BIB design is a nested BIB design with v divisible by k, and a near-resolvable BIB design is a nested BIB design with v congruent to 1 modulo k. Whist tournaments Wh$(4n)$ and Wh$(4n+1)$ are respectively a resolvable NB$(4n; n(4n-1), 2n(4n-1); 4, 2)$ and a near-resolvable NB$(4n+1; n(4n+1), 2n(4n+1); 4, 2)$. Any nested BIB design with $2k_1 = k_2 = 4$ is called a balanced doubles schedule; see Henley (1980). Resolvable or near-resolvable BIB designs are also called generalized whist tournament designs; see Abel et al. (2003). Moreover, it is known (see Morgan et al. (2001)) that partitioning the rows of a perpendicular array yields a nested BIB design.

Whereas, not so many results which relate "multiply" nested BIB designs and other designs have been published. A multiply nested BIB design is a nested block design $(V, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m)$ with parameters $(v; b_1, \ldots, b_m; k_1, \ldots, k_m)$ for which the systems $(V, \mathcal{B}_i, \mathcal{B}_{i+1})$ are

NB(v; b_i, b_{i+1}; k_i, k_{i+1}). A multiply nested BIB design is a natural generalization of a nested BIB design and is denoted by MNB(v; $b_1, \ldots, b_m$; $k_1, \ldots, k_m$); see Morgan et al. (2001). A construction of multiply nested BIB designs was presented in Sawa et al. (2007) by using pairwise additive BIB designs.

Theorem 5.1 *([Sawa et al. (2007), Theorem 2.4]) Let ℓ be a positive integer with $2 \leq \ell \leq s$. The existence of ℓ pairwise additive B(sk, b, r, k, λ) implies that of an MNB(sk; $2^{m-1}b, 2^{m-2}b, \ldots, b$; $k, 2k, \ldots, 2^{m-1}k$), where $m = \lfloor \log_2 \ell \rfloor + 1$ and $\lfloor x \rfloor$ means the greatest integer y such that $y \leq x$.*

Corollary 5.2 *(i) There exists an MNB($v = 3^n$; $2^{m-1}b$, $2^{m-2}b$, …, b; 3, 6, …, $2^{m-1}3$), where $m = \lfloor (n-1)\log_2 3 \rfloor + 1$ and $b = 3^n(3^n-1)/6$. (ii) For a prime power $q = tk+1$, there exists an MNB($v = q$; $2^{m-1}b$, $2^{m-2}b$, …, b; k, $2k$, …, $2^{m-1}k$), where $m = \lfloor \log_2 t \rfloor + 1$ and $b = (tk+1)t$. In particular when t and k are an even and an odd integer respectively, the number of blocks in each subdesign can be reduced half, that is, $b = (tk+1)t/2$.*

Proof. (i) Use Theorems 2.2 and 5.1.

(ii) Recall the proof of Theorem 2.4 in Section 3. Then, an argument similar to that used in the proof of [Sawa et al. (2007), Theorem 2.4] shows that Theorem 2.4 can be modified to the case where v is not divisible by k. Hence the result follows from Theorem 4.5. $\square$

It is expected that results obtained in Colbourn and Rosa (1999), Matsubara et al. (2006) and Sawa et al. (2007), other than [Sawa et al. (2007), Theorem 2.3] on nested designs, could hold without the restriction that v is not divisible by k. These will be discussed in a forthcoming paper.

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