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SuDoKu as an Experimental Design - Beyond the Traditional Latin Square Design

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Abstract

SuDoKu is an interesting combinatorial structure embedded within a Latin Square. It has been gaining popularity as combinatorial puzzle. Website provides plenty of examples of such SuDoKus. In the same sense as that of Mutually Orthogonal Latin Squares [MOLS], existence of Mutually Orthogonal SuDoKu Squares [MOSS] has also been recently studied. It is well-known that LSDs and their generalizations based on MOLS, viewed as experimental designs, go beyond the CRDs and RCBDs in simultaneously eliminating external sources of variation in the experimental units [eu's] in an ANOVA set-up. It is natural to examine the possibility of viewing SuDoKus and their generalizations [using MOSS] as experimental designs, going one step beyond LSDs and their generalizations. In this context, it may be noted that Subramani and Ponnuswamy (2009) [abbreviated SP(2009)] considered four possible models to accommodate the variation due to SuDoKu arrangements. It was rightly contemplated that one additional component of variation can be suitably accommodated in a SuDoKu when it is viewed as an experimental design. However, SP (2009) overlooked the fact that this can only be done by sacrificing orthogonality. Therefore, their analyses and ANOVA representations for each of the models went wrong. A detailed and corrected statistical analysis is provided along with the underlying ANOVA Table for such designs based on SuDoKu and MOSS. It is envisaged that the SuDoKus and MOSS will provide extra dimension of utility as experimental designs.

Key words ANOVA Designs, CRD, RCBD, LSD, SuDoKu, Mutually Orthogonal SuDoKu, Internal blocking.

1 Introduction

In combinatorics and in ANOVA-oriented experimental designs, a Latin Square [LS] is an $n \times n$ array filled with n different symbols, each occurring exactly once in each row and exactly once in each column. Latin Square Designs [LSDs] and their generalizations, known as Graeco Latin Square designs, based on mutually orthogonal Latin Squares, are among the basic experimental designs we are familiar with [while dealing with ANOVA set-up] in our attempt to simultaneously eliminate heterogeneity in several directions towards 'orthogonal' estimation of 'treatment contrasts'.

If each entry of an $n \times n$ Latin square is written as a triplet (r, c, s), where r represents the row, c represents the column, and s represents the symbol, we obtain a set of n^2 triplets called the orthogonal array representation of the LS. It is well-known that the definition of a Latin Square can be equivalently written in terms of an orthogonal array: A Latin Square of order n is the set of all triples (r, c, s), where $1 \leq r, c, s \leq n$, such that all ordered pairs (r, c) are distinct, all ordered pairs

(r, s) are distinct, and all ordered pairs (c, s) are distinct and each of three types has a single representation in the set. For any Latin Square, there are n^2 triplets so that choosing any two uniquely determines the third.

Again, in the Fisherian sense, three very basic experimental designs (CRD, RCBD and LSD) have been developed in order to obtain valid and reliable conclusion from ANOVA-based field experiments. Overlooking the balanced nature, treatment number and randomization, the essential difference between these designs is in controlling the number of extraneous sources of variations due to environmental factors such as fertility variation of the soil in agricultural experiments. Identifying such 'assignable' source(s) of variation and eliminating their effects from the main analysis usually leads to more reliable understanding of the nature of pure errors and consequently improves the efficiency of the experimental designs. Bolboaca et al (2009) revealed that $CRD\ error > RCBD\ error > LSD\ error$ and the precisions of the underlying experimental designs are in reverse orientation $LS > RCBD > CRD$.

Designs, such as LSDs, eliminating effects of two factors, are commonly termed as row-column designs and a systematic study of such designs provides more general and comprehensive understanding of such effects than the standard LSDs (Shah and Sinha, 1996). In a broad perspective, notions of estimability, connectedness, efficiency and optimality have been discussed in the above-cited paper. Concept of tetra-difference has been used in this context. Nevertheless, even within the framework of a standard LSD, there is a possibility for identifying an altogether different source of variation than the two extraneous sources represented by rows and columns. This becomes transparent when we use a SuDoKu as an experimental design.

In its most general form, a SuDoKu of order $n = pq$ is defined as a combinatorial arrangement going one step beyond a Latin Square of order n composed of the integers $1, 2, ..., n$ in the following sense. The whole Latin Square of order n is decomposed into pq *inner regions*, each of size $p \times q$, say such that each inner region contains all the integers $1, 2, ..., n$. In a way, the n rows of the Latin Square are broadly divided into q sets of p rows each, while the n columns of the Latin Square are broadly divided into p sets of q columns each. The $p \times q$ inner regions may also be termed as 'Internal Block Classification [IBC]'. For $n = p^2$, the inner regions are also termed as 'subsquares' of order $p \times p$. Most popular version of a SuDoKu, as a combinatorial puzzle, is of order $9 = 3^2$. Hundreds of SuDoKu puzzles for $n = 9$ have been posed wherein one is given an 'incomplete' description of the SuDoKu in terms of integers [among $1, 2, ..., 9$] placed in some of the $9^2 = 81$ cells and the aim is to 'complete' the SuDoKu by filling in the rest of the cells. We refer the readers to the websites and also to Robin (2006), Solomon (2006) and Bailey et al (2008). Another related reference is Kuhl and Denley (2012).

Speculative sight in 9×9 SuDoKu puzzle, for example, suggests the possibility of exploiting its internal structure as an experimental design, yet identifying an additional source of variation, apart from those usually attributed to rows/columns/treatments. Although the 9×9 grid with 3×3 *inner regions* is by far the most common, many variations do exist. Some such SuDoKus deal with (i) 4×4 grids with 2×2 inner regions, (ii) 25×25 grids with 5×5 inner regions, (iii) 6×6 grids with 3×2 inner regions, and so on. We do not venture into the constructional aspects of such families of SuDoKus. Our interest lies in their potential use as experimental designs, going one-step beyond the traditional LSDs. We will explain this aspect in the next section. We will closely follow the model formulations as in SP (2009) but consider a simplified version of the same.

2 SuDoKu as a Postulated Experimental Design

In the context of a SuDoKu, we will refer to the 'inner regions' as forming 'internal block classification [IBC]' having potential advantage for accommodating an additional source of variation in the eu's. For example, we display a SuDoKu of order 6×6 having 3×2 i.e., 6 IBCs [Section 5]. We say that the totality of 36 experimental units are classified equally among the 6 internal blocks. It is interesting to note that each of these internal blocks contains all the treatment symbols in the SuDoKu ! That is what makes it very special for potential use as an experimental design, going one step beyond LSD in the sense of (i) accommodating one extra component of variation and (ii) providing 'orthogonal' estimation of all treatment contrasts. In the context of an agricultural experiment, for example, these internal blocks might represent 'differential fertility situations', besides the two external sources represented by row and column components. We will refer to this feature as 'LSDs with internal blocking' and these designs originate from traditional LSDs wherein we already have the three sources of variation, viz., row-to-row variation, column-to-column variation and treatment-to-treatment variation. Therefore, SuDoKus, as experimental designs, build upon the LSDs and accommodate one more component of variation through the concept and formation of 'internal blocks'. It would be interesting and instructional to examine the contribution of this new source of variation in the context of the ANOVA Table derived thereupon. This is elaborated in the next section. It may be noted that SP(2009) considered four different models in this context and provided detailed statistical analyses of the data, along with underlying ANOVA tables and computations of Sum of Squares. However, they failed to recognize that the contribution of SuDoKu squares is nonorthogonal to those of Rows and Columns of the design. Accordingly, their analyses went wrong in the sense that the decomposition of Total Sum of Squares accordingly to all the different models turned out to be invalid. We provide a corrected version in this paper.

3 ANOVA for SuDoKu Designs

3.1 Linear Model for SuDoKu Designs

We start with a SuDoKu of order $n = pq$ consisting of n internal blocks each of order $p \times q$. Each of these internal blocks corresponds to a group of p distinct rows and q distinct columns and there are $q\times p$ such internal blocks. Regarded as an experimental design, we will designate the above as a SuDoKu design with parameters $[n, p, q]$.

The postulated linear model has the obvious representation :

$$Y_{ijkl} = \mu + t_k + r_i + c_j + b_l + e_{ijkl} \qquad (1)$$

where
Y_{ijkl} = observation recorded for treatment k associated with the row-column combination (i, j); i = designated row involving treatment k and j = designated column involving treatment k; l = designated internal block label corresponding to (i, j) combination.

All model parameters have usual interpretations. We will use the obvious notations like
Y_i = i-th row total; $Y_{.j..}$ = j-th column total; etc.

It may be noted that the row-column combination (i, j) determines the label of the internal block l as also the treatment in a given SuDoKu design.

3.2 Orthogonal Decomposition of Total Sum of Squares [TSS]

In the absence of the internal block effects $[b_l; l = 1, 2, , n]$, we have the standard decomposition :

$$TSS = SSR + SSC + SSTr + SSE \qquad (2)$$

Regarding the newly added component of variation, we observe the following : (I) Internal Block Classification [IBC] is orthogonal to the treatments; (II) Part of IBC is orthogonal to the rows; (III) Part of IBC is orthogonal to the columns.

Further to this,
(IV) $(q-1)$ row contrasts involving q groups of row totals viz.,
$RT_1 = Y_1 + Y_2 + \ + Y_p; ..; RT_q = Y_{(n-p+1)} + Y_{(n-p+2)} + ..Y_n$ are confounded with $(q-1)$ IBC contrasts [This explains (II) above].

(V) $(p-1)$ column contrasts involving p groups of column totals viz.,
$CT_1 = Y_{.1..} + Y_{.2..} + \ + Y_{.q..}; ..; CT_p = Y_{.(n--q+1)..} + Y_{.(n--q+2)..} + .. + Y_{.n..}$ are confounded with $(p-1)$ IBC contrasts [This explains (III) above].

At this stage, it is instructional to point out that the IBC contrasts are estimated only in terms of the 'tetra differences'. Vide Shah and Sinha (1996). There are pq IBC or inner block parameters and consequently, only the $(p-1)(q-1)$ linearly independent tetra-differences provide estimates of IBC contrasts - free from row or column contrasts. All others are confounded. Specifically, $(q-1)$ IBC contrasts are confounded with the row contrasts and $(p-1)$ IBC contrasts are confounded with the column contrasts. That explains the decomposition of $(pq-1)$ IBC contrasts.

Naturally, error df under the new model = error df under the LSD model - df for orthogonal estimation of IBC contrasts free from row/column contrasts
$= (n^2-1) - 3(n-1) - (p-1)(q-1) = n^2 - 4n + p + q + 1$.

This explains the nature of the decomposition of the TSS. It is now a routine task to compute different components of TSS and prepare the ANOVA Table. Further, testing of hypotheses involving (i) treatment contrasts, (ii) row-contrasts, (iii) column-contrasts and (iv) estimable IBC contrasts is also a fairly routine exercise. We skip the details.

4 Mutually Orthogonal SuDoKu Arrangements

In the same spirit as the concept of Mutually Orthogonal Latin Squares [MOLS], Mutually Orthogonal SudoKu Squares [MOSS] have been introduced. Two SuDoKus of the same order $n = pq$ and of the *same* dimension of the inner regions, are said to form a pair of MOSLS if and only if, viewed as Latin Squares, these form a pair of MOLS of order n. Golombo (2006) asked if there is a pair of MOSS of order 9. The answer is 'yes' and there is more to it. Recently, there has been a growing interest in the constructional and existential aspects of SuDoKus and MOSSs. These are discussed in Bailey et al. (2008), Lorch (2009), Pedersen and Vis (2009, 2012), SP(2009), Lorch (2010, 2013), Meng and Lu (2011), Subramani (2012) and Fontana (2011), among others.

We give an example of 2 MOSS of order 4, with 2×2 IBC [Section 5]. It is known that there are exactly 6 MOSS of order 9. In general, for SudoKu of order $n = p^2$, p being a prime number, there are exactly $(p^2 - p)$ MOSS. Again, if the order is $n = k^2$ where $k = p_1^{e_1} p_2^{e_2} \ldots . . p_t^{e_t}$ indicating a prime factorization of k, then there are at least $(q^2 - q)$ MOSS where $q = min[p_1^{e_1}, p_2^{e_2}, \ldots . . , p_t^{e_t}]$. These can be constructed by following certain well-defined rules. We provide an example of two MOSS of order 8 [Section 5].

The data analysis underlying a design based on a collection of c MOSS each of order $n = p \times q$ follows in a routine manner with the following decomposition of the TSS.

$$TSS = Row\ SS + Column\ SS + All\ MOSS\ Components\ SS\ [except\ IBC\ SS]$$
$$+ IBC\ SS[with\ df\ (p-1)(q-1)] + Error\ SS\ [by\ differencing]$$

Note that the IBC classification is orthogonal to all c factors except the basic row and column classifications! Therefore, the SS computation for IBC classification follows the same formula as before - independent of the factors attributed to the c SuDoKus. Naturally, error df is given by $(n^2 - 1) - (c+2)(n-1) - (p-1)(q-1) = (n-1)(n-c-2) + p + q - 1$.

5 Selected SuDoKus

(i) SuDoKu of order 6 with $p = 3,\ q = 2$

$$\begin{pmatrix} 1 & 6 & * & 3 & 5 & * & 4 & 2 \\ 3 & 2 & * & 4 & 1 & * & 6 & 5 \\ 5 & 4 & * & 6 & 2 & * & 1 & 3 \\ * & * & * & * & * & * & * & * \\ 4 & 1 & * & 5 & 3 & * & 2 & 6 \\ 2 & 5 & * & 1 & 6 & * & 3 & 4 \\ 6 & 3 & * & 2 & 4 & * & 5 & 1 \end{pmatrix}$$

(ii) Two MOSS of order 4 with $p = q = 2$

$$\begin{array}{cc} \text{SuDoKu 1} & \text{SuDoKu 2} \\ \begin{pmatrix} 1 & 2 & * & 3 & 4 \\ 3 & 4 & * & 1 & 2 \\ * & * & * & * & * \\ 2 & 1 & * & 4 & 3 \\ 4 & 3 & * & 2 & 1 \end{pmatrix} & \begin{pmatrix} \alpha & \beta & * & \gamma & \delta \\ \delta & \gamma & * & \beta & \alpha \\ * & * & * & * & * \\ \gamma & \delta & * & \alpha & \beta \\ \beta & \alpha & * & \delta & \gamma \end{pmatrix} \end{array}$$

(iii) Two MOSS of order 8 with $p = 4,\ q = 2$

$$\begin{array}{cc} \text{SuDoKu 1} & \text{SuDoKu 2} \\ \begin{pmatrix} 8 & 1 & * & 2 & 4 & * & 3 & 6 & * & 5 & 7 \\ 2 & 4 & * & 8 & 1 & * & 5 & 7 & * & 3 & 6 \\ 3 & 7 & * & 5 & 6 & * & 8 & 4 & * & 2 & 1 \\ 5 & 6 & * & 3 & 7 & * & 2 & 1 & * & 8 & 4 \\ * & * & * & * & * & * & * & * & * & * & * \\ 1 & 8 & * & 4 & 2 & * & 7 & 5 & * & 6 & 3 \\ 4 & 2 & * & 1 & 8 & * & 6 & 3 & * & 7 & 5 \\ 6 & 5 & * & 7 & 3 & * & 4 & 8 & * & 1 & 2 \\ 7 & 3 & * & 6 & 5 & * & 1 & 2 & * & 4 & 8 \end{pmatrix} & \begin{pmatrix} 8 & 1 & * & 2 & 4 & * & 3 & 6 & * & 5 & 7 \\ 6 & 5 & * & 7 & 3 & * & 4 & 8 & * & 1 & 2 \\ 7 & 3 & * & 6 & 5 & * & 1 & 2 & * & 4 & 8 \\ 2 & 4 & * & 8 & 1 & * & 5 & 7 & * & 3 & 6 \\ * & * & * & * & * & * & * & * & * & * & * \\ 5 & 6 & * & 3 & 7 & * & 2 & 1 & * & 8 & 4 \\ 1 & 8 & * & 4 & 2 & * & 7 & 5 & * & 6 & 3 \\ 3 & 7 & * & 5 & 6 & * & 8 & 4 & * & 2 & 1 \\ 4 & 2 & * & 1 & 8 & * & 6 & 3 & * & 7 & 5 \end{pmatrix} \end{array}$$

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